Does the Shannon Bound Really Apply to Data Structures?

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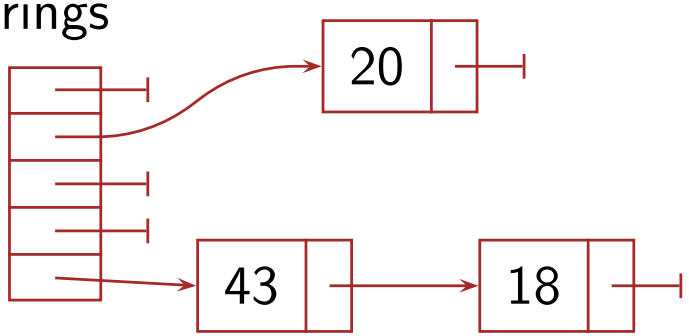
Introduction

- Consider representing a value drawn from a set of possible values
- Information theory tells how many bits are needed on the average
  - different values may use different numbers of bits
  - a white flag contains little information — except at war!
- Classic information theory uses contiguous bit strings
- Data structures are not like that
- Of course, extending classic information theory to data structures is trivial, isn't it?



  - e.g., list all bits in a sequence — but xor-pointers?
  - e.g., list visited bits in a sequence — but does that suffice?
    "the value" is the whole phonebook, not an individual name–number pair
  - well, can't do it now but *everyone knows that it is trivial*
- My claim is that it is **not trivial** but **not very hard either**
  - it is at the right level for a symposium paper and talk

Introduction

- Is it really obvious that it should be trivial?
  - McMillan's 1956 extension of the application area of Shannon's bound:

    "... for any complex x such that $|xa| < 1$ the infinite series $1 + N(1)x + N(2)x^2 + \cdots$ converges ..."
  - (the results in my paper are more trivial than that)
- The results in my paper:
  - an example demonstrating that the classic proof fails for data structures

  $\Rightarrow$ solid (even if simple) theory is needed instead of just handwaving
  - a suitable rigorous (and simple) definition of data structures
  - a theorem that extends classic theory to all non-weird data structures
  - the observation that given the above, the bound for weird d.s. is trivial ...
  - ... but non-optimality of redundant representations is not:
    unlike in classic theory, it need not hold if probabilities may be 0
  - a non-optimality theorem for redundant representations if all $p_i > 0$
- To discuss these, first we need to look at classic information theory

Classic Information Theory

- If all values are equally likely
  - 8 bits suffice to represent $2^8 = 256$ different values
  - to represent one of N values, $\lceil \lg N \rceil$ bits are needed
- If value i has probability p_i and *representation* (= *codeword*) r_i
  - short codewords for common and long codewords for rare values
  - e.g., Tallinn = 00, Tartu = 01, Pärnu = 1
  - average memory usage =

$$\sum_{i \in I} p_i |r_i|$$

- *Unique decodability*
  - a word in Finnish

(gold-, etc.)mine (entrance)hole

K A I V O S A U K K O

(water)well otter

- *Self-delimiting bit strings* (= prefix codes)
  - no codeword is a prefix of another

$\Rightarrow$ codeword can be uniquely decoded immediately after reading its last bit

Classic Information Theory

- **Shannon:** with self-delimiting bit strings at least

$$-\sum_{i \in I} p_i \lg p_i$$

  bits are needed on the average

- The only thing the proof assumes of the encoding is **Kraft's inequality**

$$\sum_{i \in I} 2^{-|r_i|} \leq 1$$

- **McMillan:** Kraft's inequality holds for *all* uniquely decodable bit string sets

  $\Rightarrow$ also the $-\sum_{i \in I} p_i \lg p_i$ bound holds

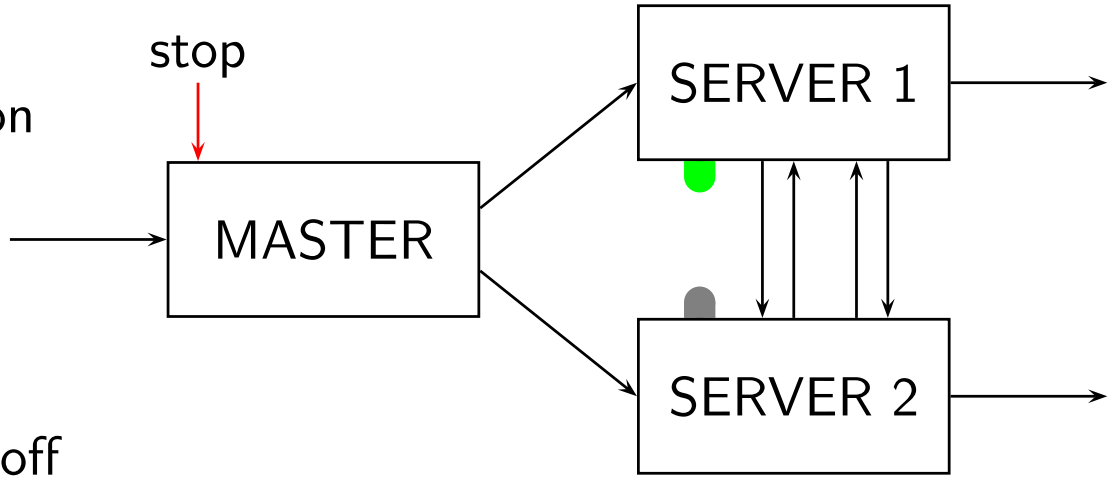
  - McMillan's proof is unexpectedly difficult!

- Unlike data structures

  - never two codewords for the same value
    (would strictly increase memory consumption, even if $p_i = 0$)
  - all codewords are contiguous bit strings with clear start and end

A Weird Data Structure

- A concurrent system
  - synchronous communication (like Ada, Occam, CSP)
  - no sub-sub tasks



- Master's view to activity bits

  00 = can switch off
  01, 10, 11 = must still wait

  $\Rightarrow$ one bit of information

- A modification saving memory
  - when a server is serving a sub-task, the other is serving an original task

  $\Rightarrow$ the activity bit of the other is certainly 1
  $\Rightarrow$ the master gets the right result independently of the bit of the first server
  $\Rightarrow$ the first server can freely use the bit for something else

- **When serving a sub-task, a server uses its activity bit for something else than denoting activity, and this does not fool the master**

A Weird Data Structure

- Let – denote that the activity bit is used for something else
  - **master does not know that**, and may accidentally read it getting 0 or 1
  - the other use determines the value absolutely freely as it needs

  $\Rightarrow$ the bit must not be counted into the memory consumption of activity info

$\Rightarrow$ Memory consumption

- 00, 01, 10, 11: two bits
- 1–, –1: one bit

- Put these numbers to Kraft's inequality

$$\sum_{i \in I} 2^{-|r_i|} = 2^{-2} + 2^{-2} + 2^{-2} + 2^{-2} + 2^{-1} + 2^{-1} = 2 > 1$$

- Leave 01, 10, and 11 out, since they are covered by 1– and –1

$$\sum_{i \in I} 2^{-|r_i|} = 2^{-2} + 2^{-1} + 2^{-1} = 1\frac{1}{4} > 1$$

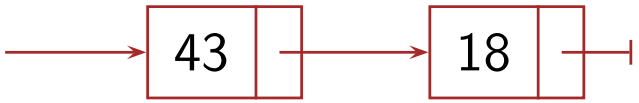
$\Rightarrow$ Kraft's inequality is violated

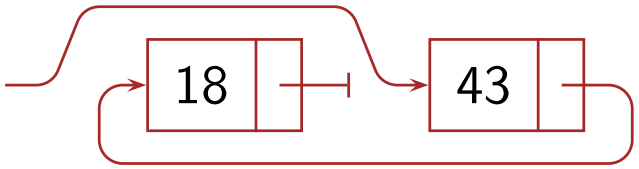
- The essence here is that 1– and –1 are two distinct representations of size 1

Formalization of Data Structures

- What is a data structure?
  - uses memory to represent one value from a set of possible values
- What is a representation?
  - uses some bits of memory and does not use the remaining bits
  - assigns 0 or 1 to each used bit

  $\Rightarrow$ can be formalized as $r = (B_0, B_1)$ such that $B_0 \cap B_1 = \emptyset$
  - the elements of B_0 and B_1 are, e.g., memory addresses
- The same value may have many representations
  - this happens all the time with data structures





- Representations of **different values** must be distinguishable from each other
  - if (B_0, B_1) and (B'_0, B'_1) represent different values, they must have a *conflicting bit*

  $\Rightarrow$ we require $B_0 \cap B'_1 \neq \emptyset$ or $B_1 \cap B'_0 \neq \emptyset$

Theorems in the Paper

- A representation scheme is *fully conflicting* iff each pair of distinct representations has a conflicting bit
- **Theorem** If it is fully conflicting, then Kraft's inequality holds.

  $\Rightarrow$ violating Kraft's inequality requires two indistinguishable representations for the same value (like 1– and –1)

  – proof of theorem: $\frac{1}{2}$ pages on the paper, induction on $|R|$, from R to $R_0, R_1, R_\perp$ to $R'_0 \cup R_\perp, R'_1 \cup R_\perp$
- **All non-weird data structures are now covered**
- Trivial observation: leaving out redundant representations does not increase average memory consumption, if the one that is kept is minimal

  – yields a fully conflicting representation scheme

  $\Rightarrow$ Shannon's bound holds
- **Also the weird data structures are now covered "bound-wise"**
- However, another important classic result deserves new analysis

Theorems in the Paper

- In classic theory, redundant representations always imply non-optimality
  - deepens the significance of the bound

$\Rightarrow$ Of course I wanted to prove the same for data structures

- Observation: it does *not* hold fully generally
- **Theorem** If at least one object has two representations, and if the probability of every representation is > 0, then the scheme is not memory-optimal.
  - proof of theorem: 1 page on the paper $+ \frac{1}{2}$ pages to support intuition, construct two fully conflicting schemes such that $\mathcal{U}(R)$ and R use the same amount of memory, and $1 \geq \sum_{r \in \mathcal{E}(R)} 2^{-|r|} > \sum_{r \in \mathcal{U}(R)} 2^{-|r|}$
- **For weird data structures, used redundancy $\Rightarrow$ non-optimality**
- Classic representations qualify as data structures
- An optimal self-delimiting representation exists in classic theory

$\Rightarrow$ **The bound is strict also with data structures**

Discussion

- A hole in the proof of a very fundamental result was revealed and fixed
  - said like that, it sounds incredible — is the paper really okay?
- Anticipated objection: It must be trivial, the paper just makes it complicated
  - I still have not seen a trivial argument
- Let's try to make it trivial: If the same value has many representations, only a shortest should be taken, and then Kraft's inequality holds
  - some justification is needed that then Kraft's inequality holds
  - to do that, we need to say what we mean by a data structure

  $=$ Definition 5 and Theorem 6 of the paper

  $\Rightarrow$ more or less the same reasoning as in the paper
- Furthermore, "redundancy $\Rightarrow$ non-optimality" was brought to data structures
  - had to add the (rather weak) assumption "used redundancy"
- **The paper sorts out an unimportant but real pathological case**

Discussion

- Is this a major result? **No!**
  - Shannon's bound for data structures was *not* changed, only its proof

  $\Rightarrow$ does not affect anyone
  - the situation where Kraft's inequality was violated is unusual
  - the redundancy theorem remains impossible to cheat
  - although the proofs are not trivial, they are not horribly difficult either (much of the paper is not proofs but telling why they are needed)
- So, is this a zero result? Not in my opinion.
  - faith in the bound and "used redundancy $\Rightarrow$ non-optimality" for data structures need not any more be based on handwaving
- Perhaps this is a result that inspires a lively after-talk discussion?
  - unless I have overrun my time slot...

Thank you for attention!