A Completeness Proof for A Predicate Logic with Undefined Truth Value

Antti Valmari
University of Jyväskylä
&
Esko Turunen
Tampere University

1 Motivation of Our Logic

All examples are on $\mathbb{R}$

It seems natural to express case analysis with propositional operators:

$$
\begin{aligned}
& 1 + \sqrt{|x|} = |x - 1| \\
\Leftrightarrow\ & \begin{cases}
(\quad x < 0 \ \wedge \ 1 + \sqrt{-x} \ = \ -x + 1 \ ) \ \vee \\
(\ 0 \le x < 1 \ \wedge \ 1 + \sqrt{x} \ = \ -x + 1 \ ) \ \vee \\
(\ 1 \le x \ \wedge \ 1 + \sqrt{x} \ = \ x - 1 \ )
\end{cases} \\
\Leftrightarrow\ & x = -1 \ \vee \ x = 0 \ \vee \ x = 4
\end{aligned}
$$

When $x = 4$, the first and last lines are clearly true ...

... but what does the second line mean?

$$4 < 0 \ \wedge \ 1 + \sqrt{-4} \ = \ -4 + 1$$

We want to write $\dfrac{6}{x - 1} = x \ \Leftrightarrow \ x = -2 \ \vee \ x = 3$, how to make it okay?

- when $x = 1$, it yieds undefined $\Leftrightarrow$ false

2 Earlier Logics

Problem: undefined should sometimes behave like F, but its negation must not yield T

Many diverse approaches exist

Example: Formal software development method Z

- each closed formula is either T or F, but we do not always know which one
- $4 < 0 \ \wedge\ 1 + \sqrt{-4} \ = \ -4 + 1 \ \Leftrightarrow\ \mathsf{F} \wedge X \ \Leftrightarrow\ \mathsf{F}$
- we do not know if $x = 1$ is a root of $\dfrac{6}{x-1} = x$
- $x \neq 1 \wedge \dfrac{6}{x-1} = x \ \Leftrightarrow\ x = -2 \vee x = 3$ is valid in Z, but ...
- ... I was taught at school $\dfrac{6}{x-1} = x \ \Leftrightarrow\ x \neq 1 \wedge 6 = x(x-1) \ \Leftrightarrow\ x = -2 \vee x = 3$

Example: Formal software development method VDM

- evaluation of an undefined term is never finished
- U denotes that the evaluation of a truth value is never finished
- because F ∧ F ⇔ T ∧ F ⇔ F, we have U ∧ F ⇔ F

⇒ Kleene's ternary logic

¬	
F	T
U	U
T	F

∧	F	U	T
F	F	F	F
U	F	U	U
T	F	U	T

∨	F	U	T
F	F	U	T
U	U	U	T
T	T	T	T

→	F	U	T
F	T	T	T
U	U	U	T
T	F	U	T

↔	F	U	T
F	T	U	F
U	U	U	U
T	F	U	T

- $4 < 0 \ \wedge \ 1 + \sqrt{-4} \ = \ -4 + 1 \ \Leftrightarrow$ F ∧ U ⇔ F
- $x \neq 1 \wedge \dfrac{6}{x-1} = x \ \Leftrightarrow \ x = -2 \vee x = 3$ is valid also in VDM
- it is undefined if $x = 1$ is a root of $\dfrac{6}{x-1} = x$

Regularity

- either $\varphi(\ldots, \mathsf{U}, \ldots)$ yields U or $\varphi(\ldots, P, \ldots)$ does not depend on P

⇒ cannot express "P yields U"

3 Guiding Principles of Our Logic

Function symbols are *strict*

- if t_i is undefined, then $f(\ldots, t_i, \ldots)$ is undefined
- e.g., $1 + \dfrac{x}{0}$ is undefined

A relation yields U if and only if at least one argument is undefined

- in particular, $t = t'$ yields U if and only if t or t' or both are undefined
- "only if"-part is only technical convenience

The negation of any undefined claim is undefined

- if $x = 0$, then both $\dfrac{1}{x} \geq 0$ and $\dfrac{1}{x} < 0$ yield U

Variables are always defined, terms may be undefined

- e.g., $x = x \Leftrightarrow$ T, but $\dfrac{1}{x} = \dfrac{1}{x} \Leftrightarrow x \neq 0$
- e.g., $\exists x : x = 0 \Leftrightarrow$ T, but $\exists x : x = \dfrac{1}{0} \Leftrightarrow$ U $\Leftrightarrow$ F

The symbol $\Rightarrow$ is not a propositional operator but a reasoning operator

- U $\Leftrightarrow$ F, so $\frac{6}{x-1} = x \;\Leftrightarrow\; x = -2 \lor x = 3$ and $x = 1$ is not a root of $\frac{6}{x-1} = x$
- $\neg\left(\frac{1}{x} > 0 \Rightarrow x > 0\right)$ is a syntax error, so we cannot derive $x \leq 0 \Rightarrow \frac{1}{x} \leq 0$
- we no longer have "$\varphi \Rightarrow \psi$ is valid if and only if $\neg\varphi \lor \psi$ is a tautology"
- actually, it is questionable whether we ever really had it
  - e.g., $x(x - |x|) = 18$
    case $x < 0$: $x(x - -x) = 18 \Leftrightarrow 2x^2 = 18 \Leftrightarrow x = -3$
    case $x \geq 0$: $x(x - x) = 18 \Leftrightarrow 0 = 18 \Leftrightarrow$ F

$\rightarrow$	F	U	T
F	T	T	T
U	U	U	T
T	F	U	T

$\Rightarrow$	F	U	T
F	$\surd$	$\surd$	$\surd$
U	$\surd$	$\surd$	$\surd$
T	–	–	$\surd$

$\leftrightarrow$	F	U	T
F	T	U	F
U	U	U	U
T	F	U	T

$\Leftrightarrow$	F	U	T
F	$\surd$	$\surd$	–
U	$\surd$	$\surd$	–
T	–	–	$\surd$

- beyond an example later on, $\Rightarrow$ and $\Leftrightarrow$ are not studied in this talk
  - their laws are studied elsewhere
  - regularity simplifies some of them

To argue that our logic is healthy, we will present a sound and complete proof system

4 "Is Defined" -Formulas

Terms, formulas, etc., are defined as usually, except $\Rightarrow$, $\Leftrightarrow$, $\rightarrow$, and $\leftrightarrow$ are left out

- $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$ may be defined as $\neg\varphi \vee \psi$ and $(\varphi \wedge \psi) \vee \neg(\varphi \vee \psi)$

Every function symbol f has an associated formula $\lfloor f \rceil$ that specifies when f is defined

- e.g., $\lfloor \sqrt{x} \rceil$ is $x \geq 0$ (literally!)
- e.g., $\left\lfloor \frac{x}{y} \right\rceil$ is $\neg(y = 0)$ (literally!)
- $\lfloor f \rceil(x_1, \ldots, x_n)$ must always be defined (but $\lfloor f \rceil(t_1, \ldots, t_n)$ need not)
  - e.g., $\lfloor f \rceil$ uses no function symbols (or only total function symbols)

The $\lfloor f \rceil$ are given (i.e., reside in axioms and proof rules)

- e.g., $\forall x : \forall y : \neg(\neg(y = 0)) \vee \frac{x}{y} \cdot y = x$
- e.g., $\{\neg(x = 0)\} \vdash \frac{x+1}{x} = \frac{x+1}{x}$

The $\lfloor f \rceil$ generate $\lfloor t \rceil$ and $\lfloor \varphi \rceil$ as follows

A function invocation is defined iff every argument is and the function itself is

$$\lfloor f(t_1, \ldots, t_n) \rceil \text{ means } \lfloor t_1 \rceil \wedge \cdots \wedge \lfloor t_n \rceil \wedge \lfloor f \rceil (t_1, \ldots, t_n)$$

- constant and variable symbols are always defined; e.g., $\lfloor x \rceil$ is $\lfloor 3 \rceil$ is T
- e.g., $\left\lfloor \frac{x}{\sqrt{y}} \right\rceil$ is $\mathsf{T} \wedge (\mathsf{T} \wedge y \geq 0) \wedge \neg(\sqrt{y} = 0)$

A relation invocation is defined iff every argument is

$$\lfloor R(t_1, \ldots, t_n) \rceil \text{ means } \lfloor t_1 \rceil \wedge \cdots \wedge \lfloor t_n \rceil$$
$$\lfloor t = t' \rceil \text{ means } \lfloor t \rceil \wedge \lfloor t' \rceil$$

Propositional rules

- $\lfloor \mathsf{F} \rceil$ is $\lfloor \mathsf{T} \rceil$ is T — and $\lfloor \mathsf{U} \rceil$ would be F
- $\lfloor \neg\varphi \rceil$ is $\lfloor \varphi \rceil$
- $\lfloor \varphi \wedge \psi \rceil$ is $(\lfloor \varphi \rceil \wedge \lfloor \psi \rceil) \vee (\lfloor \varphi \rceil \wedge \neg\varphi) \vee (\lfloor \psi \rceil \wedge \neg\psi)$
- $\lfloor \varphi \vee \psi \rceil$ is $(\lfloor \varphi \rceil \wedge \lfloor \psi \rceil) \vee (\lfloor \varphi \rceil \wedge \varphi) \vee (\lfloor \psi \rceil \wedge \psi)$

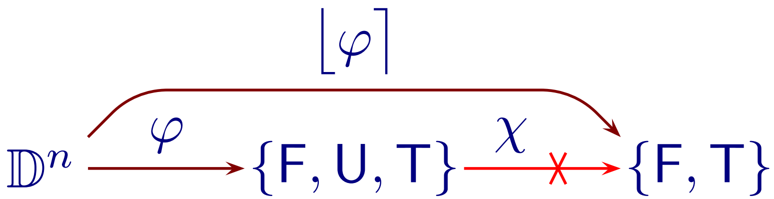
Quantifier rules

- $\lfloor \forall x : \varphi(x) \rceil$ is $(\forall x : \lfloor \varphi(x) \rceil) \vee \exists x : \lfloor \varphi(x) \rceil \wedge \neg\varphi(x)$
- $\lfloor \exists x : \varphi(x) \rceil$ is $(\forall x : \lfloor \varphi(x) \rceil) \vee \exists x : \lfloor \varphi(x) \rceil \wedge \varphi(x)$

$\lfloor \cdots \rceil$ is not an operator in the language, but an abbreviation

- given t, $\lfloor t \rceil$ can be constructed automatically
- given φ, $\lfloor \varphi \rceil$ can be constructed automatically

For each $\varphi : \mathbb{D}^n \to \{\mathsf{F}, \mathsf{U}, \mathsf{T}\}$ there is $\lfloor \varphi \rceil : \mathbb{D}^n \to \{\mathsf{F}, \mathsf{T}\}$ that yields F iff φ yields U, but there is no $\chi : \{\mathsf{F}, \mathsf{U}, \mathsf{T}\} \to \{\mathsf{F}, \mathsf{T}\}$ such that each $\lfloor \varphi \rceil$ can be expressed as $\chi(\varphi)$



5 An Example

Thanks to regularity, the following is sound:

Assume $\lfloor t \rceil \Rightarrow t = t'$ and $\lfloor t \rceil \Rightarrow \chi \Rightarrow \lfloor t \rceil \vee \neg \lfloor t' \rceil$

- if $R(t)$ is in the scope of an even number of negations, then $\varphi(R(t)) \Leftrightarrow \varphi(\chi \wedge R(t'))$
- if $R(t)$ is in the scope of an odd number of negations, then $\varphi(R(t)) \Leftrightarrow \varphi(\neg\chi \vee R(t'))$

Let

- $t = \frac{6}{x-1}(x-1)$
- $t' = 6$
- χ be $\lfloor t \rceil$ (which is $x \neq 1$)
- $\varphi(R(t))$ be $R(t)$ be $t = x(x-1)$

We get

$$\frac{6}{x-1} = x \;\Leftrightarrow\; \frac{6}{x-1}(x-1) = x(x-1) \;\;\Leftrightarrow\;\; x \neq 1 \wedge 6 = x(x-1)$$

6 Sound and Complete Proof System

Notation

- φ, ψ, χ are formulas
- Γ, Δ are sets of formulas
- x, x_i, y are variable symbols
- t, t_i, t'_i are terms

Rules about reasoning in general:

P1: $\{\varphi\} \vdash \varphi$

P2: If $\Gamma \vdash \varphi$ then $\Gamma \cup \Delta \vdash \varphi$

P3: If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \psi$

The Law of the Excluded Fourth and the concept of contradiction:

C1: $\emptyset \vdash \varphi \vee \neg\varphi \vee \neg\lfloor\varphi\rceil$ (this replaces the Law of Excluded Middle)

C2: $\{\mathsf{F}\} \vdash \varphi$

C3: $\{\varphi,\ \neg\varphi\} \vdash \mathsf{F}$

(C2 and C3) or $\{\varphi,\ \neg\varphi\} \vdash \psi$

If a formula is true, then it is also defined:

D1: $\{\varphi\} \vdash \lfloor\varphi\rceil$ (this does not exist in classical logic)

For instance

- D1: $\left\{\dfrac{\sqrt{x}}{x-1} > 0\right\} \vdash x \geq 0 \wedge \neg(x-1=0)$
- C1: $\emptyset \vdash \dfrac{\sqrt{x}}{x-1} > 0 \vee \neg\left(\dfrac{\sqrt{x}}{x-1} > 0\right) \vee \neg(x \geq 0 \wedge \neg(x-1=0))$

If the system is not contradictory

- that is, if $\Gamma \not\vdash \mathsf{F}$
- please recall that $\lfloor\neg\varphi\rceil$ is $\lfloor\varphi\rceil$

$\Rightarrow$ C1 and D1 make precisely one of φ, $\neg\varphi$, and $\neg\lfloor\varphi\rceil$ hold

$\Rightarrow$ each claim yields precise one of T, F, and U for each binding

Rules for conjunction and disjunction:

$\wedge$-**I:** $\{\varphi, \psi\} \vdash \varphi \wedge \psi$

$\wedge$-**E1:** $\{\varphi \wedge \psi\} \vdash \varphi$

$\wedge$-**E2:** $\{\varphi \wedge \psi\} \vdash \psi$

$\vee$-**I1:** $\{\varphi\} \vdash \varphi \vee \psi$

$\vee$-**I2:** $\{\psi\} \vdash \varphi \vee \psi$

$\vee$-**E:** If $\Gamma \cup \{\varphi\} \vdash \chi$ and $\Gamma \cup \{\psi\} \vdash \chi$, then $\Gamma \cup \{\varphi \vee \psi\} \vdash \chi$

Rules of equality:

=-1: $\{\lfloor t \rceil\} \vdash t = t$

=-2: If f is an n-ary function symbol and $1 \leq i \leq n$, then

$$\{t_i = t_i',\ \lfloor f(t_1, \ldots, t_n) \rceil\} \vdash f(t_1, \ldots, t_n) = f(t_1, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_n)$$

=-3: If $\varphi(x_1, \ldots, x_n)$ is a formula, $1 \leq i \leq n$ and t_i and t_i' are free for x_i in φ, then

$$\{t_i = t_i',\ \varphi(t_1, \ldots, t_n)\} \vdash \varphi(t_1, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_n)$$

For instance

- =-1: $\{x \geq 0 \wedge \neg(x - 1 = 0)\} \vdash \dfrac{\sqrt{x}}{x-1} = \dfrac{\sqrt{x}}{x-1}$
- =-2: $\{x = z + 1,\ x \geq 0 \wedge \neg(x - 1 = 0)\} \vdash \dfrac{\sqrt{x}}{x-1} = \dfrac{\sqrt{x}}{z+1-1}$

Comments

- =-1 and =-2 were tailored to not prove an undefined term equivalent to something
- by definition, $\lfloor f(t_1, \ldots, t_n) \rceil$ yields $\lfloor t_1 \rceil$, ..., $\lfloor t_n \rceil$
- by D1, $t_i = t_i'$ implies $\lfloor t_i = t_i' \rceil$, which is $\lfloor t_i \rceil \wedge \lfloor t_i' \rceil$, so $\lfloor t_i' \rceil$
- =-3 need only be assumed for relations, but proving that is too long and dull

Rules for quantifiers:

$\forall$-E: If t is free for x in φ, then $\{\lfloor t \rceil, \forall x : \varphi(x)\} \vdash \varphi(t)$

$\forall$-I: If $\Gamma \vdash \varphi(x)$ and x does not occur free in Γ, then $\Gamma \vdash \forall x : \varphi(x)$

$\exists$-I: If t is free for x in φ, then $\{\varphi(t)\} \vdash \exists x : \varphi(x)$

$\exists$-E: If $\Gamma \cup \{\varphi(y)\} \vdash \psi$ and y does not occur in Γ, $\exists x : \varphi(x)$, nor in ψ,
then $\Gamma \cup \{\exists x : \varphi(x)\} \vdash \psi$

Comments

- $\forall$-E was tailored to not prove anything about undefined terms
- variable symbols are never undefined, so $\forall$-I and $\exists$-E need not be tailored
- $\exists$-I need not be $\{\varphi(t), \lfloor t \rceil\} \vdash \exists x : \varphi(x)$
  - if t is undefined but $\varphi(t)$ is not, then by regularity $\forall x : \varphi(x)$ holds

Only 5 differences from binary logic!

7 Completeness Proof

We use Henkin's strategy: prove that every consistent theory has a model

$\Rightarrow$ if $\Gamma \not\vdash \varphi$, then $\Gamma \cup \{\neg\varphi\}$ has a model, so φ is not a semantic consequence of Γ

- therefore, we assume from now on $\Gamma \not\vdash \mathsf{F}$

Lemma There is Γ' such that

- $\Gamma' \not\vdash \mathsf{F}$
- both or neither of Γ and Γ' have a model
- infinitely many variable symbols are unused in Γ'
- for every bound x in Γ' there is an x' such that its only occurrence in Γ' is $x = x'$

Proof Replace each v_i in Γ by v_{3i} and add the $v_{3i} = v_{3i-1}$. $\square$

Choose true formulas, introduce witnesses

- let $\Gamma_0 := \Gamma'$
- for every formula φ_i, construct Γ_i

if	φ_i form	$\Gamma_i := \Gamma_{i-1} \cup$
$\Gamma_{i-1} \cup \{\lfloor \varphi_i \rceil\} \vdash$ F		$\{\neg\lfloor \varphi_i \rceil\}$
$\Gamma_{i-1} \cup \{\varphi_i\} \not\vdash$ F	is $\exists x : \psi(x)$	$\{\varphi_i,\ \psi(y)\}$
$\Gamma_{i-1} \cup \{\neg\varphi_i\} \not\vdash$ F	is $\forall x : \psi(x)$	$\{\neg\varphi_i, \neg\psi(y)\}$
$\Gamma_{i-1} \cup \{\varphi_i\} \not\vdash$ F	not $\exists x : \psi(x)$	$\{\varphi_i\}$
$\Gamma_{i-1} \cup \{\neg\varphi_i\} \not\vdash$ F	not $\forall x : \psi(x)$	$\{\neg\varphi_i\}$

- let $\Gamma_\omega := \Gamma_0 \cup \Gamma_1 \cup \cdots$

Lemma

- $\Gamma' = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_\omega$
- $\Gamma_\omega \not\vdash$ F
- for each φ, precisely one of φ, $\neg\varphi$ and $\neg\lfloor\varphi\rceil$ is in Γ_ω
- for each φ, precisely one of $\lfloor\varphi\rceil$ and $\neg\lfloor\varphi\rceil$ is in Γ_ω
- for each t, precisely one of $\lfloor t\rceil$ and $\neg\lfloor t\rceil$ is in Γ_ω
- $\Gamma_\omega \vdash \varphi$ if and only if $\varphi \in \Gamma_\omega$

Theorem Γ_ω has a model

Proof

- elements of the universe are
  - equivalence classes of terms for which $\lfloor t \rceil \in \Gamma_\omega$, induced by the $t = t'$ in Γ_ω
  - $\perp$ for the remaining terms
- nothing depends on the choice of the representative of each equivalence class
  - where necessary, use v_{3i-1} to make terms free for x

$\in \Gamma_\omega$	φ	$\neg\varphi$	$\neg\lfloor\varphi\rceil$
truth value of φ	T	F	U

- some routine arguments
- lots of dull reasoning using the proof system □

Corollary Both Γ' and Γ have a model

8 Extension to Łukasiewicz Logic

Łukasiewicz: $\mathsf{U} \xrightarrow{\text{Ł}} \mathsf{U}$ yields T and $\mathsf{U} \xleftrightarrow{\text{Ł}} \mathsf{U}$ yields T

- that P yields U can be expressed as $(P \xrightarrow{\text{Ł}} \neg P) \wedge (\neg P \xrightarrow{\text{Ł}} P)$
- all truth functions $\{\mathsf{F}, \mathsf{U}, \mathsf{T}\}^n \to \{\mathsf{F}, \mathsf{U}, \mathsf{T}\}$ can be expressed

This reduces to the earlier case by replacing each

$$\varphi \xrightarrow{\text{Ł}} \psi$$

by

$$\neg\varphi \vee \psi \vee \neg(\lfloor\varphi\rceil \vee \lfloor\psi\rceil)$$

9 Conclusions

Key ideas

- $\Rightarrow$ and $\Leftrightarrow$ are employed to express school reasoning
  - cannot be interpreted as propositional operators
- $\frac{1}{0}$, etc., are not treated as values
  - variables are never undefined, terms may be
- the intuitive notion "is defined" is encoded as mechanical rules
  - "is defined" is itself always defined
- regularity simplifies things
- for each φ, the model contains precisely one of φ, $\neg\varphi$ and $\neg\lfloor\varphi\rceil$
  - correspondingly φ yields T, F or U

Many practical reasoning laws have been developed

- would be a topic for another talk

Thank you for attention! Questions?