Scattering theory

Solutions to Exercises #9, 10.12.2007

1. Let $\varphi \in C_0^\infty(\mathbf{R})$, and assume $\mathrm{supp}(\varphi) \subset [-R, R]$. Then

$$\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}\,dx = \int_{\varepsilon\leq|x|\leq R} \frac{\varphi(x)-\varphi(0)}{x}\,dx$$

since $1/x$ is an odd function. We have

$$\Big|\frac{\varphi(x)-\varphi(0)}{x}\Big| \leq \frac{\|\nabla\varphi\|_{L^\infty}|x|}{|x|} = \|\nabla\varphi\|_{L^\infty}.$$

This shows that $\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\varphi(x)/x\,dx$ is well defined, and we obtain a linear map p.v. $1/x$ on $C_0^\infty(\mathbf{R})$ satisfying

$$|\langle \text{p.v.}\,\frac{1}{x}, \varphi\rangle| \leq 2R\|\nabla\varphi\|_{L^\infty}, \quad \varphi \in C_0^\infty([-R,R]).$$

Thus p.v. $1/x \in \mathscr{D}'(\mathbf{R})$.

We have $\log|x| \in L^1_{\mathrm{loc}}(\mathbf{R})$ since the function is continuous outside 0 and

$$\int_0^1 |\log\,x|\,dx = \int_0^\infty te^{-t}\,dt < \infty$$

by the substitution $x = e^{-t}$. Thus $\log|x| \in \mathscr{D}'(\mathbf{R})$, and for $\varphi \in C_0^\infty([-R,R])$ one has

$$\begin{aligned}\langle \frac{d}{dx}\log|x|, \varphi\rangle &= -\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x|\leq R}\log|x|\;\varphi'(x)\,dx \\ &= (\varphi(\varepsilon)-\varphi(-\varepsilon))\log\,\varepsilon + \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}\,dx.\end{aligned}$$

Since $|(\varphi(\varepsilon)-\varphi(-\varepsilon))\log\,\varepsilon| \leq 2\|\nabla\varphi\|_{L^\infty}\varepsilon|\log\,\varepsilon|$ and

$$\lim_{\varepsilon\to 0}\varepsilon|\log\,\varepsilon| = \lim_{t\to\infty} te^{-t} = 0$$

by taking $\varepsilon = e^{-t}$, we obtain $d/dx(\log|x|) = \text{p.v.}\,1/x$.

2. If $\varepsilon > 0$ then $1/(x \pm i\varepsilon)$ is a C^∞ function, and if $\varphi \in C_0^\infty([-R, R])$ then

$$\begin{aligned}\langle \frac{1}{x+i\varepsilon} + \frac{1}{x-i\varepsilon}, \varphi\rangle &= 2\int \frac{x}{x^2+\varepsilon^2}\varphi(x)\,dx = 2\lim_{\delta\to 0}\int_{|x|\geq\delta} \frac{x}{x^2+\varepsilon^2}\varphi(x)\,dx \\ &= 2\langle \text{p.v.}\frac{1}{x}, \varphi\rangle + 2\lim_{\delta\to 0}\int_{\delta\leq|x|\leq R}\Big(\frac{x}{x^2+\varepsilon^2} - \frac{1}{x}\Big)\varphi(x)\,dx \\ &= 2\langle \text{p.v.}\frac{1}{x}, \varphi\rangle - 2\varepsilon^2\lim_{\delta\to 0}\int_{\delta\leq|x|\leq R}\frac{1}{x(x^2+\varepsilon^2)}\varphi(x)\,dx \\ &= 2\langle \text{p.v.}\frac{1}{x}, \varphi\rangle - 2\lim_{\delta\to 0}\int_{\delta\leq|t|\leq R/\varepsilon}\frac{1}{t(t^2+1)}(\varphi(\varepsilon t) - \varphi(0))\,dx,\end{aligned}$$

where in the last equality we used the substitution $x = \varepsilon t$ and the fact that $1/t(t^2+1)$ is odd in t. The integrand in the last integral is bounded by $\|\nabla\varphi\|_{L^\infty}\varepsilon|t|/|t|(t^2+1) = \|\nabla\varphi\|_{L^\infty}\varepsilon/(t^2+1)$. Since $1/(t^2+1)$ is integrable in $\mathbf{R}$, we conclude that the limit of the last integral as $\delta \to 0$ and $\varepsilon \to 0$ is zero. This shows the claim.

3. If $a > -1$ then x_+^a is in $L^1_{\text{loc}}(\mathbf{R})$ and hence in $\mathscr{D}'(\mathbf{R})$. For $a > 0$ and $\varphi \in C_0^\infty(\mathbf{R})$ we have

$$\langle \frac{d}{dx}x_+^a, \varphi\rangle = -\lim_{\delta\to 0}\int_\delta^\infty x^a\varphi'(x)\,dx = \lim_{\delta\to 0}(\delta^a\varphi(\delta) + a\int_\delta^\infty x^{a-1}\varphi(x)\,dx).$$

Since $a > 0$ we may take the limit as $\delta \to 0$, and $d/dx(x_+^a) = ax_+^{a-1}$.

4. Let $\tilde\mu \in C^1(\overline{\mathbf{R}_+}; \mathbf{R})$ and $(1+t)|\tilde\mu'(t)| \leq N\tilde\mu(t)$, $t \geq 0$. If $t \geq 0$ define $a(t) = (1+t)^N\tilde\mu(t)$, and compute

$$a'(t) = (1+t)^{N-1}(N\tilde\mu(t) + (1+t)\tilde\mu'(t)) \geq 0.$$

This shows that $(1+s)^N\tilde\mu(s) \leq (1+t)^N\tilde\mu(t)$ if $0 \leq s \leq t$, which gives one direction of the inequality. The other direction follows by considering $a(t) = (1+t)^{-N}\tilde\mu(t)$.