Scattering theory

Solutions to Exercises #4, 12.10.2007

1. Let μ be a finite positive Borel measure on $\mathbf{R}$. Define $P = \{x\,;\,\mu(\{x\}) \neq 0\}$ to be the set of pure points of μ (by the lectures, P is countable and so has Lebesgue measure zero). Let $\mu_{\mathrm{pp}}(E) = \mu(E \cap P)$ and $\mu_{\mathrm{cont}}(E) = \mu(E \cap P^c)$ for Borel sets E. These are finite positive Borel measures, and on P^c one has the Lebesgue decomposition $\mu_{\mathrm{cont}} = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$ where μ_{ac} is absolutely continuous and μ_{sing} is singular with respect to Lebesgue measure, and there is a Borel set $S \subset P^c$ with Lebesgue measure zero such that μ_{sing} is concentrated on S (i.e. $\mu_{\mathrm{sing}}(E) = \mu_{\mathrm{sing}}(E \cap S)$ for all Borel E). Let $C = \mathbf{R} \setminus (P \cup S)$. Then μ_{pp}, μ_{ac} and μ_{sing} are finite positive Borel measures concentrated on P, C, and S, respectively.

   Define H'_{pp} as the closure of $\{\chi_P f\,;\, f \in C_c(\mathbf{R})\}$ in H, similarly for H'_{ac} and H'_{sing}. Then for any $f \in H$ there exist $f_j \in C_c(\mathbf{R})$ with $\|f_j - f\| \to 0$, so we may write f as

   $$f = \lim f_j = \lim(\chi_P f_j + \chi_C f_j + \chi_S f_j) = \lim \chi_P f_j + \lim \chi_C f_j + \lim \chi_S f_j.$$

   One has $H = H'_{\mathrm{pp}} \oplus H'_{\mathrm{ac}} \oplus H'_{\mathrm{sing}}$ since clearly the spaces are orthogonal. It remains to show that $f \in H'_{\mathrm{pp}}$ iff the spectral measure μ_f is pure point, i.e. $H'_{\mathrm{pp}} = H_{\mathrm{pp}}$. The cases for H'_{ac} and H'_{sing} are similar. Since A is multiplication by λ, it follows from the proof of Prop. 1.7.9 in the lectures that $d\mu_f = |f|^2\,d\mu$. Now $f \in H'_{\mathrm{pp}}$ iff $f = \lim \chi_P f_j$ for $f_j \in C_c(\mathbf{R})$. For such f one has

   $$\mu_f(E) = \int_E |f|^2\,d\mu = \lim \int_E \chi_P |f_j|^2\,d\mu = \int_{E\cap P} |f|^2\,d\mu = \int_E |f|^2\,d\mu_{\mathrm{pp}}$$

   so μ_f is pure point. Conversely, let $d\mu_f = |f|^2\,d\mu$ be pure point. Note that

   $$|f|^2\,d\mu = |f|^2\,d\mu_{\mathrm{pp}} + |f|^2\,d\mu_{\mathrm{ac}} + |f|^2\,d\mu_{\mathrm{sing}}.$$

   Thus $|f|^2\,d\mu - |f|^2\,d\mu_{\mathrm{pp}}$ is both pure point and continuous, hence must be zero, and so $f = 0$ μ-a.e. on P^c. If $f_j \in C_c(\mathbf{R})$ and $f_j \to f$ in H, then $(1 - \chi_P) f_j \to 0$ so $\chi_P f_j \to f$ in H, and therefore $f \in H'_{\mathrm{pp}}$.

2. Let $H = \ell^2$ and $(Lu)_n = u_{n+1}$, $(Ru)_n = u_{n-1}$. These are bounded operators and $(Lu, v) = \sum u_{n+1}\bar{v}_n = \sum u_n \bar{v}_{n-1} = (u, Rv)$ so $L^* = R$ and $R^* = L$. Thus $A = L + R$ is bounded and self-adjoint. Let $U : H \mapsto$

$L^2(0,1)$, $(u_n) \mapsto \sum u_n e^{2\pi i n x}$. This map is unitary by Parseval's theorem, and ULU^{-1} and URU^{-1} are given by multiplication with $e^{-2\pi i x}$ and $e^{2\pi i x}$, respectively. Thus

$$UAU^{-1}f(x) = 2\cos(2\pi x)f(x), \quad f \in L^2(0,1).$$

If φ is bounded Borel then $\varphi(A)$ is given by [1]

$$U\varphi(A)U^{-1}f(x) = \varphi(2\cos(2\pi x))f(x), \quad f \in L^2(0,1).$$

If $u \in H$ and $f = Uu$ then for $\varphi \in C_c(\mathbf{R})$

$$\begin{aligned} \int \varphi \, d\mu_u &= (\varphi(A)u, u) = (U\varphi(A)U^{-1}f, f) = \int_0^1 \varphi(2\cos(2\pi x))|f(x)|^2 \, dx \\ &= \int_0^{1/2} \varphi(2\cos(2\pi x))|f(x)|^2 \, dx + \int_{1/2}^1 \varphi(2\cos(2\pi y))|f(y)|^2 \, dy. \end{aligned}$$

We change variables by $x(\lambda) = y(\lambda) = \frac{1}{2\pi}\cos^{-1}(\frac{\lambda}{2})$ in these integrals to get

$$\int \varphi \, d\mu_u = \frac{1}{4\pi}\int_{-2}^{2} \varphi(\lambda)\frac{|f(x(\lambda))|^2}{\sqrt{1-\lambda^2/4}} \, d\lambda + \frac{1}{4\pi}\int_{-2}^{2} \varphi(\lambda)\frac{|f(y(\lambda))|^2}{\sqrt{1-\lambda^2/4}} \, d\lambda.$$

It follows that $d\mu_u(\lambda) = \frac{1}{4\pi(1-\lambda^2/4)^{1/2}}[|f(x(\lambda))|^2 + |f(y(\lambda))|^2]\chi_{(-2,2)}(\lambda) \, d\lambda$.

3. We first show that the resolvent $R(z)$ is analytic in $\rho(A)$. Let $z_0 \in \rho(A)$ and let $M = \|R(z_0)\|$, so that $\|(A - z_0)u\| \geq \frac{1}{M}\|u\|$ for $u \in \mathscr{D}(A)$. If $|z - z_0| < 1/2M$ then $\|(A - z)u\| \geq \|(A - z_0)u\| - |z - z_0|\|u\| \geq \frac{1}{2M}\|u\|$, so $B(z_0, 1/2M) \subset \rho(A)$ by Ex. 3, Problem 1, and

$$\|R(z)\| \leq 2M, \quad |z - z_0| < \frac{1}{2M}.$$

This shows that $\rho(A)$ is open. The resolvent identity $R(z) - R(z_0) = (z - z_0)R(z)R(z_0)$ from the lectures implies

$$\|R(z) - R(z_0)\| \leq 2M^2|z - z_0|, \quad |z - z_0| < \frac{1}{2M},$$

[1] See proof of Theorem 1.7.6 in the lectures, which says that $\varphi(A)$ corresponds to $\varphi(M_a)$, $a = 2\cos(2\pi x)$, where M_a is multiplication by a in $L^2(0,1)$. One has $\varphi(M_a)f(x) = \varphi(a(x))f(x)$ since the spectral projection for M_a is $E_\lambda f = \chi_{(-\infty,\lambda)}(M_a)f = \chi_{\{a<\lambda\}}f$ by Example 1.4.1 in the lectures. From this fact we obtain $s(M_a)f(x) = s(a(x))f(x)$ for simple functions s, thus for general φ.

and for $u, v \in H$

$$\frac{(R(z)u, v) - (R(z_0)u, v)}{z - z_0} \to (R(z_0)^2 u, v) \quad \text{as } z \to z_0.$$

Therefore $z \mapsto (R(z)u, v)$ is analytic in $\rho(A)$.

Let λ be an isolated point of $\sigma(A)$, so $\overline{B(\lambda, \delta)} \cap \sigma(A) = \{\lambda\}$ for some $\delta > 0$, and define

$$(P_\lambda u, v) = -\frac{1}{2\pi i} \oint_{\Gamma_\lambda} (R(z)u, v)\, dz, \quad u, v \in H$$

whenever Γ_λ is a simple closed curve in $B(\lambda, \delta) \setminus \{\lambda\}$ which goes around λ once counterclockwise (i.e. $\text{Ind}(\Gamma_\lambda, \lambda) = 1$). Since $z \mapsto (R(z)u, v)$ is analytic in $B(\lambda, \delta) \setminus \{\lambda\}$, Cauchy's theorem (see e.g. Rudin, Real and complex analysis, Theorem 10.35) implies that the definition is independent of the choice of Γ_λ. By choosing $\Gamma_\lambda = \partial B(\lambda, \delta/2)$, we obtain

$$|(P_\lambda u, v)| \leq C_{\lambda,\delta} \|u\| \|v\|,$$

and the integral defines P_λ as a bounded linear operator on H.

It remains to show that P_λ is an orthogonal projection onto $\ker(A - \lambda)$, i.e. $P_\lambda^2 = P_\lambda^* = P_\lambda$ and $\mathscr{R}(P - \lambda) = \ker(A - \lambda)$. To do this we write $P_\lambda u$ as an H-valued Bochner integral

$$P_\lambda u = -\frac{1}{2\pi i} \oint_{\Gamma_\lambda} R(z)u\, dz$$

and note that the integral can be approximated by Riemann sums in the H norm. Then, choosing $\Gamma_\lambda = \partial B(\lambda, \delta/4)$ and $\Gamma'_\lambda = \partial B(\lambda, \delta/2)$ we have

$$\begin{aligned}
(P_\lambda^2 u, v) &= \frac{1}{(2\pi i)^2} \oint_{\Gamma_\lambda} \oint_{\Gamma'_\lambda} (R(z)R(w)u, v)\, dw\, dz \\
&= \frac{1}{(2\pi i)^2} \oint_{\Gamma_\lambda} \oint_{\Gamma'_\lambda} \frac{((R(z) - R(w))u, v)}{z - w}\, dw\, dz.
\end{aligned}$$

Since $\oint_{\Gamma'_\lambda} (z - w)^{-1}\, dw = -2\pi i$ and $\oint_{\Gamma_\lambda} (z - w)^{-1}\, dz = 0$ we have $P_\lambda^2 = P_\lambda$.

Choosing $\Gamma_\lambda = \partial B(\lambda, r)$ gives

$$(P_\lambda u, v) = -\frac{1}{2\pi i} \int_0^{2\pi} (R(\lambda + re^{i\theta})u, v)\, ire^{i\theta}\, dr,$$

$$(u, P_\lambda v) = \overline{(P_\lambda v, u)} = -\frac{1}{2\pi i} \int_0^{2\pi} \overline{(R(\lambda + re^{i\theta})v, u)}\, ire^{-i\theta}\, dr.$$

Since $R(z)^* = R(\bar{z})$ one gets $\overline{(R(\lambda + re^{i\theta})v, u)} = (R(\lambda + re^{-i\theta})u, v)$, and $P_\lambda^* = P_\lambda$ follows by changing variables $\theta \mapsto -\theta$ in the second integral.

Finally, we show $\ker(A - \lambda) = \mathscr{R}(P_\lambda)$. If $u \in \ker(A - \lambda)$ then $(A - z)u = (\lambda - z)u$ so $R(z)u = (\lambda - z)^{-1}u$. We see that $u \in \mathscr{R}(P_\lambda)$ since $u = P_\lambda u$:

$$P_\lambda u = -\frac{1}{2\pi i}\oint_{\Gamma_\lambda} (\lambda - z)^{-1} u \, dz = -\frac{1}{2\pi i}\int_0^{2\pi} \frac{u}{-re^{i\theta}} ire^{i\theta} \, d\theta = u.$$

On the other hand, if $u = P_\lambda v$ then $(A - \lambda)u = 0$ since

$$(A - \lambda)P_\lambda u = -\frac{1}{2\pi i}\oint_{\Gamma_\lambda} (z - \lambda)R(z)u \, dz = 0.$$

Here we used the fact that $\|(z-\lambda)R(z)\| \leq C$ near λ [2], so λ is a removable singularity for the analytic function $z \mapsto (z-\lambda)R(z)$ and the integral over Γ_λ is zero by Cauchy's theorem.

4. Let $\lambda \in \sigma(A)$ be isolated. If λ is not an eigenvalue, then $\ker(A-\lambda) = \{0\}$ so $P_\lambda \equiv 0$ by Problem 3. Then Morera's theorem would imply that $R(z)$ is analytic (in particular bounded) near λ, which is impossible since λ was in the spectrum.

   Let $\lambda \in \sigma_d(A)$. Then $0 < \dim \ker(A - \lambda) < \infty$ by definition. Define $A_\lambda = A - \lambda|_{\mathscr{D}(A_\lambda)}$ where $\mathscr{D}(A_\lambda) = \mathscr{D}(A) \cap \ker(A - \lambda)^\perp$. Then $A_\lambda : \mathscr{D}(A_\lambda) \to \ker(A - \lambda)^\perp$ is self-adjoint [3]. Since $\ker(A_\lambda) = \{0\}$ we see that 0 is not an eigenvalue of A_λ, so 0 cannot be an isolated point of $\sigma(A_\lambda)$ by the first part. Since $\sigma(A_\lambda) \subset (-\lambda + \sigma(A))$ [4] we must have $0 \in \rho(A_\lambda)$, so indeed A_λ has a bounded inverse.

   Suppose now that $0 < \dim \ker(A-\lambda) < \infty$ and $A_\lambda : \mathscr{D}(A_\lambda) \to \ker(A-\lambda)^\perp$ is bijective and has a bounded inverse. Clearly λ is an eigenvalue of A with finite multiplicity. We need to show that λ is an isolated point of $\sigma(A)$, and to do this it is enough to construct the resolvent $R(z)$ for z near λ, $z \neq \lambda$. Let $f \in H$ and consider the equation $(A - z)u = f$. We have a splitting of H into closed subspaces,

$$H = \ker(A - \lambda) \oplus \ker(A - \lambda)^\perp.$$

[2] This follows from the estimate $\|R(z)\| \leq \operatorname{dist}(z, \sigma(A))^{-1}$, which follows from the functional calculus fact that $\|\varphi(A)\| = \|\varphi\|_{L^\infty(\sigma(A))}$ when we consider the choice $\varphi(x) = (x - z)^{-1}$.

[3] If $v \in \ker(A - \lambda)$ then $((A - \lambda)u, v) = (u, (A - \lambda)v) = 0$ for $u \in \mathscr{D}(A)$ so $A - \lambda$ maps into $\ker(A-\lambda)^\perp$. One has $(A_\lambda u, v) = (u, (A-\lambda)v)$ for $u \in \mathscr{D}(A_\lambda), v \in \mathscr{D}(A)$, which shows that A_λ is self-adjoint.

[4] Proof: if $z+\lambda \in \rho(A)$ then $\|(A-z-\lambda)u\| \geq c\|u\|$ for $u \in \mathscr{D}(A)$, hence for $u \in \mathscr{D}(A_\lambda)$, so $z \in \rho(A_\lambda)$.

Let P be the orthogonal projection onto $\ker(A-\lambda)$, and write $f = f_1+f_2$, $u = u_1 + u_2$ where $f_1 = Pf$, $u_1 = Pu$. We would like to solve

$$\begin{aligned}(A-z)u_1 &= f_1,\\ (A-z)u_2 &= f_2.\end{aligned}$$

Since $(A-z)Pf = (\lambda-z)Pf = (\lambda-z)f_1$ we may take $u_1 = (\lambda-z)^{-1}Pf$. Also, for $u_2 \in \mathscr{D}(A) \cap \ker(A-\lambda)^{\perp}$ one has

$$(A-z)u_2 = (A_\lambda + (\lambda - z))u_2 = (I + (\lambda - z)A_\lambda^{-1})A_\lambda u_2$$

and $I + (\lambda-z)A_\lambda^{-1}$ is invertible for z near λ by Neumann series. Thus we may take $u_2 = A_\lambda^{-1}(I + (\lambda - z)A_\lambda^{-1})^{-1} f_2$. The operator

$$\tilde{R}(z)f = (\lambda - z)^{-1}Pf + A_\lambda^{-1}(I + (\lambda - z)A_\lambda^{-1})^{-1}(I - P)f$$

is then bounded on H if z is close to λ, and $(A-z)\tilde{R}(z) = I$. One sees that $\tilde{R}(z)$ maps H onto $\mathscr{D}(A)$ by using the splitting above, so $A - z$ is bijective and has a bounded inverse. Thus points $z \neq \lambda$ near λ are in $\rho(A)$, so λ is an isolated point of $\sigma(A)$ as desired.