Scattering theory

Solutions to Exercises #2, 28.9.2007

1. Let A be a densely defined, symmetric, and semibounded operator. First assume that $(Au, u) \geq \|u\|^2$ for $u \in \mathscr{D}(A)$. Define $(u, v)_D = (Au, v)$ for $u, v \in \mathscr{D}(A)$. The assumptions on A ensure that this is an inner product in $\mathscr{D}(A)$, and the corresponding norm satisfies $\|u\|_D \geq \|u\|$.

   Let D be the completion of $(\mathscr{D}(A), \|\cdot\|_D)$. That is, $D = C/\sim$ where C is the set of Cauchy sequences in $(\mathscr{D}(A), \|\cdot\|_D)$ and $(u_j) \sim (v_j)$ iff $\|u_j - v_j\|_D \to 0$, and where $\|[u_j]\|_D = \lim \|u_j\|_D$. Since any Cauchy sequence in $(\mathscr{D}(A), \|\cdot\|_D)$ converges in H, there is a natural linear map from D to H given by $[u_j] \mapsto u$ if $\|u_j - u\| \to 0$. We may identify D with a subspace of H if this map is injective, and this is the case: if (u_j) is Cauchy in $(\mathscr{D}(A), \|\cdot\|_D)$ and $\|u_j - u\| \to 0$, then $\|u_j - u\|_D = \lim_{k\to\infty} \|u_j - u_k\|_D \to 0$ as $j \to \infty$.

   Let $\mathscr{D}(A_D) = D \cap \mathscr{D}(A^*)$ and define A_D as the restriction of A^* to $\mathscr{D}(A_D)$. Since $\mathscr{D}(A) \subset \mathscr{D}(A_D)$, we have that $\mathscr{D}(A_D)$ is a dense linear subspace of H. We will show that A_D is self-adjoint in three steps.

   Step 1: A_D is symmetric. This follows from

   $$(A_D u, v) = (u, v)_D, \quad u \in \mathscr{D}(A_D), v \in D.$$

   To show this take sequences $u_j, v_k \in \mathscr{D}(A)$ with $\|u_j - u\|_D \to 0$ and $\|v_k - v\|_D \to 0$. Then

   $$(A_D u, v) = \lim_k (A^* u, v_k) = \lim_k (u, A v_k) = \lim_k \lim_j (u_j, A v_k) = (u, v)_D.$$

   Step 2: $\mathscr{R}(A_D) = H$. We let $f \in H$ and find $u \in \mathscr{D}(A_D)$ satisfying $A_D u = f$. Since

   $$|(f, v)| \leq \|f\| \, \|v\| \leq \|f\| \|v\|_D, \quad v \in D,$$

   there is a unique $u \in D$ with $(u, v)_D = (f, v)$ for all $v \in D$. Since $(u, Aw) = (u, w)_D = (f, w)$ for $w \in \mathscr{D}(A)$, one has $u \in \mathscr{D}(A^*)$. Consequently $u \in \mathscr{D}(A_D)$ and $A_D u = f$.

   Step 3: A_D self-adjoint. In fact, any symmetric operator with full range is self-adjoint. Let $v \in \mathscr{D}(A_D^*)$, so $(A_D u, v) = (u, A_D^* v)$ for $u \in \mathscr{D}(A_D)$. Choose $\tilde{v} \in \mathscr{D}(A_D)$ with $A_D \tilde{v} = A_D^* v$. Then $(A_D u, \tilde{v}) = (u, A_D \tilde{v}) =$

$(u, A_D^* v)$ by symmetry, so $(A_D u, v - \tilde{v}) = 0$ for $u \in \mathscr{D}(A_D)$. One has $v = \tilde{v}$ since $\mathscr{R}(A_D) = H$, and it follows that A_D is self-adjoint.

We have proved the Friedrichs extension theorem when $(Au, u) \geq \|u\|^2$. If $(Au, u) \geq c\|u\|^2$ we define $\tilde{A} = A + (1-c)I$ so $(\tilde{A}u, u) \geq \|u\|^2$, and let $\tilde{A}_D$ be the Friedrichs extension of $\tilde{A}$. Then $\tilde{A}_D - (1-c)I$ is a self-adjoint extension of A.

2. Let $H = L^2(I)$, $I = (-1, 1)$, and define A with domain $\mathscr{D}(A) = C_0^\infty(I)$ by $Au = -u''$. The symmetry of A follows by integration by parts (see Ex. 1, Problem 5). The Poincaré inequality $\|u\| \leq 2\|u'\|$ for $u \in C_0^\infty(I)$ [1] implies
$$(Au, u) = -\int_{-1}^{1} u''\bar{u}\,dt = \int_{-1}^{1} |u'|^2\,dt \geq \frac{1}{4}\|u\|^2.$$
Therefore, A is semibounded with lower bound $\frac{1}{4}$. We use Problem 1 to determine the Friedrichs extension of A (note that the first part of the proof works when A has a positive lower bound). One has $\|u\|_D = (Au, u)^{1/2} = \|u'\|$, and D is the completion of $C_0^\infty(I)$ with respect to this norm. One has
$$D = \{u \in H^1(I)\,;\, u(\pm 1) = 0\} = H_0^1(I).^2$$
One has $A^*u = -u''$ with domain $\mathscr{D}(A^*) = H^2(I)$ (Ex. 1, Problem 5). Therefore, the Friedrichs extension is given by $A_D : u \mapsto -u''$ with domain $\mathscr{D}(A_D) = H^2(I) \cap H_0^1(I)$. Thus the Friedrichs extension corresponds to Dirichlet boundary conditions.

3. Let $H = L^2(\mathbf{R})$ and let $A : u \mapsto i(x^2u' + xu)$ with $\mathscr{D}(A) = C_0^\infty(\mathbf{R})$. The symmetry of A is an integration by parts: for $u, v \in C_0^\infty(\mathbf{R})$
$$\begin{aligned}(Au, v) &= \int i(x^2u' + xu)\bar{v}\,dx = \int i(-2xu\bar{v} - x^2u\bar{v}' + xu\bar{v})\,dx \\ &= \int u\overline{i(x^2v' + xv)}\,dx = (u, Av).\end{aligned}$$
We wish to compute the defect indices $n_\pm(A) = \dim\ker(A^* \pm i)$. Here A is symmetric but not closed, but closable operators have the property

[1] Proof: $\|u\|^2 = \int_I u\bar{u}\,dt = -\int_I (u'\bar{u} + u\bar{u}')t\,dt = -2\int_I \mathrm{Re}\,(u\bar{u}')t\,dt \leq 2\|u\|\|u'\|$.

[2] In fact, if (u_j) is a Cauchy sequence in $(C_0^\infty(I), \|\cdot\|_D)$, then (u_j) is Cauchy in $H^1(I)$ by the Poincaré inequality, hence converges in $H^1(I)$ to some $u \in H^1(I)$. By Sobolev embedding one also has uniform convergence in $\bar{I}$, so $u(\pm 1) = 0$. Conversely, if $u \in H^1(I)$ and $u(\pm 1) = 0$, one can produce $u_j \in C_0^\infty(I)$ with $u_j \to u$ in $H^1(I)$ by using suitable cutoffs and mollifiers, see e.g. Evans, Partial differential equations, Sec. 5.5.

that $(\bar{A})^* = A^*$ [3] so in fact $n_\pm(A) = n_\pm(\bar{A})$. We will show that $n_\pm(A) = 1$ so by a theorem in the lectures $\bar{A}$ has a self-adjoint extension, and then also A has one.

We obtain by using distributional derivatives that

$$\begin{aligned} v \in \mathscr{D}(A^*) &\Leftrightarrow \exists\, v^* \in L^2 : (u, v^*) = (i(x^2u' + xu), v)\ \forall\, u \in C_0^\infty \\ &\Leftrightarrow \exists\, v^* \in L^2 : (u, v^*) = (u, i(x^2v' + xv))\ \forall\, u \in C_0^\infty. \end{aligned}$$

Thus $\mathscr{D}(A^*) = \{v \in L^2\,;\, x^2v' + xv \in L^2\}$ and $A^*v = i(x^2v' + xv)$. If $v = v_\pm \in \ker(A^* \pm i)$ then

$$x^2v' + (x \pm 1)v = 0, \quad v \in L^2, x^2v' + xv \in L^2.$$

Working in $x \neq 0$ we have $v' + \frac{x\pm1}{x^2}v = 0$, so $(ve^{\log|x| \mp 1/x})' = 0$ and

$$v_\pm(x) = \begin{cases} C_{\pm,p}|x|^{-1}e^{\pm 1/x}, & x > 0, \\ C_{\pm,n}|x|^{-1}e^{\pm 1/x}, & x < 0, \end{cases}$$

for some constants $C_{\pm,p}$ and $C_{\pm,n}$. This gives the general distributional solution to $(A^* \pm i)v_\pm = 0$ in $x \neq 0$. Since $v_\pm \in L^2$ we must have $C_{+,p} = C_{-,n} = 0$, and $C_{+,n}$ and $C_{-,p}$ are free parameters. Thus $n_\pm(A) = \dim \ker (A^* \pm i) = 1$.

4. Let A be self-adjoint, and let E be the projection-valued measure for A. We write $d\mu_x(\lambda) = d(E_\lambda x, x)$, so μ_x is a finite positive Borel measure on $\mathbf{R}$ for each $x \in H$. If g is a bounded Borel function, we know that $(g(A)x, x) = \int g(\lambda)\, d\mu_x(\lambda)$ defines a bounded operator $g(A)$ on H satisfying $g(A)^* = \bar{g}(A)$, and $\|g(A)x\|^2 = \int |g|^2\, d\mu_x$. By polarization, this implies $(g(A)x, h(A)x) = \int g\bar{h}\, d\mu_x = ((g\bar{h})(A)x, x)$ for bounded Borel functions g, h, which again implies $g(A)h(A) = (gh)(A)$. We obtain for $\varphi \in C_0^\infty(\mathbf{R})$

$$\int \varphi\, d\mu_{g(A)x} = (\varphi(A)g(A)x, g(A)x) = ((\varphi|g|^2)(A)x, x) = \int \varphi|g|^2\, d\mu_x,$$

so $d\mu_{g(A)x} = |g|^2 d\mu_x$ if g is bounded Borel.

Let now g be a Borel function on $\mathbf{R}$, not necessarily bounded. Let $\chi_n = \chi_{\{|g|\leq n\}}$ and $g_n = \chi_n g$. Define

$$\mathscr{D}(g(A)) = \{x \in H\,;\, \int |g|^2\, d\mu_x < \infty\}.$$

[3] Since $A \subset \bar{A}$, $(\bar{A})^* \subset A^*$ by Ex. 1, Problem 2. Let $v \in \mathscr{D}(A^*)$, so $(Au, v) = (u, A^*v)$ for $u \in \mathscr{D}(A)$. We obtain $(\bar{A}u, v) = (u, A^*v)$ for $u \in \mathscr{D}(\bar{A})$ by approximating u with $u_j \in \mathscr{D}(A)$ so that $u_j \to u$ and $Au_j \to \bar{A}u$, and we get $A^* \subset (\bar{A})^*$.

If $x \in H$, we want to show that $\chi_n(A)x \in \mathscr{D}(g(A))$ and $\chi_n(A)x \to x$. The first fact follows since $\int |g|^2 \, d\mu_{\chi_n(A)x} = \int \chi_n |g|^2 \, d\mu_x < \infty$, and for the second fact we use

$$\begin{aligned}\|\chi_n(A)x - x\|^2 = \|(1-\chi_n)(A)x\|^2 = \int |1-\chi_n(\lambda)|^2 \, d\mu_x = \int_{\{|g|>n\}} d\mu_x \\ \leq \frac{1}{n^2} \int |g|^2 \, d\mu_x \to 0.\end{aligned}$$

We see that $\mathscr{D}(g(A))$ is a dense linear subspace of H since $\int |g|^2 \, d\mu_{cx} = |c|^2 \int |g|^2 \, d\mu_x$ and since

$$\int |g_n|^2 \, d\mu_{x+y} = \|g_n(A)(x+y)\|^2 \leq 2(\|g_n(A)x\|^2 + \|g_n(A)y\|^2).$$

If $x \in \mathscr{D}(g(A))$ then $(g_n(A)x)$ is Cauchy in H since

$$\|g_m(A)x - g_n(A)x\|^2 = \int |g_m - g_n|^2 \, d\mu_x \to 0, \quad m,n \to \infty.$$

We may define $g(A)x = \lim g_n(A)x$. Then $g(A)$ is a linear operator with domain $\mathscr{D}(g(A))$, and it satisfies

$$(g(A)x, x) = \int g \, d\mu_x.$$

In fact this condition determines $g(A)$ uniquely. It remains to show that $g(A)$ is self-adjoint if g is real. Let $y \in \mathscr{D}(g(A)^*)$, i.e. for some $y^* \in H$ one has

$$(g(A)x, y) = (x, y^*) \quad \text{for } x \in \mathscr{D}(g(A)).$$

Then using that $(g_n(A)x, y) = (x, g_n(A)y)$ we get $(x, g_n(A)y - y^*) \to 0$ for $x \in \mathscr{D}(g(A))$, hence $g_n(A)y \to y^*$ weakly and so $\|g_n(A)y\|^2 \leq C$. This shows that $y \in \mathscr{D}(g(A))$ and $g(A)$ is self-adjoint.