Scattering theory

Solutions to Exercises #1, 21.9.2007

1. The norm $\|\cdot\|_A$ is induced by the inner product $(u,v)_A = (u,v)+(Au,Av)$ on $\mathscr{D}(A)$, and $\mathscr{D}(A)$ becomes an inner product space.

   "A closed $\implies$ $(\mathscr{D}(A), \|\cdot\|_A)$ Hilbert": Let (u_j) be a Cauchy sequence in $(\mathscr{D}(A), \|\cdot\|_A)$. Then $\forall\, \varepsilon\ \exists\, M$ such that

   $$\|u_j - u_k\|_A^2 = \|u_j - u_k\|^2 + \|A(u_j - u_k)\|^2 < \varepsilon^2 \quad \text{whenever } j,k \geq M.$$

   Therefore (u_j) and (Au_j) are Cauchy in H and converge to $u \in H$ and $v \in H$, respectively. Since A is closed one has $u \in \mathscr{D}(A)$ and $v = Au$, and $\|u_j - u\|_A \to 0$ as $j \to \infty$. This shows that $(\mathscr{D}(A), \|\cdot\|_A)$ is Hilbert.

   "$(\mathscr{D}(A), \|\cdot\|_A)$ Hilbert $\implies$ A closed": Let (u_j) be a sequence in $\mathscr{D}(A)$ with $u_j \to u$ and $Au_j \to v$ in H. We need to show that $u \in \mathscr{D}(A)$ and $Au_j \to Au$. One has that (u_j) and (Au_j) are Cauchy in H, which implies that (u_j) is Cauchy in $(\mathscr{D}(A), \|\cdot\|_A)$. Since this space is complete there exists $\tilde{u} \in \mathscr{D}(A)$ with $\|u_j - \tilde{u}\|_A \to 0$. This implies convergence in H, and since limits are unique one has $u = \tilde{u} \in \mathscr{D}(A)$ and $Au_j \to Au$.

2. Recall that $v \in \mathscr{D}(A^*)$ iff $\exists\, v^* \in H$ such that $(u, v^*) = (Au, v)\ \forall\, u \in \mathscr{D}(A)$ (and then $A^*v = v^*$).

   (i) Let $v_j \in \mathscr{D}(A^*)$ with $v_j \to v$ and $A^*v_j \to v^*$ in H. Then

   $$\begin{aligned} &(u, A^*v_j) = (Au, v_j)\ \forall\, u \in \mathscr{D}(A) \\ \implies &(u, v^*) = (Au, v)\ \forall\, u \in \mathscr{D}(A). \end{aligned}$$

   This shows that $v \in \mathscr{D}(A^*)$, $A^*v = v^*$, and $A^*v_j \to A^*v$.

   (ii) Let $A \subset B$, show $B^* \subset A^*$. One has

   $$\begin{aligned} v \in \mathscr{D}(B^*) &\implies (u, B^*v) = (Bu, v)\ \forall\, u \in \mathscr{D}(B) \\ &\implies (u, B^*v) = (Au, v)\ \forall\, u \in \mathscr{D}(A) \end{aligned}$$

   and therefore $v \in \mathscr{D}(A^*)$ and $B^*v = A^*v$.

3. If A is symmetric then $(Au, u) = (u, Au) = \overline{(Au, u)}$, so $(Au, u) \in \mathbf{R}$ for $u \in \mathscr{D}(A)$. Conversely, let $Q(u) = (Au, u)$, and assume that $Q(u) \in \mathbf{R}$ for $u \in \mathscr{D}(A)$. Note that $Q(cu) = Q(u)$ if $|c| = 1$. By polarization

   $$\begin{aligned} 4(Au, v) &= Q(u+v) - Q(u-v) + iQ(u+iv) - iQ(u-iv), \\ 4(Av, u) &= Q(u+v) - Q(u-v) - iQ(u+iv) + iQ(u-iv). \end{aligned}$$

   This shows that $(Au, v) = \overline{(Av, u)} = (u, Av)$ for $u, v \in \mathscr{D}(A)$.

4. (i) Since A is closable, there is a closed operator $B \supset A$. Therefore, $\overline{\mathscr{G}(A)} \subset \mathscr{G}(B)$. Define

$$\mathscr{D}(\bar{A}) = \{x \in H\,;\, (x,y) \in \overline{\mathscr{G}(A)} \text{ for some } y \in H\}.$$

Then $\mathscr{D}(\bar{A})$ is a subspace of H, and for $x \in \mathscr{D}(\bar{A})$ we define $\bar{A}x = y$ if $(x,y) \in \overline{\mathscr{G}(A)}$. This is well defined since $\overline{\mathscr{G}(A)} \subset \mathscr{G}(B)$, and $(x,y), (x,\tilde{y}) \in \overline{\mathscr{G}(A)}$ implies $y = Bx = \tilde{y}$. One has $A \subset \bar{A} \subset B$, and since this is valid for any closed $B \supset A$ we have the desired unique closed extension.

(ii) A symmetric implies $A \subset A^*$, so A is closable by Exercise 2, part (i). If $x, y \in \mathscr{D}(\bar{A})$ then $(x, \bar{A}x), (y, \bar{A}y) \in \overline{\mathscr{G}(A)}$, so there are sequences $(x_j), (y_j)$ in $\mathscr{D}(A)$ with $x_j \to x$, $Ax_j \to \bar{A}x$, and $y_j \to y$, $Ay_j \to \bar{A}y$. By symmetry $(Ax_j, y_j) = (x_j, Ay_j)$, and taking limits shows that $(\bar{A}x, y) = (x, \bar{A}y)$ so $\bar{A}$ is symmetric. Finally, if A has a self-adjoint extension B then $\bar{A} \subset B$ so $\bar{A}$ has a self-adjoint extension, and if $\bar{A}$ has a self-adjoint extension then clearly A has one.

5. Let $A : u \mapsto u''$ with domain $\mathscr{D}(A) = C_0^2(I)$ in $L^2(I)$, where $I = (-1,1)$. Then A is densely defined since C_0^2 functions are dense in L^2. Also, A is symmetric by an integration by parts: for $u, v \in C_0^2(I)$

$$(Au, v) = \int_{-1}^{1} u''\bar{v}\,dt = -\int_{-1}^{1} u'\bar{v}'\,dt = \int_{-1}^{1} u\bar{v}''\,dt = (u, Av). \qquad (1)$$

We have

$$\mathscr{D}(A^*) = \{v \in L^2(I)\,;\, \exists\, w \in L^2(I) \text{ s.t. } (u'', v) = (u, w) \text{ for } u \in C_0^2(I)\}.$$

By looking at $u \in C_0^\infty(I)$ one sees that $v \in \mathscr{D}(A^*)$ if and only if $v \in L^2(I)$ and the distributional derivative v'' is in $L^2(I)$. Therefore one has $\mathscr{D}(A^*) = \{v \in L^2(I)\,;\, v'' \in L^2(I)\} = H^2(I)$ [1], and also $A^*v = v''$ for $v \in \mathscr{D}(A^*)$.

[1] The last equality may be seen as follows. Let $v, v'' \in L^2(I)$. We want to show $|\langle v', \varphi\rangle| \leq C\|\varphi\|_{L^2(I)}$ for $\varphi \in C_0^\infty(I)$, which will imply $v' \in L^2(I)$. If $\varphi \in C_0^\infty(I)$ is given, we decompose $\varphi = \varphi_0 + c\psi$ where $c = \int_{-1}^1 \varphi\,dt$ and $\psi \in C_0^\infty(I)$ is a fixed function with $\int_{-1}^1 \psi\,dt = 1$. Then $\int_{-1}^1 \varphi_0\,dt = 0$, and we have $\varphi_0 = \Phi'$ where $\Phi \in C_0^\infty(I)$ is defined by $\Phi(t) = \int_{-1}^t \varphi_0(s)\,ds$. Then

$$\langle v', \varphi\rangle = \langle v', \Phi' + c\psi\rangle = -\langle v'', \Phi\rangle - c\langle v, \psi'\rangle$$

which implies $|\langle v', \varphi\rangle| \leq C\|\varphi\|_{L^2(I)}$ since $|c| \leq C\|\phi\|_{L^2(I)}$ and $\|\Phi\|_{L^2(I)} \leq C\|\varphi\|_{L^2(I)}$.

To obtain a self-adjoint extension of A, we impose boundary conditions on the domain. A straightforward choice is Dirichlet boundary conditions. Define $A_D : u \mapsto u''$ with domain

$$\mathscr{D}(A_D) = \{u \in H^2(I)\,;\, u(\pm 1) = 0\} = H^2(I) \cap H^1_0(I).$$

Note that $H^2(I) \subset C^1(\bar{I})$ by Sobolev embedding, so the pointwise values exist. Then A_D is a densely defined operator, which is symmetric because (1) is valid for $u, v \in \mathscr{D}(A_D)$. Also

$$\begin{aligned} v \in \mathscr{D}(A_D^*) &\Leftrightarrow \exists\, w \in L^2 \text{ s.t. } (u'', v) = (u, w) \text{ for } u \in H^2 \cap H^1_0 \\ &\Leftrightarrow v \in L^2, v'' \in L^2, (u, v'') = (u'', v) \text{ for } u \in H^2 \cap H^1_0 \\ &\Leftrightarrow v \in H^2, (u, v'') = (u'', v) \text{ for } u \in H^2 \cap H^1_0. \end{aligned}$$

The condition $(u, v'') = (u'', v)$ means that $(u\bar{v}')(1) - (u\bar{v}')(-1) = (u'\bar{v})(1) - (u'\bar{v})(-1)$ for $u \in H^2 \cap H^1_0$. Since $u(\pm 1) = 0$ and $u'(\pm 1)$ can be arbitrary, this implies $v(\pm 1) = 0$. It follows that $\mathscr{D}(A_D^*) = \mathscr{D}(A_D)$ and A_D is self-adjoint.