Assume A is self-adjoint in H and $x \in H$. Recall μ_x is a bounded Borell-measure defined by

$$\mu_x(\Omega) = (x, E_\Omega x)$$

where

$$A = \int \lambda \, dE_\lambda .$$

Note that then

$$(x, Ax) = \int \lambda \, d\mu_x(\lambda) .$$

Note also that μ_x is positive measure:

If P is ON-projection, then

$$P^* = P \text{ and } P^2 = P$$

especially

$$(x, Px) = (x, P^2 x) = (P^* x, Px) = \|Px\|^2 \geq 0$$

Thus

$$\mu_x(\Omega) = (x, E_\Omega x) \geq 0 , \quad \forall x \in H$$

and $\Omega \in B(R)$.

Positive measures can be decomposed as

$$\mu = \mu_{pp} + \mu_{cont}$$

where $\mu_{pp}(\Omega) = \mu(P \cap \Omega)$.

Thus

$$\mu_{cont}(\Omega) = \mu(\Omega) - \mu(P \cap \Omega) \geq 0$$

and thus μ_{cont} is a positive measure, as well.

Finally recall Lebesgue;

$$\mu_{cont} = \mu_{ac} + \mu_{sing}$$

We want to show the important

1.7.5 Definition If A is self adjoint, define $x \in H_{ac} \overset{df}{\iff}$

μ_x is abs. cont.

and respectively H_{pp} and H_{sing}

<u>1.7.6 Theorem</u> $H = H_{pp} \oplus H_{ac} \oplus H_{sing}$
Each of these subspaces are <u>invariant</u> under A. $A|_{H_{pp}}$ has a complete set of eigenvectors, $A|_{H_{ac}}$ has only abs cont. spectral measures (μ_x is ac for all $x \in H_{ac}$) and $A|_{H_{sing}}$ has only singular spectral measures.

We start by defining the spaces L^2_x and H_x for all $x \in H$.
We let

$$L^2_x = L^2(\mathbb{R}, \mu_x)$$

Note that if $\varphi \in L^2_x$ then

$$\int |\varphi(\lambda)|^2 \, d(x, E_\lambda x) < \infty$$

and

$$\varphi(A)x \in H$$

is well-defined. Moreover

$$\| \varphi(A) u \|^2 = \int |\varphi(\lambda)|^2 \, d\mu_x$$

Thus we obtain an isometric map

$$L^2_x \ni \varphi \mapsto \varphi(A)x \in H$$

The range is closed subspace of H denoted by H_x. $\{\varphi(A)x \mid \varphi \in C_0\} \subset H_x$ is dense

Lemma If $y \perp H_x$, then

$$H_y \perp H_x$$

Proof If $\varphi, \psi \in C_0$

$$(\varphi(A)x, \psi(A)y) = (\psi(A)^* \varphi(A)x, y)$$

$$= (\bar{\psi}(A)\varphi(A)x, y) = (\bar{\psi}\varphi(A)x, y)$$

by the functional calculus. But

$$(\bar{\psi}\varphi(A)x, y) = 0$$

since $\bar{\psi}\varphi(A)x \in H_x$.

1.7.7 Theorem Let A be self-adjoint in H. Then there exist at most countable set

$$\{x_1, x_2, \dots\}$$

such that

$$H = \bigoplus_i H_{x_i}$$

<u>Proof</u> Assume $\{y_i \mid i \in \mathbb{N}\}$ is a dense set in H. Define

$$x_1 = y_1$$

$$x_2 = P_2 y_2 ,$$

where P_2 is ON-projection to $H_{x_1}^{\perp}$.

Finally, set

$$x_n = P_n y_n$$

where P_n is ON-projection to $(H_{x_1} \oplus \cdots \oplus H_{x_{n-1}})^{\perp}$

Set

$$H_0 = \bigoplus_j H_{x_j}$$

Now H_0 is dense and $\forall n$

$$y_n = P_n y_n + (I - P_n) y_n$$

$$= x_n + \underbrace{(I - P_n) y_n}_{\substack{\cap \\ H_{x_1} \oplus \cdots \oplus H_{x_{n-1}}}} \in H_0$$

Thus H_0 contains all y_n and $\Rightarrow H_0 = H$

$\boxtimes$

<u>1.7.8 Corollary</u> The map

$$U : \oplus L^2_{x_j} \to H = \oplus H_{x_j}$$

defined by

$$U : (\varphi_j) \longrightarrow \sum \varphi_j(A) x_j$$

is unitary and

$U^{-1} A U$ is multiplication with λ in $\oplus L^2_{x_j}$.

<u>Proof</u> We know already that

$$U : L^2_{x_j} \to H_{x_j}$$

is isometry and surjection (by the def. of H_{x_j}) and thus unitary. Hence by Theorem 1.7.7

$U : \oplus L_{x_j} \to H$ is unitary.

If $U_j = U|_{L^2_{x_j}} : L^2_{x_j} \to H_{x_j}$, then

$$U_j^{-1} A U_j \varphi = U_j^{-1} A \varphi(A) x_j = U_j^{-1} (\lambda \varphi(\cdot))(A) x_j$$

$$= \lambda \varphi(\cdot) \quad , \text{ for all } \varphi \in C_0$$

This proves the claim for U_j and hence for U.

Remark (i) $x \in H_x$

(ii) H_x is invariant for A bounded

Proof Ex

Let us prove first Theorem 1.7.5 in a special case

1.7.9 **Proposition** Assume μ is a point Borell-measure on $\mathbb{R}$ and $A : L^2_\mu \to L^2_\mu$ defined by

$$A\varphi(\lambda) = \lambda\varphi(\lambda)$$

$$\mathcal{D}(A) = \{ \varphi \in L^2_\mu \mid \lambda\varphi(\lambda) \in L^2_\mu \}$$

Then

$$L^2_\mu = H_{ac} \oplus H_{pp} \oplus H_{sing}$$

Proof We have

$$\mu = \mu_{ac} + \mu_{pp} + \mu_{sing} \quad \text{and}$$

$$L^2_\mu = L^2_{\mu_{ac}} \oplus L^2_{\mu_{pp}} \oplus L^2_{\mu_{sing}}$$

What is left to prove

(*) $\varphi \in L^2_{ac} \Leftrightarrow \mu_\varphi$ is ac

and similarly for L^2_{pp} and L^2_{sing}

To see this assume

(1) $\varphi \in L^2_{\mu_{ac}} \iff \int |\varphi(\lambda)|^2 d\mu = \int |\varphi(\lambda)|^2 d\mu_{ac}$

Now

$$\mu_\varphi(\Omega) = (\varphi, E_\Omega \varphi)$$

But

$$E_\Omega \varphi = \chi_\Omega \varphi$$

(since A is multiplication with λ)

Thus

$$\mu_\varphi(\Omega) = \int_\Omega |\varphi|^2 d\mu \overset{(1)}{=} \int_\Omega |\varphi|^2 d\mu_{ac}$$

which proves that μ_φ is abs. cont.

Similarily with μ_{pp} and μ_{sing}. $\square$

Proof of Theorem 1.7.6 The claim

$$H = H_{ac} \oplus H_{pp} \oplus H_{sing}$$

follows first for $H = \bigoplus_j L^2_{x_j}$ from Prop. 1.7.9. For the general case we need

If $U: H_1 \to H_2$ is unitary, then

$$x \in H_{ac} \text{ for } A: H_1 \to H_1 \iff$$

$$Ux \in H_{ac} \text{ for } UAU^{-1}: H_2 \to H_2$$

and similarly for H_{pp} and H_{sing}.

If φ is bounded Borel measurable, then

(1) $\varphi(B) = U\varphi(A)U^{-1}$ for $B := UAU^{-1}$.

To prove (1) denote α_A be the spectral map of Theor 1.6.1 for A. Now define

$$\beta(\varphi) = U\alpha_A(\varphi)U^{-1}:$$

$$\beta : L^\infty_B(\mathbb{R}) \to L(H_2)$$

Then it is easy to check that all the properties of Theorem 1.6.1 are valid for β. Thus, by uniqueness, $\beta = \alpha_B$ = spectral map for B in H_2. Thus

$$\varphi(B) = \alpha_B(\varphi) = \beta(\varphi) = U\alpha_A(\varphi)U^{-1}$$
$$= U\varphi(A)U^{-1}$$

and (1) follows. Now the claim follows: If $y = Ax$, we get

$$(y, \varphi(B)y) = (Ux, (U\varphi(A)U^{-1})Ux)$$
$$= (x, \varphi(A)x), \quad \text{for all } \varphi \in L^\infty_B(\mathbb{R}).$$

This proves

$$\mu_y(B) = \mu_x(A)$$

i.e. the spectral measure of x w.r.t. A is the same as the spectral measure of y w.r.t. B. This proves

$$H = H_{ac} \oplus H_{pp} \oplus H_{sing}.$$

The other claims are left as exercise.

$\square$