For the converse we need

1.4.6 Theorem Let $K = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbb{R}^n$ be a cube and X open set s.t. $K \subset X$.

Then $\exists\ \varphi \in C_0^\infty(X)$ s.t

$$\sum_{g \in \mathbb{Z}^n} \varphi(x-g) = 1$$

Proof Choose $\psi \in C_0^\infty(X)$ s.t.

$$\psi(x) = 1 \quad , \quad \forall x \in K$$

Now

$$\Phi(x) = \sum_{g \in \mathbb{Z}^n} \psi(x-g) \geq 1 \quad , \quad \forall x$$

and is periodic. Thus

$$\varphi(x) = \frac{\psi(x)}{\Phi(x)}$$

satisfies

$$\sum \varphi(x+g) = \sum \frac{\psi(x+g)}{\Phi(x+g)} = \sum \frac{\psi(x+g)}{\Phi(x)}$$

$$= \frac{\Phi(x)}{\Phi(x)} = 1 \quad , \quad \forall x \in \mathbb{R}^n$$

$\square$

Next we continue with proof of Theorem 14.4.2

Choose $\varphi \in C_0^\infty(\Omega)$ s.t.

$$\sum \varphi(x-y_k) \equiv 1 \quad \text{with}$$

$\sqrt{n}\, y_k \in \mathbb{Z}^n$. Let further, $u \in C^\infty$ with

$$\| u \|_{B^*_{P_0}} = 1 \quad \text{and}$$

$$u_k(x) = \varphi(x-y_k)\, u .$$

If $y_k \in X_j$ we have by assumption

$$\| V u_k \|_{L^2}^2 \leq M_j^2 \, \| P_0(D) \left(\varphi(x-y_k) u(x) \right) \|_{L^2}^2$$

But the Leibnitz rule reads:

$$P_0(D)(fg) = \sum_\alpha D^\alpha f \; \frac{P_0^{(\alpha)} g}{\alpha !} \qquad (\text{Ex.!})$$

Thus

(1) $$\| V u_k \|_{L^2}^2 \leq M_j^2 \, \| D_x^\alpha \varphi(x-y_k) \, \frac{P_0^{(\alpha)} u(x)}{\alpha !} \|_{L^2}^2 .$$

But
$$|Vu|^2 = \left| \sum V(\varphi(x-y_k) u) \right|^2 = \left| \sum V u_k \right|^2$$

(2) $$\leq C_n \sum_k |V u_k|^2$$

$$\left(\text{Note} \quad \sum_{j=1}^N |c_j| \leq \sqrt{N} \left(\sum_{j=1}^N |c_j|^2 \right)^{1/2} \right)$$

Next we notice that

$$\left.\begin{array}{l} x \in X_j \\ Vu_k \neq 0 \end{array}\right\} \Rightarrow y_k \in \tilde{X}_j := X_{j-1} \cup X_j \cup X_{j+1}$$

Thus it follows from (1) and (2) that

$$\| Vu \|^2_{L^2(X_j)} \leq C \left(M_{j-1}^2 + M_j^2 + M_{j+1}^2 \right) \| \sum_\alpha P_0^{(\alpha)}(D) u \|^2_{L^2(\tilde{X}_j)}$$

$$\leq C' R_j \| P_0^{(\alpha)}(D) u \|^2_{B^*} \left(M_{j-1} + M_j + M_{j+1} \right)^2$$

$$\leq C' R_j \left(M_{j-1} + M_j + M_{j+1} \right)^2$$

and hence

$$\| Vu \|_B = \sum_j \left(R_j \| Vu \|^2_{L^2(X_j)} \right)^{1/2}$$

$$= C \sum_j \left(R_j^2 \left(M_{j-1} + M_j + M_{j+1} \right)^2 \right)^{1/2}$$

$$\leq C' \sum_j R_j M_j$$

Olle

$$u_J = \sum_{|y_k| \geq R_J + 1} u_k$$

nun

(14.4.5) $$\| V u_J \|_B \leq \sum_J^\infty R_j M_j \to 0 \quad \text{as } J \to \infty$$

We need to show that

$$V : C^\infty \cap \{ U \mid \| U \|_{B^*_{p_0}} \le 1 \} \longrightarrow B$$

compactly. Thus assume $u^\nu \in C^\infty$ with

$$\| u^\nu \|_{B^*_{p_0}} \le 1 .$$

Denote now

$$U^\nu_k = \varphi(\cdot - y_k)\, u^\nu ,$$

$u^\nu_k(\cdot + y_k) \in C^\infty_0(\Omega)$, thus we may assume

$$V(x,D)\, U^\nu_k \quad \text{conv. in } L^2$$

for every k fixed. By (14.4.5)

$$\| VU^\nu - VU^\mu \|_B \le \| \sum VU^\nu_k - VU^\mu_k \|_B$$

$$\le \sum_{|y_k| \le J} \| VU^\nu_k - VU^\mu_k \|_B + C \sum_J^\infty R_j M_j$$

Thus VU^ν is Cauchy $\Rightarrow$ convergent in B

Finally we show that $C^\infty \cap B^*_{p_0}$ is dense in $B^*_{p_0}$:

Let $u \in B^*_{P_0}$, $\|u\|_{B^*_{P_0}} \leq 1$ and $\varepsilon > 0$.

Now

$$\forall_k \ \exists \ v_k \in C_0^\infty(\Omega + d_{y_k,?}) \ \text{s.t.}$$

$$\|u_k - v_k\|_{B^*_{P_0}} \leq \varepsilon/2^k$$

(this since $\|u_k - v_k\|_{B^*_{P_0}} \leq c_k \sum \|P_0^{(\alpha)}(D)(u_k - v_k)\|_{L^2}$

$\leq C_k \|u_k - v_k\|_{H^N}$, where $N = \deg P_0$.)

Then

$$v = \sum v_k \in C^\infty \text{ and}$$

$$\|u - v\|_{B^*_{P_0}} \leq \sum_k \|u_k - v_k\|_{B^*_{P_0}}$$

$$\leq \sum_k \frac{\varepsilon}{2^k} < \varepsilon.$$

$\blacksquare$

Examples ① Potential scattering:

Assume $P_0(D) = -\Delta^2$ and

$V(x, D) = V(x)$ is of order zero.

We want to get sufficient cond. for V to be short range.

By theorem we need to assume

$$\{ V(x+y)u \mid u \in C_0^\infty(\Omega),\ \|\Delta u\|_{L^2} \le 1 \}$$

is precomp. $\forall y \in \mathbb{R}^n$. But this is clearly equivalent (by Poincaré)

$$V(x+y) H_0^2(\Omega) \hookrightarrow L^2 \quad \text{is compact.}$$

We may assume $y = 0$. Sobolev embedding

(*) $\quad H_0^2 \hookrightarrow L^p \quad$ is compact

if

$$2 - n/2 > -n/p \iff p < \frac{2n}{n-2}$$

Now if $f \in H_0^2(\Omega)$, we have

$$Vf \in L^2, \text{ if}$$

$$\int |V|^2 |f|^2 < \left(\int |V|^{2q'}\right)^{1/q'} \left(\int |f|^{2q}\right)^{1/q} < \infty$$

This is true for

$$q < \frac{n}{n-2} \iff q' > n/2$$

Thus (*) holds if $V \in L^p$, $p > n$.

Infinity behaviour?

$$\|V(x+y)\,u\|_{L^2} \le \overbrace{\sup_{\substack{y\in X_j\\ x\in\Omega^j}} |V(x+y)|}^{M_j}\,\|u\|_{L^2}$$

$$\le C M_j \|\Delta u\|_{L^2} = C M_j \| P_0(D) u\|_{L^2}$$

Now if, say,

$$|V(x)| \le \frac{C}{|x|(\log|x|)^{1+\varepsilon}}, \quad \text{for large } |x|$$

we have

$$M_j \le \frac{C}{2^j j^{1+\varepsilon}}$$

and we have

$$\sum M_j R_j = \sum \frac{C}{j^{1+\varepsilon}} < \infty$$

Thus a sufficient condition that V is short range for Δ-operator is

(1) $\quad V \in L^p_{loc}$, for some $p > n$

and

(2) $\quad |V(x)| \le \dfrac{C}{|x|(\log|x|)^{1+\varepsilon}}$ for large $|x|$.

Especially, if (2) is replace with

(2') $\quad |V(x)| \le \dfrac{C}{|x|^{1+\varepsilon}}$ for large $|x|$.

Example ②: $V(x, D) = A(x) \cdot \nabla + V(x)$

We assume first $V \equiv 0$.

i) Local condition:

$$\{ V(x+y, D) u, \ \|\Delta u\|_{L^2} \leq 1 \} \hookrightarrow L^2$$

comp. We may again assume $y = 0$.

$\Leftrightarrow$

$$A(x, D) : H^2_0(\Omega) \longrightarrow L^2 \quad \text{compact}$$

For suffic. cond it is enough: $\exists p$

(i) $H^2_0(\Omega) \hookrightarrow W^{1,p}$ compactly

(ii) $A : L^p_{loc} \longrightarrow L^2_{loc}$ boundedly

First (i) is true, if

$$2 - n/2 > 1 - n/p \Leftrightarrow p < \frac{2n}{n-1}$$

Finally (ii) is true if with

$$A \in L^q, \quad q > 2n$$

ii) In finity

$$\| A(x+y) \cdot \nabla u \|_{L^2}$$

$$\leq M_\delta \|\nabla u\|_{L^2} \leq M_\delta \|\Delta u\|_{L^2}$$

with

$$M_j = \sup_{x \in X_j} |A(x)|$$

Thus

$$|A(x)| \leq \frac{c}{|x| (\log |x|)^{1+\varepsilon}}$$

is sufficient. $\square$