"b) ⇒ a)": Now $\chi(D)u$ sat. c)

for every $\chi \in S$. But

$$(P_0(D)-\lambda)\,\chi(D)u = \chi(\xi)\hat{f}$$

Thus $\hat{f}\big|_{M_\lambda} \equiv 0$.

We has thus shown c) ⇒ b), a) ⇒ c) and b) ⇒ a)

$\square$

The following theorem is used to prove that the scattering matrix is unitary.

<u>14.3.8 Theorem</u> If $u \in B^*$ solves $(P_0(D)-\lambda)u = f$, $\in B$

$\lambda \in \mathbb{R} \setminus Z(P_0)$, then

(1) $u = R_0(\lambda \mp i0)f + u_\pm$; $\hat{u}_\pm = v_\pm \, dS$

where $v_\pm \in L^2(M_\lambda)$. Moreover

(2) $$\int_{M_\lambda} (|v_+|^2 - |v_-|^2)\,|P_0'|\,dS = 2(2\pi)^{n+1}\,\mathrm{Im}\,(u,f)$$

Finally

$$P_0^{(\alpha)}(D)u \in B^* \quad \forall \alpha \iff$$

$$|P_0'|\,v_\pm \in L^2(M_\lambda)$$

Proof Take

$$v = \big(R_0(\lambda+i0) + R_0(\lambda-i0)\big) f/2$$

Then

$$(P_0(D)-\lambda)\, v = f$$

Hence

$$u_0 = u - v \quad \text{solves}$$

$$u_0 \in B^* \quad \text{and} \quad (P_0(D)-\lambda)\, u_0 = 0$$

$$\Rightarrow \quad \hat{u}_0 = v_0\, dS \,, \quad v_0 \in L^2(M_\lambda)$$

Now

$$\operatorname{Im}(u,f) = \overset{=0}{\operatorname{Im}(v,f)} + \operatorname{Im}(u_0,f) \tag{3}$$

Note $\forall \varepsilon > 0$

$$\frac{1}{P(\xi)-\lambda-i\varepsilon} + \frac{1}{P(\xi)-\lambda+i\varepsilon} = \frac{P(\xi)-\lambda}{P(\xi)-\lambda-\varepsilon^2} \in \mathbb{R}.$$

But by Plancherel

$$\operatorname{Im}(u_0,f) = \frac{1}{(2\pi)^n}\int_{M_\lambda} v_0\, \hat{f}\, dS \tag{4}$$

Recall $f \in B \Rightarrow \hat{f} \in L^2(M_\lambda)$ and

$$\int_{M_\lambda} |\hat{f}|^2\, dS \le C \|f\|_B^2$$

Since

$$\frac{1}{P(\xi)-\lambda-i0} - \frac{1}{P(\xi)-\lambda+i0} = 2\pi i\, \delta(P(\xi)-\lambda)$$

we have

$$v = (R_0(\lambda+i0) + R_0(\lambda-i0))\frac{f}{2} = R_0(\lambda\mp i0)f$$

$$\pm \pi i\, \mathcal{F}^{-1}\left(\hat{f}\,\frac{dS}{|P_0'|}\right)$$

Thus $u = u_0 + v$ has the form

$$u = R_0(\lambda\mp i0)f + u_\pm$$

where $\hat{u}_\pm = v_\pm\, dS$ and

$$v_\pm = v_0 \pm \pi i \frac{\hat{f}}{|P_0'|}.$$

Moreover

$$|v_+|^2 - |v_-|^2 = 4\pi\, \mathrm{Im}\, \frac{v_0\, \bar{\hat{f}}}{|P_0'|}$$

[Check $|a+ib|^2 - |a-ib|^2 = 4\,\mathrm{Re}\, a\bar{b}$, ok]

This (3) and (4) prove formula (2)

(and that $(|v_+|^2 - |v_-|^2)|P_0'| \in L^1(M_\lambda)$)

Finally

Theorem 14.3.2

$$P_0^{(\alpha)}(D)u \in B^* \Leftrightarrow P_0^{(\alpha)}(D)u_\pm \in B^*$$

$$\Leftrightarrow \quad P_0^{(\alpha)} v_\pm \in L^2(M_\lambda) \Leftrightarrow P_0' v_\pm \in L^2(M_\lambda)$$

if P_0 is simply characteristic

$\square$

We did not exactly prove that $v_\pm \in L^2$. This follows from the facts that P_0 has simple characteristics and from $|\tilde{P}_0(\xi)| \to \infty$ as $|\xi| \to \infty$. $\Rightarrow$

$$|P_0'(\xi)| \to \infty \quad \text{as} \quad |\xi| \to \infty \quad \text{in } M_\lambda$$

since

$$|\tilde{P}'(\xi)| \leq |P_0(\xi)| + |P_0'(\xi)|$$

Thus

$$|P_0'(\xi)| \geq c > 0 \quad \text{for} \quad |\xi| \in M_\lambda$$

$$\Rightarrow \quad \hat{f}/P_0' \in L^2 \Rightarrow v_\pm \in L^2.$$

14.3.9 Corollary If $(u,f) \in \mathbb{R}$, then u is λ-outgoing $\Leftrightarrow$ u is λ-incoming

Proof We have

$$\int_{M_\lambda} (|v_+|^2 - |v_-|^2)\,|P_0'|\,dS = c\,\mathrm{Im}(u,f) = 0$$

$$\Rightarrow \left(v_+ = 0 \Leftrightarrow v_- = 0 \right)$$

$$u = R_0(\lambda \mp i0) f + u_\pm$$

$$\hat{u}_\pm = v_\pm\, dS$$

$\square$

14.3.10 Corollary If $f \in \mathcal{B}$, $\lambda \notin \mathbb{R} \setminus Z(P_0)$

$u = R_0(\lambda \pm i0) f$, then

$$c \int_{M_\lambda} |\hat{f}|^2 \frac{dS}{|P_0'|} = \pm \mathrm{Im}(u,f),$$

$$c = \frac{1}{2}(2\pi)^{1-n}$$

Proof $u = R_0(\lambda + i0) f \Rightarrow u_- = 0 \Rightarrow v_- = 0$

$$v_\pm = v_0 \pm \pi i\, \hat{f}/|P_0'| \Rightarrow v_0 = \pi i\, \hat{f}/|P'|$$

and $$\int_{M_\lambda} (|v_+|^2 - |v_-|^2)\,|P_0'|\,dS = c\,\mathrm{Im}(u,f)$$

$\square$

14.4 SHORT RANGE POTENTIALS

We study

$$P = P_0(D) + V$$

where $V = V(x, D)$ is a diff. operator.

Formally the resolvent R for P satisfies

$$R_0(z) = R(z) + R_0(z) V R(z) = R(z) + R(z) V R_0(z)$$

Proof

(1) $$P - z = P_0 - z + V$$

$$R(z) = (P - z)^{-1}, \quad R_0(z) = (P_0 - z)^{-1}$$

Multiply (1) from left by R_0 and from right by R $\Rightarrow$

$$R_0(z) = R(z) + R_0(z) V R(z)$$

The last formula follows by multiplying other way around.

$\square$

Thus we expect

(2) $$R(z) = R_0(z)(1 + V R_0(z))^{-1}$$

To be able to use Fredholm theory we want VR_0 is compact in appropriate function space.

Recall Theorem 14.3.2 $\Rightarrow$

$$R_0(z) : B \longrightarrow \{ u \mid P_0^{(\alpha)}(D)u \in B^*, \ \forall \alpha \},$$

$$=: B^*_{P_0}$$

Define

$$\|u\|_{B^*_{P_0}} = \sum_\alpha \| P_0^{(\alpha)}(D) u \|_{B^*}$$

$$= \sum_\alpha \| (\partial_\xi^\alpha P_0)(D) u \|_{B^*}$$

<u>14.4.1 Def'</u> $V(x, D)$ is <u>short range</u>, (for $P_0(D)$) if

$$V(x,D) \left(\{ u \in B^*_{P_0} \cap C^\infty \mid \|u\|_{B^*_{P_0}} \leq 1 \} \right) \subset B$$

and is precompact.

Let $\Omega \subset\subset \mathbb{R}^n$ be the unit ball in $\mathbb{R}^n$