We showed last time (Hö 14.3.3)

$u \in B^*$, $(P_0(D) - \lambda) u = 0$, $\lambda \in \mathbb{R} \setminus Z(P_0)$

$\iff$ $\hat{u} = v \, dS$, $v \in L^2(M_\lambda)$

Moreover

(1) $\quad \|u\|_{B^*} \sim \|v\|_{L^2}$

For $\hat{f} \in C_0^\infty$ we have also shown

(2) $$\int R(\lambda + i0) f \, \overline{R(\lambda \pm i0) f} \, \varphi(x/R) \frac{dx}{R}$$

$$\xrightarrow[R \to \infty]{} A_\pm$$

where $A_- = 0$ and

$$A_+ = \frac{1}{(2\pi)^{n-1}} \int_{M_\lambda} |\hat{f}|^2 \Big(\int_{s>0} \varphi(s p') \, ds \Big) \frac{dS}{|p'|}$$

<u>Remark</u> If $P_0(\xi) = \xi^2$, we have

$p'(\xi) = \nabla p(\xi) = 2\xi$ and $|p'(\xi)| = 2|\xi| = 2k$

for $\xi \in M_\lambda = S_k = \{|\xi| = k\}$, $\lambda = k^2$

Thus

$$A_+ = \frac{1}{(2\pi)^{n-1}} \frac{1}{4k} \int_{|\xi| = k} |\hat{f}(\xi)|^2 \Big(\int_0^\infty \varphi(t\xi) \, dt \Big) dS(\xi)$$

From now on we use Hörmander for numbering theorems
This generalizes to an important

Theorem 14.3.4 Assume $\varphi \in C_0$;

$$Q_j < P_0 \quad , \quad j = 1, 2$$

$$\lambda \in \mathbb{R} \setminus Z(P_0) \qquad \text{and}$$

$$\left.\begin{array}{l}(P_0(D) - \lambda) u = 0 \\ u \in B^*\end{array}\right\} \Leftrightarrow \quad \hat{u} = v \, dS$$

Then for $f_1, f_2 \in B$

$$(1) \qquad \int Q_1(D) R(\lambda + i0) f_1 \overline{Q_2(D) R(\lambda \pm i0) f_2} \, \varphi(\cdot/R) \frac{dx}{R}$$

$$\to A_\pm$$

where $A_- = 0$ and

$$(2) \qquad A_+ = \frac{1}{(2\pi)^{n-1}} \int_{M_\lambda} Q_1 \hat{f}_1 \overline{Q_2 \hat{f}_2} \left(\int_0^\infty \varphi(t \hat{P}_0') \, dt \right) \frac{dS}{|P_0'|^2}$$

$$(3) \qquad \int Q_1(D) R(\lambda + i0) f_1 \, \bar{u} \, \varphi(\cdot/R) \, dx/R \to$$

$$\to i (2\pi)^{-n} \int_{M_\lambda} Q_1 \hat{f} \, \bar{v} \left(\int_{-\infty}^{\infty} \varphi(t \hat{P}_0') \, dt \right) \frac{dS}{|P_0'|}$$

note

and

$$(4) \qquad \int |u|^2 \varphi(\cdot/R) \frac{dx}{R} \to \frac{1}{(2\pi)^{n-1}} \int_{M_\lambda} |v|^2 \left(\int_{\mathbb{R}} \varphi(t \hat{P}_0') \, dt \right) \frac{dS}{|P_0'|}$$

Proof Density + Theorem 14.2.2 (i.e. (1))

Details omitted.

Remarks 1 Similar result holds also for $R(\lambda - i0) f_1$.

Remark 2 Intuitively: contribution to $R_0(\lambda + i0) f$ that originate from

$$\xi \in M_\lambda$$

are concentrated to the direction $P_0'(\xi)$ and contribution to u spread equally to both direction $\pm P_0'(\xi)$.

14.3.5 Definition $u \in B^*$ is λ-outgoing if $\lambda \in \mathbb{R} \setminus Z(P_0)$, if $\exists\, f \in B$ s.t.

$$u = R(\lambda + i0) f$$

and u is λ-incoming if

$$u = R(\lambda - i0) f$$

We have shown

f is both λ-incoming and λ-outgoing

$\Leftrightarrow \quad \hat{f} = 0$ on M_λ

These solutions are characterized by

<u>14.3.6 Theorem</u> Let $u \in B^*$ and $\lambda \in \mathbb{R} \setminus Z(P_0)$.

Assume $(P_0(D) - \lambda) u = f \in B$. Then TFAE

a) u is both in- and outgoing

b) $u \in \mathring{B}^*$

c) $Q(D) u \in \mathring{B}^*$, for $Q < P_0$

<u>Proof</u> c) $\Rightarrow$ b) is trivial.

"a) $\Rightarrow$ c)": We use Theorem 14.3.4 (1) i.e.

$$\int |Q(D)u|^2 \varphi(\cdot/R) \frac{dx}{R} =$$

$$\int Q(D) R(\lambda + i0) f \, \overline{Q(D) R(\lambda - i0) f} \, \varphi(\cdot/R) \frac{dx}{R}$$

$$\to 0$$

By taking $\varphi \geq 0$ and $\varphi > 0$ the claim follows by definition of $\mathring{B}^*$.

"c) ⟹ a)" Take $\psi \in C_0^\infty$ with $\psi(x) = 1$ for $|x| \le 1$. Let

$$u_R(x) = \psi(x/R)\, u(x).$$

Then

$$(1) \quad (P_0(D) - \lambda)\, u_R(x) = \psi(x/R) f(x) + \sum_{\alpha \neq 0} R^{-|\alpha|} (\partial^\alpha \psi)(x/R) P_0^{(\alpha)}(D) u$$

Lemma If $u \in \mathring{B}^*$ then

$$\left\| \tfrac{1}{R} \varphi(x/R)\, u \right\|_B \to 0 \quad \text{as } R \to \infty$$

Proof Assume supp $\varphi \subset B(0,1)$.

Then

$$\| \varphi(x/R)\, u/R \|_B \le c \sum_{2^j \le R} 2^{+j} \int_{X_j} |u|^2 \frac{1}{R^2}\, dx$$

$$\le c R^{-1} \sum_{2^j \le R} \int_{X_j} |u|^2\, dx = c R^{-1} \int_{|x| \le R} |u|^2 dx \to 0$$

$\square$

Thus the sum in (1) $\to 0$ in B.

On the other hand

$$\psi(x/R) f \to f \quad \text{in } B$$

This and (1) show $\hat{f}(\xi) = 0$ for $\xi \in M_\lambda$

⟹ a) holds.