KSLUENTO 2

In last lecture we had

$$\partial p / \partial \xi_1 > 0$$

and

$$F : (\xi_1, \xi') = (p(\xi), \xi')$$

$$F^{-1} : (\lambda, \xi') = (\eta(\xi', \lambda), \xi')$$

both local isomorphisms.

$$u_z = \mathcal{F}^{-1}\left[\frac{\chi \hat{f}}{p - z - i0} \right], \quad \operatorname{Im} z \geq 0$$

Lemma 2.4.5

(1) $\| u_z(x_1, \cdot) \|_{L^2} \leq C \, \| f(t, \cdot) \|_{L^2} * H(x_1)$

$+$ small as $x_1 \to \infty$

Idea: For $\eta = \eta(\lambda, \xi')$ get

$$\frac{\chi(\xi)}{p(\xi) - \lambda - i0} = \frac{\chi(\xi)}{\xi_1 - \eta - i0} \, \frac{\xi_1 - \eta}{p(\xi) - \lambda}$$

Denote

$$\chi(\xi) \frac{\xi_1 - \eta}{p(\xi) - \lambda} = g(\xi)$$

Then

$$\frac{g(\xi)}{\xi_1 - \eta - i0} = \frac{g(\eta, \xi')}{\xi_1 - \eta - i0} + \frac{g(\xi) - g(\eta, \xi')}{\xi_1 - \eta}$$

$\swarrow \mathcal{F}^{-1}$

H_1

2.4.6 Corollary If $\rho \in C^2$, then if u_z is as in Lemma 2.4.5

$$\| u_z \|_{B^*} \le C \| f \|_B$$

Proof Use Lemma 2.4.5 with $n = 0$ to get

(1) $\| u_z(x_1, \cdot) \|_{L^2} \le C \int \| f(t, \cdot) \|_{L^2} \, dt$

Recall that (Theorem 2.3.7)

(2) $\int_{-\infty}^{\infty} \| u(x_1, \cdot) \|_{L^2} \, dx_1 \le \sqrt{2} \, \| u \|_B$

(3) $\| u \|_{B^*} \le \sqrt{2} \sup_{x_1} \| u(x_1, \cdot) \|_{L^2}$

Thus

$$\| u_2 \|_{B^*} \overset{(3)}{\leq} \sqrt{2} \sup_{x_1} \| u_2(x_1, \cdot) \|_{L^2}$$

$$\overset{(1)}{\leq} C \int \| f(t, \cdot) \|_{L^2} \, dt \overset{(2)}{\leq} C \| f \|_B$$

$\square$

2.4.7 Corollary If $\varphi \in C_0^0(\mathbb{R}^n)$ and $\lambda \in X_1 \subset \mathbb{R}$ we have

$$\int |u_\lambda(x)|^2 \varphi(x/R) \, dx/R$$

$$\longrightarrow \frac{1}{(2\pi)^{n-1}} \int_{p(\xi)=\lambda} |\mathcal{X} \hat{f}|^2 \left(\int_{s>0} \varphi(sp') \, ds \right) \frac{dS}{|p'|}$$

as $x_1 \to \infty$. Here dS is the surface measure on $\{p(\xi) = \lambda\}$

Proof We will use Theorem 2.4.5 and $u_\infty(x', \lambda)$ defined there i.e.

$$\hat{u}_\infty(\xi', \lambda) = i \frac{\mathcal{X} \hat{f}}{\partial p / \partial \xi_1} (\eta, \xi')$$

especially

(1) $\left(e^{i x_1 \eta(D', \lambda)} u_\infty \right)(-x', \lambda)$

is the Fourier transform of

(2) $$e^{ix_1\eta}\left(\frac{i x \hat{f}}{\partial p/\partial \xi_1}\right)(\eta, \xi')$$

We recall Corollary 2.2.6 (Parseval's theorem):

If $V_{x_1} = \mathcal{F}_{n-1}(e^{ix_1F}v)$, $v \in L^2(\mathbb{R}^{n-1})$, $F \in C^1(\mathbb{R}^{n-1})$ and $\varphi \in C_0(\mathbb{R}^n)$

then

(3) $$\int_{x_1>0} |V_{x_1}(\xi)|^2 \varphi(\xi/R)\, dx_1\, d\xi/R \longrightarrow \int_{x_1>0} |v(x')|^2 \varphi(x_1F'(x'), x_1)\, dx_1$$

as $R \to \infty$. We take $F = \eta$.

Now by for every $M > 0$

$$\int_{0<x_1<M} |u_\lambda(x_1, x')|^2 \varphi(x_1 R^{-1}, x' R^{-1})\, dx/R$$

$\longrightarrow 0$ as $R \to \infty$

Hence by Theorem 4.2.5

we may substitute $u_\lambda(x_1, x')$

by $e^{ix_1\eta(D',\lambda)} u_\infty(x',\lambda)$ and use (3) to obtain

$$\int_{x_1>0} |u_\lambda(x)|^2 \varphi(x/R)\, dx/R \longrightarrow$$

(*)

$$C_n \int_{t>0} \left|\frac{\mathcal{X}\hat{f}}{\partial p/\partial \xi_1}(\eta, \xi')\right|^2 \varphi\left(t, -t\frac{\partial \eta}{\partial \xi'}\right) \cdot dt\, d\xi'.$$

Also by Theorem 4.2.5, (2) the integral over $x_1 < 0 \longrightarrow 0$.

Finally

$$p(\eta, \xi') = \lambda$$

so the integration in (*) is over the surface $p(\xi) = \lambda$. Moreover

$$(4) \qquad \frac{\partial p}{\partial \xi_1}(\eta, \xi') \frac{\partial \eta}{\partial \xi'} + \frac{\partial p}{\partial \xi'} = 0$$

Compare: Surface

$$f(x_1, x_2) = \lambda \Rightarrow df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial f}{\partial x_2} = 0$$

Finally

(5) $$\frac{d\xi'}{\partial p/\partial \xi_1} = \frac{dS}{|p'|}$$ (Exercise)

and we can set $f = s\frac{\partial p}{\partial \xi_1}$ in (*) to get

$$\int |u_\lambda(x)|^2 \varphi(x/R)\frac{dx}{R} \longrightarrow$$

$$C_n \int_{p(\xi)=\lambda} \int_{s>0} |\chi(\hat{f})|^2 \underbrace{\varphi\left(s\frac{\partial p}{\partial \xi_1}, -s\frac{\partial p}{\partial \xi_1}\frac{\partial \mu}{\partial \xi'}\right)}_{\substack{\text{by } (4) \\ \varphi(sp')}} ds \frac{dS}{|p'|}$$

by (5).

□

We are now ready to prove

2.4.8 Theorem Assume $p \in C^2(X)$, $X \Subset \mathbb{R}^n$ and $p' \neq 0$ in X. Then if $f \in B$ and $\chi \in C_0^\infty(X)$ we may extend the function

$$z \longmapsto u_z = \mathcal{F}^{-1}\left(\frac{\chi \hat{f}}{p - z}\right), \quad \operatorname{Im} z \neq 0$$

to $\mathbb{C}^\pm = \{\pm \operatorname{Im} z \geq 0\}$ as a weak* cont. function with values in B^* and

$$\|u_z\|_{B^*} \leq C \|f\|_B,$$

$f \in B$, $z \in \mathbb{C}^\pm$.

Proof For $f \in S$ and χ of suff. small support the claim follows from Corollary 2.4.6. By partition of unity the claim is shown without the smallness assumptions of the support of χ.

Now

$$z \longmapsto u_z \quad \text{is cont}$$

with values in S' we see

$$\| u_z \|_{B^*} \leq C \| f \|_B ,$$

for $f \in S$. Finally since S is dense in B the claim follows

$\square$

2.4.9 Theorem Assume $p, f, \mathcal{X}$ and u_z are as in Theorem 2.4.8, then for every $\varphi \in C_0^0(\mathbb{R}^n)$

$$\int | u_{\lambda \pm i0}(x) |^2 \varphi(x/R) \, dx/R$$

$$\longrightarrow \frac{1}{(2\pi)^{n-1}} \int_{p=\lambda} | \mathcal{X} \hat{f} |^2 \left(\int_{\pm s > 0} \varphi(s p') \, ds \right) \frac{dS}{|p'|}$$

and

$$\int u_{\lambda + i0}(x) \, \overline{u_{\lambda - i0}(x)} \, \varphi(x/R) \frac{dx}{R}$$

$$\longrightarrow 0$$

as $R \longrightarrow \infty$.