Scattering theory
Exercises #7, 09.11.2007
(return by 16.11.2007)

In the following exercises, if you get stuck, it may be useful to consult Hörmander, Analysis of Linear Partial Differential Operators, vol. 1.

1. Let V be a k-dimensional linear subspace of $\mathbf{R}^n$, and let $d\sigma_V$ be the Euclidean surface measure on V. Show that the Fourier transform of $d\sigma_V$ is $(2\pi)^k d\sigma_{V^\perp}$.

2. Compute the Fourier transform of the surface measure of the unit sphere in $\mathbf{R}^3$.

3. Let $u = d\sigma$ be the surface measure of the unit sphere in $\mathbf{R}^n$. Show that

$$\hat{u}(\xi) = (2\pi)^{n/2}|\xi|^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(|\xi|),$$

   where $J_s(r)$ are Bessel functions of the first kind.

4. Let M be a C^2 hypersurface in $\mathbf{R}^n$, and $M_\varepsilon = M + B(0,\varepsilon)$. If $f \in C_0(M)$ and $\tilde{f}$ is any function in $C_0(\mathbf{R}^n)$ satisfying $\tilde{f}|_M = f$, show that

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{M_\varepsilon} \tilde{f}(x)\, dx = \int_M f\, dS$$

   where dS is surface measure on M, defined as $dS(x') = \sqrt{1 + |\nabla h(x')|^2}\, dx'$ in a coordinate patch where M is given as the graph $\{(x', h(x'))\}$.