Scattering theory

Exercises #4, 5.10.2007

(return by 12.10.2007)

1. Let μ be a finite positive Borel measure on $\mathbf{R}$ and $H = L^2(\mathbf{R}, \mu)$. Define $Af(\lambda) = \lambda f(\lambda)$ with domain $\mathscr{D}(A) = \{f \in H\,;\, \lambda f \in H\}$. Show in detail that the decomposition $\mu = \mu_{\text{pp}} + \mu_{\text{ac}} + \mu_{\text{sing}}$ induces a splitting
$$H = H_{\text{pp}} \oplus H_{\text{ac}} \oplus H_{\text{sing}}$$
where $f \in H$ is in H_{pp} iff the spectral measure $d\mu_f$ is pure point, etc.

2. Let $H = \ell^2 = \{(u_n)_{n=-\infty}^{\infty}\,;\, \sum |u_n|^2 < \infty\}$. Define the left and right shift operators by $(Lu)_n = u_{n+1}$ and $(Ru)_n = u_{n-1}$. Show that $A = L + R$ is a bounded self-adjoint operator, and find a unitary map $U : H \to L^2(0,1)$ and a function $a(x)$ such that $UAU^* f(x) = a(x) f(x)$ for $f \in L^2(0,1)$. Compute the spectral measures $d\mu_u(\lambda) = d(E_\lambda u, u)$.

The following exercises consider the isolated points of $\sigma(A)$, where A is a self-adjoint operator in a Hilbert space H, and also the *discrete spectrum* $\sigma_d(A) := \{\lambda \in \sigma(A)\,;\, \lambda \text{ is an isolated eigenvalue with } \dim\ker(A-\lambda) < \infty\}$.

3. If λ is an isolated point of $\sigma(A)$, define the Riesz projection P_λ by
$$(P_\lambda u, v) = -\frac{1}{2\pi i}\oint_{\Gamma_\lambda} (R(z)u, v)\,dz, \quad u, v \in H,$$
where $\Gamma_\lambda = \partial B(\lambda, \varepsilon) \subset \mathbf{C}$ and $\overline{B(\lambda,\varepsilon)} \cap \sigma(A) = \{\lambda\}$. Show that the definition is independent of ε, and that P_λ is an orthogonal projection onto $\ker(A - \lambda)$. (Hint: show that $z \mapsto (R(z)u, v)$ is analytic in $\rho(A)$, and use basic results in complex analysis.)

4. Use the Riesz projections to show that any isolated point of $\sigma(A)$ is an eigenvalue, and that $\lambda \in \sigma_d(A)$ if and only if $0 < \dim\ker(A-\lambda) < \infty$ and $A - \lambda|_{\mathscr{D}(A)\cap\ker(A-\lambda)^\perp}$ has a bounded inverse.