Scattering theory
Exercises #3, 28.9.2007
(return by 5.10.2007)

1. Let A be self-adjoint, and assume that for some $M > 0$ one has
$$\|(A-\lambda)u\| \geq M\|u\|, \quad u \in \mathscr{D}(A).$$
Show that $\lambda \in \rho(A)$. Further, show that $\{z \in \mathbf{C}\,;\, |z-\lambda| < M\} \subset \rho(A)$. (Hint: remember that A has no residual spectrum.)

2. We say that λ is an approximate eigenvalue of A if there is a sequence $(u_j) \subset \mathscr{D}(A)$, $\|u_j\| = 1$, such that $\|(A-\lambda)u_j\| \to 0$ as $j \to \infty$. If A is self-adjoint, show that $\sigma(A)$ is precisely the set of approximate eigenvalues.

3. Let $H = L^2(\mathbf{R})$ and define A by $Au = -u''$ with $\mathscr{D}(A) = H^2(\mathbf{R})$. Show that A is self-adjoint and $\sigma(A) = [0,\infty)$. (Hint: use $(Au,u) \geq 0$ and Problems 1 and 2.)

4. Let $H = L^2(I)$, $I = (0,1)$, and define for $z \in \mathbf{C}$ the operator $A_z : u \mapsto iu'$ with domain $\mathscr{D}(A_z) = \{u \in H^1(I)\,;\, u(1) = zu(0)\}$. Determine the values of z for which A_z is self-adjoint. What is $\sigma(A_z)$ in this case?