Scattering theory

Exercises #2, 21.9.2007

(return by 28.9.2007)

1. Fill in the details of the proof of Friedrichs extension theorem (any symmetric semibounded operator has a self-adjoint extension).

2. Let $H = L^2(I)$ with $I = (-1, 1)$, and define A with domain $\mathscr{D}(A) = C_0^\infty(I)$ by $Au = -u''$. Show that A is symmetric and semibounded, and determine the Friedrichs extension of A. (Hint: use the Poincaré inequality $\|u\|_{L^2(I)} \leq C\|u'\|_{L^2(I)}$ for $u \in C_0^\infty(I)$).

3. Let $H = L^2(\mathbf{R})$, and define A with domain $\mathscr{D}(A) = C_0^\infty(\mathbf{R})$ by $(Au)(x) = i(x^2u'(x)+xu(x))$. Show that A is symmetric and determine the defect indices $n_\pm(A)$. Does A have a self-adjoint extension?

4. Let A be a self-adjoint operator on $\mathscr{D}(A) \subset H$, let E be the projection valued measure for A given in the spectral theorem, and let g be a complex Borel function on the real line. Show that

$$\mathscr{D}(g(A)) = \{x \in H\,;\, \int_{-\infty}^{\infty} |g(\lambda)|^2\, d(x, E_\lambda x)\}$$

is a dense linear subspace of H. Prove that the formula

$$(x, g(A)x) = \int_{-\infty}^{\infty} g(\lambda)\, d(x, E_\lambda x), \quad x \in \mathscr{D}(g(A)),$$

defines a linear operator with domain $\mathscr{D}(g(A))$. If g is real, show that $g(A)$ is self-adjoint. (Hint: if g is bounded, then $g(A)$ is a bounded operator on H and $\|g(A)x\|^2 = \int |g(\lambda)|^2\, d(x, E_\lambda x)$. The functions $\chi_n = \chi_{\{|g|\leq n\}}$ may be useful.)