Calderón problem

Solutions to Exercises #2, 16.6.2008

4. Let $P_0 = -h^2\Delta$ and $P_{0,\varphi} = e^{\varphi/h}P_0e^{-\varphi/h} = (hD)^2 - 1 + 2i\alpha \cdot hD$, and let $u \in C_c^\infty(\Omega)$. Theorem 5.3 in the lectures states that

$$h\|u\| \leq C\|P_{0,\varphi}u\|. \tag{1}$$

To have a bound for $\|Du\|$ we compute

$$\|hDu\|^2 = (hDu|hDu) = ((hD)^2u|u) = (P_{0,\varphi}u|u) + \|u\|^2 - 2i(\alpha \cdot hDu|u),$$

which implies upon using the inequality $|ab| \leq \delta a^2 + \frac{1}{4\delta}b^2$ that

$$\|hDu\|^2 \leq \frac{1}{2}\|P_{0,\varphi}u\|^2 + \frac{1}{2}\|hDu\|^2 + C\|u\|^2.$$

Moving one term one the other side, we have

$$\|hDu\|^2 \leq \|P_{0,\varphi}u\|^2 + C\|u\|^2.$$

This and the original Carleman estimate (1) imply

$$h^2\|hDu\|^2 \leq h^2\|P_{0,\varphi}u\|^2 + C\|P_{0,\varphi}u\|^2 \leq C\|P_{0,\varphi}u\|^2,$$

and we obtain the required estimate in the case $q = 0$,

$$h\|u\| + h\|hDu\| \leq C\|P_{0,\varphi}u\|.$$

If q is nonzero one may proceed as in the proof of Theorem 5.3 to add a potential if h is small enough.

5. Let $P_\varphi = e^{\varphi/h}h^2(-\Delta + q)e^{-\varphi/h}$, so that $P_\varphi^* = P_{0,-\varphi} + h^2\bar{q}$. Applying Theorem 5.6 to P_φ^*, we have the Carleman estimate

$$h^3((\alpha \cdot \nu)\partial_\nu v|\partial_\nu v)_{\partial\Omega_+} + h^2\|v\|^2 \leq C\|P_\varphi^* v\|^2 - Ch^3((\alpha \cdot \nu)\partial_\nu v|\partial_\nu v)_{\partial\Omega_-},$$

which is valid for $v \in C^\infty(\overline{\Omega})$ satisfying $v|_{\partial\Omega} = 0$. Define

$$M = \{v \in C^\infty(\overline{\Omega})\,;\, v|_{\partial\Omega} = 0, \partial_\nu v|_{\partial\Omega_-} = 0\}.$$

Let $D = P_\varphi^* M$ be a subspace of $L^2(\Omega)$, and consider the linear functional

$$L : D \to \mathbf{C}, \quad L(P_\varphi^* v) = (v|f), \quad \text{for } v \in M.$$

This is well defined by the Carleman estimate, which also implies

$$|L(P_\varphi^* v)| \leq \|v\|\|f\| \leq \frac{C}{h}\|f\|\|P_\varphi^* v\|.$$

Thus L is a bounded linear functional on D.

The Hahn-Banach theorem ensures that there is a bounded linear functional $\hat{L} : L^2(\Omega) \to \mathbf{C}$ satisfying $\hat{L}|_D = L$ and $\|\hat{L}\| \leq Ch^{-1}\|f\|$. By the Riesz representation theorem, there is $\tilde{r} \in L^2(\Omega)$ such that

$$\hat{L}(w) = (w|\tilde{r}), \quad w \in L^2(\Omega),$$

and $\|\tilde{r}\| \leq Ch^{-1}\|f\|$. Then, for $w \in C_c^\infty(\Omega)$, by the definition of weak derivatives we have

$$(w|P_\varphi \tilde{r}) = (P_\varphi^* w|\tilde{r}) = \hat{L}(P_\varphi^* w) = L(P_\varphi^* w) = (w|f),$$

which shows that $P_\varphi \tilde{r} = f$ in the weak sense. The function $r = h^2\tilde{r}$ satisfies $e^{\varphi/h}(-\Delta + q)e^{-\varphi/h} r = f$ in Ω, and $\|r\| \leq Ch\|f\|$.

It remains to show that $\tilde{r}|_{\partial\Omega_+} = 0$. For this we use the fact (stated in the lectures) that $\tilde{r}$ is in the space $H_\Delta(\Omega) = \{u \in L^2(\Omega)\,;\, \Delta u \in L^2(\Omega)\}$, and that there is a well defined bounded trace operator $H_\Delta(\Omega) \mapsto H^{-1/2}(\partial\Omega)$. An integration by parts (which can be justified by using properties of $H_\Delta(\Omega)$) gives for $v \in M$ that

$$(v|P_\varphi \tilde{r}) - (P_\varphi^* v|\tilde{r}) = h^2(\partial_\nu v|\tilde{r})_{\partial\Omega}.$$

The left hand side is $(v|f) - (v|f) = 0$ and $\partial_\nu v|_{\partial\Omega_-} = 0$, which implies

$$(\partial_\nu v|\tilde{r})_{\partial\Omega_+} = 0, \quad v \in M.$$

Since we may choose $\partial_\nu v$ to be any smooth function in a slightly smaller set than $\partial\Omega_+$, we obtain $\tilde{r}|_{\partial\Omega_+} = 0$ as required.

6. Let $P_0 = P_0(hD) = (hD)^2$ and $\varphi(x) = \alpha \cdot x$, and consider the convexified weight $\varphi_\varepsilon(x) = \varphi(x) + \frac{h}{\varepsilon}\frac{\varphi^2}{2}$. We first prove a Carleman estimate for the operator

$$P_{0,\varphi_\varepsilon} = P_{0,\varphi_\varepsilon}(hD) = e^{\varphi_\varepsilon/h} P_0(hD) e^{-\varphi_\varepsilon/h} = P_0(hD + i\nabla\varphi_\varepsilon).$$

As in the proof of Theorem 5.3, we decompose $P_{0,\varphi_\varepsilon} = A_\varepsilon + iB_\varepsilon$ where $A_\varepsilon = (hD)^2 - (\nabla\varphi_\varepsilon)^2$ and $B_\varepsilon = \nabla\varphi_\varepsilon \circ hD + hD \circ \nabla\varphi_\varepsilon$ are self-adjoint. Also, since $\nabla\varphi_\varepsilon = (1 + \frac{h}{\varepsilon}\varphi)\alpha$, we have

$$A_\varepsilon = (hD)^2 - (1 + \frac{h}{\varepsilon}\varphi)^2, \quad B_\varepsilon = 2(1 + \frac{h}{\varepsilon}\varphi)\alpha \cdot hD + \frac{h^2}{i\varepsilon}.$$

If $u \in C_c^\infty(\Omega)$, we compute

$$\|P_{0,\varphi_\varepsilon} u\|^2 = \|A_\varepsilon u\|^2 + \|B_\varepsilon u\|^2 + (i[A_\varepsilon, B_\varepsilon]u|u).$$

Now A_ε and B_ε have variable coefficients and $[A_\varepsilon, B_\varepsilon]$ does not vanish. A direct computation shows that

$$i[A_\varepsilon, B_\varepsilon] = \frac{4h^2}{\varepsilon}\Big(\sum_{j,k=1}^n \alpha_j \alpha_k hD_j hD_k + (1 + \frac{h}{\varepsilon}\varphi)^2\Big).$$

The inner product becomes

$$(i[A_\varepsilon, B_\varepsilon]u|u) = \frac{4h^2}{\varepsilon}\|\alpha \cdot hDu\|^2 + \frac{4h^2}{\varepsilon}\|(1 + \frac{h}{\varepsilon}\varphi)u\|^2.$$

Note that this expression is positive instead of zero. If ε is fixed and h is so small that $1 + \frac{h}{\varepsilon}\varphi \geq 1/2$ for $x \in \overline{\Omega}$, we obtain

$$\|P_{0,\varphi_\varepsilon} u\|^2 \geq \frac{h^2}{\varepsilon}\|u\|^2.$$

This is in a sense stronger than the result in Theorem 5.3, since we may choose ε to be very small (but fixed).

If $A \in L^\infty(\Omega)^n$ is a vector field and if $q \in L^\infty(\Omega)$, consider the operator

$$P_{\varphi_\varepsilon} := e^{\varphi_\varepsilon/h} h^2(-\Delta + A \cdot D + q)e^{-\varphi_\varepsilon/h} = P_{0,\varphi_\varepsilon} + hA \cdot hD + ihA \cdot \nabla\varphi_\varepsilon + h^2 q.$$

We have

$$\frac{h}{\sqrt{\varepsilon}}\|u\| \leq \|P_{0,\varphi_\varepsilon} u\| \leq \|P_{\varphi_\varepsilon} u\| + Ch\|hDu\| + Ch\|u\|.$$

On the other hand, the argument in Exercise 4 gives

$$\|hDu\| \leq \|P_{0,\varphi_\varepsilon} u\| + C\|u\| \leq \|P_{\varphi_\varepsilon} u\| + Ch\|hDu\| + C\|u\|,$$

which implies for h small that

$$\|hDu\| \leq C\|P_{\varphi_\varepsilon} u\| + C\|u\|.$$

If $h \ll \varepsilon \ll 1$, combining these estimates gives

$$\frac{h}{\sqrt{\varepsilon}}(\|u\| + \|hDu\|) \leq C\|P_{\varphi_\varepsilon} u\|.$$

It remains to prove an estimate for P_φ instead of P_{φ_ε}. But

$$P_{\varphi_\varepsilon} = e^{\varphi^2/2\varepsilon} P_\varphi e^{-\varphi^2/2\varepsilon}$$

where $e^{\pm\varphi^2/2\varepsilon}$ is uniformly bounded in $\overline{\Omega}$ together with its derivatives. Thus the last Carleman estimate easily implies

$$h(\|u\| + \|hDu\|) \leq C\|P_\varphi u\|.$$