Calderón problem

Exercises #2, 23.04.2008

(return by 06.05.2008)

The exercises can be returned to me personally (room B414 or lectures), in my box at the mail room, or by email.

1. Let $f : \overline{\Omega} \to \mathbf{C}$ be continuous, where $\Omega \subseteq \mathbf{R}^n$ is bounded. Show that there is a modulus of continuity ω such that $|f(x) - f(y)| \leq \omega(|x - y|)$.

2. Assuming the claim in Ex. 3, determine $\Lambda_\gamma f$.

3. Let $\gamma \equiv 1$ in the unit disc $\mathbb{D} \subseteq \mathbf{R}^2$. Show that the solution in $H^1(\mathbb{D})$ to $\nabla \cdot \gamma \nabla u = 0$ in $\mathbb{D}$, with $u|_{\partial\mathbb{D}} = f \in H^{1/2}(\partial\mathbb{D})$, is given by
$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}(k) e^{ik\theta}.$$

Below, $q \in L^\infty(\Omega)$, and $\varphi(x) = \alpha \cdot x$ where α is a unit vector in $\mathbf{R}^n$.

4. (H^1 Carleman estimate) Show that there are $C > 0$ and $h_0 > 0$ such that for any h with $0 < h \leq h_0$, one has
$$\|u\| + \|hDu\| \leq Ch\|e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}u\|, \quad u \in C_c^\infty(\Omega).$$

5. (Solvability with vanishing data on part of boundary) Show that there are $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, the equation
$$\begin{cases} e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}r = f & \text{in } \Omega, \\ r = 0 & \text{on } \partial\Omega_+, \end{cases}$$
has a solution $r \in L^2(\Omega)$ for any $f \in L^2(\Omega)$, with $\|r\| \leq Ch\|f\|$. (Hint: use test functions which vanish, along with their normal derivative, on suitable parts of the boundary.)

6. (Large first order perturbations) Let $A = (A_1, \ldots, A_n) \in L^\infty(\Omega)^n$ be a vector field. Show that there are $C > 0$ and $h_0 > 0$ such that for any h with $0 < h \leq h_0$, one has
$$\|u\| + \|hDu\| \leq Ch\|e^{\varphi/h}(-\Delta + A \cdot \nabla + q)e^{-\varphi/h}u\|, \quad u \in C_c^\infty(\Omega).$$
(Hint: use the convexified weight $\varphi_\varepsilon = \varphi + \frac{h}{\varepsilon}\frac{\varphi^2}{2}$, where $\varepsilon > 0$ is small but fixed.)