Calderón problem

Exercises #1, 02.04.2008

(return by 15.04.2008)

The exercises can be returned to me personally (room B414 or lectures), in my box at the mail room, or by email.

1. If $(a_j)_{j=1}^{\infty} \subseteq \mathbf{C}$, let $s_N = \sum_{j=1}^{N} a_j$ be the partial sums. The partial sums are said to *Cesàro converge* to s if
$$\frac{s_1 + \ldots + s_N}{N} \to s \qquad \text{as } N \to \infty.$$
   (a) Show that if the partial sums converge to s in the usual sense, then they also Cesàro converge to s.
   (b) Show, by example, that the converse to (a) is not true.

2. Show that the Fejér kernel is an approximate identity.

3. Let $\gamma \in C^2(\overline{\Omega})$ and $u \in H^1(\Omega)$, and let $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$. Give a suitable weak definition, involving integration against test functions, for the following identity:
$$-\nabla \cdot \gamma \nabla(\gamma^{-1/2} u) = \gamma^{1/2}(-\Delta + q)u$$
   Prove that the identity holds in this weak sense.

4. (H^2 bounds for G_ζ) If $\zeta \in \mathbf{C}^n$ and $\zeta \cdot \zeta = 0$, $|\zeta| \geq 1$, follow the proof of Theorem 3.7 and show that G_ζ maps $L^2(\Omega)$ into $H^2(\Omega)$, with the estimate
$$\|G_\zeta f\|_{H^2(\Omega)} \leq C_0 |\zeta| \|f\|_{L^2(\Omega)}, \qquad f \in L^2(\Omega),$$
   where C_0 is a constant only depending on Ω and n.

5. (First order perturbations) Give an analog of Theorem 3.8 for the equation
$$(D^2 + 2\zeta \cdot D + A \cdot (D + \zeta) + q)r = f \quad \text{in } \Omega,$$
   where $q \in L^\infty(\Omega)$, and $A = (A_1, \ldots, A_n) \in L^\infty(\Omega)^n$ is a vector field in Ω whose L^∞ norm is sufficiently small.

6. (Additional decay for G_ζ) If $\zeta \in \mathbf{C}^n$ and $\zeta \cdot \zeta = 0$, $|\zeta| \geq 1$, and if $f \in L^2(\Omega)$ is a *fixed* function, show that
$$\|G_\zeta f\|_{H^1(\Omega)} \to 0, \qquad |\zeta| \to \infty.$$
   (Hint: decompose f as the sum of a smooth function and a small L^2 function.)