Topics in differential geometry

Analysis on manifolds

Lecture notes, Spring 2022

Mikko Salo

Department of Mathematics and Statistics
University of Jyväskylä

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Preface

This course is an introduction to analysis on manifolds. The topic may be viewed as an extension of multivariable calculus from the usual setting of Euclidean space to more general spaces, namely Riemannian manifolds. In particular we will explain how to compute derivatives, integrate, and solve differential equations (mostly Laplace and heat equation) on Riemannian manifolds. If time permits we will also use analytic methods to discuss certain landmark results in geometry, such as the uniformization theorem for Riemann surfaces, the Hodge theorem, the Weyl law for eigenvalues of the Laplacian, or the Gauss-Bonnet theorem.

- Title: Topics in differential geometry – analysis on manifolds
- Lectures: Tue 14-16 and Thu 14-16, room MaD381 (22.03.–05.05.).
- Exercise sessions: Tue 12-14, room MaD355 (29.03.-10.05.).
- Language: instruction in English, completion in English or Finnish.
- Prerequisites: multivariable calculus, functional analysis (partial differential equations are also helpful). Familiarity with smooth or Riemannian manifolds is helpful but not strictly necessary. The course is suitable as a continuation for Riemannian geometry in period 3, but it is also suitable for advanced students, PhD students and postdocs in analysis, geometry or PDEs (the required geometric background will be reviewed in the beginning).
- The course can be taken for credit (MATS5150, 5 cr) by attending the lectures and by returning written solutions to at least 50 % of the exercises in each exercise set.
- Lecture notes and exercises will be provided on the course webpage.
- Instructor: Mikko Salo, room MaD359, mikko.j.salo@jyu.fi
- Exercise sessions: Suman Kumar Sahoo and Pu-Zhao Kow

References. We will not follow any single textbook, and the main reference for the course are these lecture notes. However, the following textbooks may be useful:

- I. Madsen, J. Tornehave, *From calculus to cohomology.* Cambridge University Press, 1997.
- J. Jost, *Riemannian geometry and geometric analysis.* 4th edition, Springer, 2005.
- J.M. Lee, *Riemannian manifolds.* An introduction to curvature. Springer, 1997.
- P. Petersen, *Riemannian geometry.* 2nd edition, Springer, 2006.
- M.E. Taylor, *Partial differential equations I. Basic theory.* Springer, 1996.
- M.E. Taylor, *Partial differential equations II. Qualitative studies of linear equations.* Springer, 1996.

CHAPTER 1

Introduction

Exercise. (Warm-up) What kinds of quantities and operations appear in relation to analysis (or multivariable calculus) in a bounded open set $U \subset \mathbb{R}^n$?

Some possible answers:

- Functions: continuity, partial derivatives, integrals, L^p spaces, Taylor expansions, Fourier or related expansions
- Vector fields: gradient, curl, divergence, flows
- Measures, distributions
- Laplace operator, Laplace, heat and wave equations
- Integration by parts formulas (Gauss, divergence, Green)
- Tensor fields, differential forms
- Distance, distance-minimizing curves (line segments), area, volume, perimeter

Imagine similar concepts on a hypersurface (e.g. double torus in $\mathbb{R}^3$)!

This course is an introduction to analysis on manifolds. The first part of the course title has the following Wikipedia description:

"**Mathematical analysis** is a branch of mathematics that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic functions. These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis. Analysis may be distinguished from geometry; however, it can be applied to any space of mathematical objects that has a definition of nearness (a topological space) or specific distances between objects (a metric space)."

Following this description, our purpose will be to study in particular differentiation, integration, and differential equations on spaces that are more general than the standard Euclidean space $\mathbb{R}^n$. Different classes of spaces allow for different kinds of analysis:

- *Topological spaces* are a good setting for studying continuous functions and limits, but in general they do not have enough structure to allow studying derivatives.
- The smaller class of *metric spaces* admits certain notions of differentiability, but in particular higher order derivatives are not always well defined.
- *Differentiable manifolds* are modeled after pieces of Euclidean space and allow differentiation and integration, but they do not have a canonical Laplace operator and thus the theory of differential equations is limited.

The class of spaces studied in this course will be that of *Riemannian manifolds.* These are differentiable manifolds with an extra bit of structure, a Riemannian metric, that allows to measure lengths and angles of tangent vectors. Adding this extra structure leads to a very rich theory where many different parts of mathematics come together. We mention a few related aspects, and some of these will be covered during this course (the more advanced topics that will be covered will be chosen according to the interests of the audience):

1. *Calculus.* Riemannian manifolds are differentiable manifolds, hence the usual notions of multivariable calculus on differentiable manifolds apply (derivatives, vector and tensor fields, integration of differential forms).
2. *Metric geometry.* Riemannian manifolds are metric spaces: there is a natural distance function on any Riemannian manifold such that the corresponding metric space topology coincides with the usual topology. Distances are realized by certain distinguished curves called geodesics, and these can be studied via a second order ODE (the geodesic equation).
3. *Measure theory.* Any oriented Riemannian manifold has a canonical measure given by the volume form. The presence of this measure allows one to integrate functions and to define L^p spaces on Riemannian manifolds.
4. *Differential equations.* There is a canonical Laplace operator on any Riemannian manifold, and all the classical linear partial differential equations (Laplace, heat, wave) have natural counterparts.

5. *Spectral theory.* One can talk about the eigenvalues of the canonical Laplace operator and prove fundamental results such as the Weyl law describing their asymptotics.
6. *Dynamical systems.* The geodesic flow on a closed Riemannian manifold is a Hamiltonian flow on the cotangent bundle, and the geometry of the manifold is reflected in properties of the flow (such as complete integrability or ergodicity).
7. *Conformal geometry.* The notions of conformal and quasiconformal mappings make sense on Riemannian manifolds, and there is enough underlying structure to provide many tools for studying them.
8. *Topology.* There are several ways of describing topological properties of the underlying manifold in terms of analysis. In particular, Hodge theory characterizes the cohomology of the space via the Laplace operator acting on differential forms, and Morse theory describes the topological type of the space via critical points of a smooth function on it.
9. *Curvature.* The notion of curvature is fundamental in mathematics, and Riemannian manifolds are perhaps the most natural setting for studying curvature. Related concepts include the Riemann tensor, the Ricci tensor, and scalar curvature. There has been recent interest in lower bounds for Ricci curvature and their applications.
10. *Inverse problems.* Many interesting inverse problems have natural formulations on Riemannian manifolds, such as integral geometry problems where one tries to determine a function from its integrals over geodesics, or spectral rigidity problems where one tries to determine properties of the underlying space from knowledge of eigenvalues of the Laplacian.
11. *Geometric analysis.* There are many branches of mathematics that are called geometric analysis. One particular topic is that of geometric evolution equations, where geometric quantities evolve according to a certain PDE. One of the most famous such equations is Ricci flow, where a Riemannian metric is deformed via its Ricci tensor. This was recently used by Perelman to complete Hamilton's program for proving the Poincaré and geometrization conjectures.

Notation. Throughout these notes we will apply the *Einstein summation convention*: repeated indices in lower and upper position are summed. For instance, the expression

$$a_{jkl}b^j c^k$$

is shorthand for

$$\sum_{j,k} a_{jkl}b^j c^k.$$

The summation indices run typically from 1 to n, where n is the dimension of the manifold in question.

CHAPTER 2

Calculus in Euclidean space

Let U be any nonempty open subset of $\mathbb{R}^n$ (not necessarily bounded, and could be equal to $\mathbb{R}^n$). We fix standard Cartesian coordinates $x = (x_1, \ldots, x_n)$ and will use these coordinates throughout this chapter. We may sometimes write x^j instead of x_j, and we will also denote by η_j or η^j the jth coordinate of a vector $\eta \in \mathbb{R}^n$.

2.1. Functions and Taylor expansions

Let $C(U)$ be the set of continuous functions on U. For partial derivatives, we will write

$$\partial_j f = \frac{\partial f}{\partial x_j},$$
$$\partial_{j_1 \cdots j_k} f = \frac{\partial^k f}{\partial x_{j_1} \cdots \partial x_{j_k}}.$$

We denote by $C^k(U)$ the set of k times continuously differentiable real valued functions on U. Thus

$$C^k(U) = \{f : U \to \mathbb{R}\,;\, \partial_{j_1 \cdots j_l} f \in C(U) \text{ whenever } l \leq k \text{ and } j_1, \ldots, j_l \in \{1, \ldots, n\}\}.$$

Recall that if $f \in C^k(U)$, then $\partial_{j_1 \cdots j_k} f = \partial_{j_{\sigma(1)} \cdots j_{\sigma(k)}} f$ where σ is any permutation of $\{1, \ldots, k\}$.

We also denote by $C^\infty(U)$ the infinitely differentiable functions on U, that is,

$$C^\infty(U) = \bigcap_{k \geq 0} C^k(U).$$

Theorem 2.1. *(Taylor expansion) Let $f \in C^k(U)$, let $x_0 \in U$, and assume that $B(x_0, r) \subset U$. If $x \in B(x_0, r)$, then*

$$f(x) = \sum_{l=0}^{k} \frac{1}{l!} \left[\sum_{j_1, \ldots, j_l = 1}^{n} \partial_{j_1 \cdots j_l} f(x_0)(x - x_0)_{j_1} \ldots (x - x_0)_{j_l} \right] + R_k(x; x_0)$$

where $|R_k(x;x_0)| \le \eta(|x-x_0|)|x-x_0|^k$ *for some function* η *with* $\eta(s) \to 0$ *as* $s \to 0$.

Remark. The Taylor expansion of order 2 is given by

$$f(x) = f(x_0)+\nabla f(x_0)\cdot(x-x_0)+\frac{1}{2}\nabla^2 f(x_0)(x-x_0)\cdot(x-x_0)+R_2(x;x_0)$$

where $\nabla f = (\partial_1 f, \ldots, \partial_n f)$ is the gradient of f and $\nabla^2 f(x) = (\partial_{jk} f(x))_{j,k=1}^n$ is the Hessian matrix of f.

Proof. Considering $g(y) := f(x_0+y)$, we may assume that $x_0 = 0$. Assume that $B(0,r) \subset U$, fix $x \in B(0,r)$, and define

$$h : (-1-\varepsilon, 1+\varepsilon) \to U, \quad h(t) := g(tx)$$

where $\varepsilon > 0$ satisfies $(1+\varepsilon)|x| < r$. Then $h(t)$ is a C^k function for $|t| < 1+\varepsilon$, and repeated use of the fundamental theorem of calculus gives

$$\begin{aligned}
h(t) &= h(t) - h(0) + h(0) = h(0) + \int_0^t h'(s)\,ds \\
&= h(0) + h'(0)t + \int_0^t (h'(s) - h'(0))\,dt \\
&= h(0) + h'(0)t + \int_0^t \int_0^s h''(u)\,du\,ds \\
&= h(0) + h'(0)t + h''(0)\frac{t}{2} + \int_0^t \int_0^s (h''(u) - h''(0))\,du\,ds \\
&= \ldots \\
&= h(0) + h'(0)t + \ldots + h^{(k)}(0)\frac{t^k}{k!} \\
&\quad + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} (h^{(k)}(t_k) - h^{(k)}(0))\,dt_k \cdots dt_1. \qquad (2.1)
\end{aligned}$$

Here we used that $\int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-1}} dt_k \cdots dt_1 = \frac{t^k}{k!}$.

We compute

$$\begin{aligned}
h'(t) &= \partial_j f(tx) x_j, \\
h''(t) &= \partial_{jl} f(tx) x_j x_l, \\
&\vdots \\
h^{(k)}(t) &= \partial_{j_1 \cdots j_k} f(tx) x_{j_1} \ldots x_{j_k}.
\end{aligned}$$

Applying (2.1) with $t = 1$ gives the result in the statement, where

$$R_k(x) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} [\partial_{j_1 \cdots j_k} f(t_k x) - \partial_{j_1 \cdots j_k} f(0)]\, x_{j_1} \ldots x_{j_k}\, dt_k \cdots dt_1.$$

The bound for R_k follows since $\partial_{j_1 \cdots j_k} f$ is uniformly continuous on compact sets. □

At this point it may be good to mention another convenient form of the Taylor expansion, which we state but will not use. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set natural numbers. Then $\mathbb{N}^n$ consists of all n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ where the α_j are nonnegative integers. Such an n-tuple α is called a multi-index. We write $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. For partial derivatives, the notation

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

will be used. We also write $\alpha! = \alpha_1! \cdots \alpha_n!$.

Theorem 2.2. *(Taylor expansion, multi-index version) Let $f \in C^k(U)$, let $x_0 \in U$, and assume that $B(x_0, r) \subset U$. If $x \in B(x_0, r)$, then*

$$f(x) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(x_0)}{\alpha!} (x - x_0)^\alpha + R_k(x_0; x)$$

where R_k satisfies similar bounds as before.

Proof. Exercise. □

Exercise 2.1. Prove Theorem 2.2 for $k = 2$ by using Theorem 2.1.

Exercise 2.2. Prove Theorem 2.2 in general.

Exercise 2.3. Find an expression for $R_k(x; x_0)$ in Theorem 2.2 as a single integral.

2.2. Tensor fields

If $f \in C^k(U)$, if $x \in U$ and if $v \in \mathbb{R}^n$ is such that $|v|$ is sufficiently small, we write the Taylor expansion given in Theorem 2.1 in the form

$$f(x + v) = \sum_{l=0}^{k} \frac{1}{l!} \left[\sum_{j_1, \ldots, j_l = 1}^{n} \partial_{j_1 \cdots j_l} f(x) v_{j_1} \ldots v_{j_l} \right] + R_k(x + v; x).$$

The first few terms are

$$f(x + v) = f(x) + \partial_j f(x) v_j + \frac{1}{2} \partial_{jk} f(x) v_j v_k + \ldots.$$

Looking at the terms of various degrees motivates the following definition.

Definition. An *m-tensor field* in U is a collection of functions $u = (u_{j_1 \cdots j_m})_{j_1,\ldots,j_m=1}^n$ where each $u_{j_1 \cdots j_m}$ is in $C^\infty(U)$. The tensor field u is called *symmetric* if $u_{j_1 \cdots j_m} = u_{j_{\sigma(1)} \cdots j_{\sigma(m)}}$ for any $j_1, \ldots, j_m$ and for any σ which is a permutation of $\{1, \ldots, m\}$.

Remark. This definition is specific to $\mathbb{R}^n$, since we are deliberately not allowing any other coordinate systems than the Cartesian one. Later on we will consider tensor fields on manifolds, and their transformation rules under coordinate changes will be an important feature (these will decide whether the tensor field is covariant, contravariant or mixed). However, upon fixing a local coordinate system all tensor fields will look essentially like the ones defined above.

Examples.

1. The 0-tensor fields in U are just the scalar functions $u \in C^\infty(U)$.
2. The 1-tensor fields in U are of the form $u = (u_j)_{j=1}^n$ where $u_j \in C^\infty(U)$. Thus 1-tensor fields are exactly the vector fields in U; the tensor $(u_j)_{j=1}^n$ is identified with $(u_1, \ldots, u_n)$.
3. The 2-tensor fields in U are of the form $u = (u_{jk})_{j,k=1}^n$ where $u_{jk} \in C^\infty(U)$. Thus 2-tensor fields can be identified with smooth matrix functions in U. The 2-tensor field is symmetric iff the matrix is symmetric.
4. If $f \in C^\infty(U)$, then we have for any $m \geq 0$ an m-tensor field $u = (\partial_{j_1 \cdots j_m} f)_{j_1,\ldots,j_m=1}^n$ consisting of partial derivatives of f. This tensor field is symmetric since the mixed partial derivatives can be taken in any order.

Again by looking at the terms in the Taylor expansion, one can also think that an m-tensor $u = (u_{j_1 \cdots j_m})_{j_1,\ldots,j_m=1}^n$ acts on a vector $v \in \mathbb{R}^n$ by the formula

$$v \mapsto u_{j_1 \cdots j_m}(x) v^{j_1} \cdots v^{j_m}.$$

The last expression can be interpreted as a multilinear map acting on the m-tuple of vectors $(v, \ldots, v)$.

Definition. If $m \geq 0$, an *m-linear map* is any map

$$L : \overbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}^{m \text{ copies}} \to \mathbb{R}$$

such that L is linear in each of its variables separately.

The following theorem is almost trivial, but for later purposes it will be good to know that a tensor field can be thought of in two ways: either as a collection of coordinate functions, or as a map on U that takes values in the set of multilinear maps.

THEOREM 2.3. *(Tensors as multilinear maps) If $u = (u_{j_1 \cdots j_m})_{j_1,\ldots,j_m=1}^n$ is an m-tensor field on $U \subset \mathbb{R}^n$, then for any $x \in U$ there is an m-linear map $u(x)$ defined via*

$$u(x)(v_1, \ldots, v_m) = u_{j_1 \cdots j_m}(x) v_1^{j_1} \cdots v_m^{j_m}, \quad v_1, \ldots, v_m \in \mathbb{R}^n,$$

and it holds that $u_{j_1 \cdots j_m}(x) = u(x)(e_{j_1}, \ldots, e_{j_m})$. Conversely, if T is a function that assigns to each $x \in U$ an m-linear map $T(x)$, and if the functions $u_{j_1 \cdots j_m} : x \mapsto T(x)(e_{j_1}, \ldots, e_{j_m})$ are in $C^\infty(U)$ for each $j_1, \ldots, j_m$, then $(u_{j_1 \cdots j_m})$ is an m-tensor field in U.

PROOF. Exercise. □

EXERCISE 2.4. Prove Theorem 2.3.

2.3. Vector fields and differential forms

Let $U \subset \mathbb{R}^n$ be an open set. We wish to consider vector fields on U and certain operations related to vector fields.

DEFINITION. A C^k *vector field* in U is a map $F = (F_1, \ldots, F_n) : U \to \mathbb{R}^n$ such that all the component functions F_j are in $C^k(U)$. The set of vector fields on U is denoted by $C^k(U, \mathbb{R}^n)$.

Recall from Section 2.2 that vector fields are the same as 1-tensor fields (this is special to Euclidean space, on manifolds one needs to distinguish between covariant and contravariant tensors). If $u \in C^\infty(U)$, the *gradient* of u gives rise to a vector field in U:

$$\mathrm{grad} : C^\infty(U) \to C^\infty(U, \mathbb{R}^n), \quad \mathrm{grad}(u) = (\partial_1 u, \ldots, \partial_n u).$$

If $F \in C^\infty(U, \mathbb{R}^n)$, the *divergence* of F gives rise to a function in U:

$$\mathrm{div} : C^\infty(U, \mathbb{R}^n) \to C^\infty(U), \quad \mathrm{div}(F) = \partial_1 F_1 + \ldots + \partial_n F_n.$$

The following basic identity suggests that in order to define the Laplace operator on a space, it may be enough to have a reasonable definition of divergence and gradient.

LEMMA 2.4. *div* $\circ$ *grad* $= \Delta$.

PROOF. $\operatorname{div}(\operatorname{grad}(u)) = \partial_1(\partial_1 u) + \ldots + \partial_n(\partial_n u) = \Delta u$. ☐

We will consider further operations on vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Curl in $\mathbb{R}^2$. Let $U \subset \mathbb{R}^2$ be open. If $F \in C^\infty(U, \mathbb{R}^2)$, the curl of F is the function

$$\operatorname{curl}(F) := \partial_1 F_2 - \partial_2 F_1.$$

Thus curl : $C^\infty(U, \mathbb{R}^2) \to C^\infty(U)$.

Curl in $\mathbb{R}^3$. Let $U \subset \mathbb{R}^3$ be open. If $F \in C^\infty(U, \mathbb{R}^3)$, the *curl* of F is the vector field

$$\begin{aligned} \operatorname{curl}(F) := \nabla \times F &= \begin{vmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1). \end{aligned}$$

Thus curl : $C^\infty(U, \mathbb{R}^3) \to C^\infty(U, \mathbb{R}^3)$.

LEMMA 2.5. *In two dimensions, one has*

$$\mathit{curl} \circ \mathit{grad} = 0.$$

In three dimensions, one has

$$\mathit{curl} \circ \mathit{grad} = 0, \quad \mathit{div} \circ \mathit{curl} = 0.$$

PROOF. If $U \subset \mathbb{R}^2$ and $u \in C^\infty(U)$, we have

$$\operatorname{curl}(\operatorname{grad}(u)) = \partial_1(\partial_2 u) - \partial_2(\partial_1 u) = 0.$$

If $U \subset \mathbb{R}^3$ and $u \in C^\infty(U)$, we have

$$\operatorname{curl}(\operatorname{grad}(u)) = (\partial_2\partial_3 u - \partial_3\partial_2 u, \partial_3\partial_1 u - \partial_1\partial_2 u, \partial_1\partial_2 u - \partial_2\partial_1 u) = 0.$$

Moreover, for $F \in C^\infty(U, \mathbb{R}^3)$ we have

$$\begin{aligned} \operatorname{div}(\operatorname{curl}(F)) &= \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1) \\ &= 0. \end{aligned}$$ ☐

The previous lemma can be described in terms of two sequences: if $U \subset \mathbb{R}^2$ consider

$$C^\infty(U) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^2) \xrightarrow{\text{curl}} C^\infty(U), \tag{2.2}$$

and if $U \subset \mathbb{R}^3$ consider

$$C^\infty(U) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U). \tag{2.3}$$

In both sequences, the composition of any two subsequent operators is zero. This suggests that there may be further structure which underlies these situations and might extend to higher dimensions. This is indeed the case, and the calculus of *differential forms* (or *exterior algebra*) will reveal this structure. We will next discuss this calculus in a simple case.

EXERCISE 2.5. If $U \subset \mathbb{R}^3$ is open and $F \in C^\infty(U, \mathbb{R}^3)$, show that

$$\text{curl}(\text{curl}(F)) - \text{grad}(\text{div}(F)) = (-\Delta F_1, -\Delta F_2, -\Delta F_3).$$

Differential forms. The purpose will be to rewrite for instance (2.3) as a sequence

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U) \tag{2.4}$$

where $\Omega^k(U)$ will be the set of *differential k-forms* on $U \subset \mathbb{R}^3$, and d will be a universal operator that reduces to grad, curl, and div in the respective degrees.

Let $U \subset \mathbb{R}^n$ be open. Motivated by (2.2) and (2.3), we define

$$\Omega^0(U) := C^\infty(U)$$

and

$$\Omega^1(U) := C^\infty(U, \mathbb{R}^n).$$

Thus $\Omega^0(U)$ is the set of smooth functions in U, and any $\alpha \in \Omega^1(U)$ can be identified with a vector field $\alpha = (\alpha_j)_{j=1}^n$ where $\alpha_j \in C^\infty(U)$. We write formally

$$\alpha = (\alpha_j)_{j=1}^n = \alpha_j \, dx^j.$$

REMARK. For the purposes of this section it is enough to think of dx^j as a formal object. However, the proper way to think of dx^j would be as a 1-form (the exterior derivative of the function $x^j : U \to \mathbb{R}$), i.e. as a map that assigns to each $x \in U$ the linear map $dx^j|_x : T_x M \to \mathbb{R}$ that satisfies $dx^j|_x(e_k) = \delta_k^j$ where $\{e_1, \ldots, e_n\}$ is the standard basis of $T_x M \approx \mathbb{R}^n$.

To define $\Omega^k(U)$ for $k \geq 2$, first define the set of ordered k-tuples

$$\mathcal{I}_k := \{(i_1, \ldots, i_k) \,;\, 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}.$$

If $I \in \mathcal{I}_k$, we consider the formal object

$$dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}.$$

Then $\Omega^k(U)$ will be thought of as the set

$$\Omega^k(U) = \{\alpha_I \, dx^I \, ; \, \alpha_I \in C^\infty(U)\}$$

where the sum is over all $I \in \mathcal{I}_k$. The number of elements in $\mathcal{I}_k$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. We can make the above formal definition rigorous:

Definition. If $U \subset \mathbb{R}^n$, define for $0 \leq k \leq n$

$$\Omega^k(U) := C^\infty(U, \mathbb{R}^{\binom{n}{k}}).$$

The elements of $\Omega^k(U)$ are called *differential k-forms* on U, and any k-form $\alpha \in \Omega^k(U)$ can be written as

$$\alpha = (\alpha_I)_{I \in \mathcal{I}_k} = \alpha_I \, dx^I$$

where $\alpha_I \in C^\infty(U)$ for each I.

Remark. Note that since $\binom{n}{k} = \binom{n}{n-k}$, the set $\Omega^{n-1}(U)$ can be identified with the set of vector fields on U, and $\Omega^n(U)$ with $C^\infty(U)$. In fact one has

$$\Omega^{n-1}(U) = \{\sum_{j=1}^n \alpha_j \, dx^1 \wedge \ldots \wedge \widehat{dx^j} \wedge \ldots \wedge dx^n \, ; \, \alpha_j \in C^\infty(U)\},$$
$$\Omega^n(U) = \{f \, dx^1 \wedge \ldots \wedge dx^n \, ; \, f \in C^\infty(U)\}$$

where $\widehat{dx^j}$ means that dx^j is omitted from the wedge product.

The above definition is correct, but to keep things simple we have avoided a detailed discussion of the *wedge product* $\wedge$. To define the d operator in (2.4) properly we need to say a little bit more. The wedge product is an associative product on elements of the form dx^I, satisfying

$$dx^j \wedge dx^k = -dx^k \wedge dx^j,$$

and more generally if $J = (j_1, \ldots, j_k)$ is a k-tuple with $j_1, \ldots, j_k \in \{1, \ldots, n\}$ (not necessarily ordered), we should have

$$dx^{j_1} \wedge \cdots \wedge dx^{j_k} = (-1)^{\mathrm{sgn}(\sigma)} dx^{j_{\sigma(1)}} \wedge \cdots \wedge dx^{j_{\sigma(k)}}$$

where σ is any permutation of $\{1, \ldots, k\}$. This implies two conditions:

- $dx^{j_1} \wedge \cdots \wedge dx^{j_k} = 0$ if $(j_1, \ldots, j_k)$ contains a repeated index.

- $dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ can be expressed as $\pm dx^I$ for a unique $I \in \mathcal{I}_k$ if $(j_1, \ldots, j_k)$ contains no repeated index.

With this understanding we make the following definition.

DEFINITION. The *exterior derivative* is the map $d : \Omega^k(U) \to \Omega^{k+1}(U)$ defined by

$$d(\alpha_I \, dx^I) := \partial_j \alpha_I \, dx^j \wedge dx^I.$$

EXAMPLES.

(1) If $f \in \Omega^0(U)$ (so $f \in C^\infty(U)$), then df is the gradient of f written as a 1-form:

$$df = \partial_j f \, dx^j.$$

(2) If $\alpha \in \Omega^1(U)$, so $\alpha = \alpha_k \, dx^k$ for some $\alpha_j \in C^\infty(U)$, then

$$d\alpha = \partial_j \alpha_k \, dx^j \wedge dx^k = \sum_{1 \le j < k \le n} (\partial_j \alpha_k - \partial_k \alpha_j) \, dx^j \wedge dx^k.$$

(3) Any $u \in \Omega^n(U)$ satisfies $du = 0$ since $dx^{j_1} \wedge \ldots \wedge dx^{j_{n+1}} = 0$ whenever $j_1, \ldots, j_{n+1} \in \{1, \ldots, n\}$ (there will be a repeated index).

The second example above gives an n-dimensional analogue of the curl operator, as also suggested by the following lemma:

LEMMA 2.6. *(The exterior derivative in two and three dimensions)*

1. Let $U \subset \mathbb{R}^2$. If $f \in \Omega^0(U)$, then

$$df = (\mathrm{grad}(f))_j \, dx^j.$$

If $\alpha = F_1 \, dx^1 + F_2 \, dx^2 \in \Omega^1(U)$ and $F = (F_1, F_2)$, then

$$d\alpha = (\mathrm{curl}(F)) \, dx^1 \wedge dx^2.$$

2. Let $U \subset \mathbb{R}^3$. If $f \in \Omega^0(U)$, then

$$df = (\mathrm{grad}(f))_j \, dx^j.$$

If $\alpha = F_j \, dx^j \in \Omega^1(U)$ and $F = (F_1, F_2, F_3)$, then

$$d\alpha = (\mathrm{curl}(F))_j \, dx^{\hat{j}}$$

where

$$dx^{\hat{1}} := dx^2 \wedge dx^3, \quad dx^{\hat{2}} := dx^3 \wedge dx^1, \quad dx^{\hat{3}} := dx^1 \wedge dx^2.$$

Finally, if $u = F_j \, dx^{\hat{j}} \in \Omega^2(U)$ and $F = (F_1, F_2, F_3)$, then

$$du = (\mathrm{div}(F)) \, dx^1 \wedge dx^2 \wedge dx^3.$$

PROOF. Exercise (partly contained in the examples above). □

EXERCISE 2.6. Complete the proof of Lemma 2.6.

Let us now verify that $d \circ d$ is always zero.

LEMMA 2.7. *$d \circ d = 0$ on $\Omega^k(U)$ for any k with $0 \leq k \leq n$.*

PROOF. If $\alpha = \alpha_I \, dx^I \in \Omega^k(U)$, we compute

$$d\alpha = \sum_{k=1}^{n} \sum_{I \in \mathcal{I}_k} \partial_k \alpha_I \, dx^k \wedge dx^I$$

and

$$d(d\alpha) = \sum_{j,k=1}^{n} \sum_{I \in \mathcal{I}_k} \partial_{jk} \alpha_I \, dx^j \wedge dx^k \wedge dx^I$$

By the properties of the wedge product, we get

$$d(d\alpha) = \sum_{1 \leq j < k \leq n} \sum_{I \in \mathcal{I}_k} (\partial_{jk} \alpha_I - \partial_{kj} \alpha_I) \, dx^j \wedge dx^k \wedge dx^I.$$

This is $= 0$ since the mixed partial derivatives are equal. □

If $U \subset \mathbb{R}^n$ is open, we therefore have a sequence

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U) \tag{2.5}$$

and the composition of any two subsequent operators is zero. This gives the desired generalization of (2.2) and (2.3) to any dimension. In fact we have obtained much more: as we will see during this course, differential forms turn out to be an object of central importance in many kinds of analysis on manifolds.

Differential forms as tensors. It will be useful to interpret differential forms as tensor fields satisfying an extra condition.

DEFINITION. An m-tensor field $(u_{j_1 \cdots j_m})_{j_1,\ldots,j_m=1}^n$ in $U \subset \mathbb{R}^n$ is called *alternating* if $u_{j_{\sigma(1)} \cdots j_{\sigma(m)}} = (-1)^{\mathrm{sgn}(\sigma)} u_{j_1 \cdots j_m}$ for any $j_1, \ldots, j_m$ and for any σ which is a permutation of $\{1, \ldots, m\}$.

We understand that 0-tensor fields and 1-tensor fields are always alternating. A 2-tensor field $u = (u_{jk})_{j,k=1}^n$ is alternating iff $u_{kj} = -u_{jk}$ for any j, k, i.e. the matrix (u_{jk}) is skew-symmetric at each point. An m-tensor field $u = (u_{j_1 \cdots j_m})$ is alternating iff $u_{j_1 \cdots j_m}$ changes sign when any two indices are interchanged (since any permutation can be expressed as the product of transpositions). Note that for an alternating tensor, $u_{j_1 \cdots j_m} = 0$ whenever $(j_1, \ldots, j_m)$ contains a repeated index.

THEOREM 2.8. *If $U \subset \mathbb{R}^n$ is open and $0 \leq k \leq n$, the set $\Omega^k(U)$ can be identified with the set of alternating k-tensor fields on U.*

PROOF. Consider the map

$$T : \Omega^k(U) \to \{\text{alternating } k\text{-tensors}\}, \quad \alpha_I \, dx^I \mapsto (\tilde{\alpha}_{j_1 \cdots j_k})$$

where

$$\tilde{\alpha}_{j_1 \cdots j_k} := \begin{cases} 0, & (j_1, \ldots, j_k) \text{ contains a repeated index,} \\ \frac{1}{\sqrt{k!}}(-1)^{\mathrm{sgn}(\sigma)} \alpha_I, & (j_1, \ldots, j_k) \text{ contains no repeated index,} \end{cases}$$

where σ is the permutation of $\{1, \ldots, k\}$ such that $I = (j_{\sigma(1)}, \ldots, j_{\sigma(k)})$ is the unique element of $\mathcal{I}_k$ containing the same entries as $(j_1, \ldots, j_k)$. (The constant $\frac{1}{\sqrt{k!}}$ is a harmless normalizing factor which will be useful later.) Then $(\tilde{\alpha}_{j_1 \cdots j_k})$ is alternating by construction. It is clear that T is injective, and surjectivity follows since any alternating tensor is uniquely determined by the elements $\tilde{\alpha}_I$ where $I \in \mathcal{I}_k$. $\square$

EXERCISE 2.7. Prove that an m-tensor field u, considered as an m-linear form, is alternating if and only if $u(v_1, \ldots, v_m) = 0$ whenever $v_j = v_k$ for some pair (j, k) with $j \neq k$.

Cohomology. By Lemma 2.7, we observe that

$$u = d\alpha \text{ for some } \alpha \in \Omega^{k-1}(U) \implies du = 0.$$

This may be rephrased as follows:

$$\mathrm{Im}(d|_{\Omega^{k-1}(U)}) \text{ is a linear subspace of } \mathrm{Ker}(d|_{\Omega^k(U)}).$$

We express this in one more way: if $u \in \Omega^k(U)$, we say that u is *closed* if $du = 0$ and that u is *exact* if $u = d\alpha$ for some $\alpha \in \Omega^{k-1}(U)$. Thus, any exact differential form is closed. The question of whether any closed form is exact depends on the topological properties of U. To study this property we make the following definition.

DEFINITION. The *de Rham cohomology groups* of U are defined by

$$H^k_{\mathrm{dR}}(U) = \mathrm{Ker}(d|_{\Omega^k(U)})/\mathrm{Im}(d|_{\Omega^{k-1}(U)}), \quad 0 \leq k \leq n.$$

By this definition each $H^k_{\mathrm{dR}}(U)$ is in fact a (quotient) vector space, not just a group. Any closed k-form is exact iff $H^k_{\mathrm{dR}}(U) = \{0\}$. This happens for all $k \geq 1$ at least when U has very simple topology.

LEMMA 2.9. *(Poincaré lemma) If $U \subset \mathbb{R}^n$ is open and star-shaped with respect to some $x_0 \in U$ (meaning that for any $x \in U$ the line segment between x_0 and x lies in U), then*

$$H^k_{\mathrm{dR}}(U) = \begin{cases} \mathbb{R}, & k = 0, \\ \{0\}, & 1 \leq k \leq n. \end{cases}$$

PROOF. For simplicity we only do the proof for $n = 2$, see [**MT**] for the general case (which is somewhat more involved). Assume that U is star-shaped with respect to 0. We have

$$H^0_{\mathrm{dR}}(U) = \mathrm{Ker}(d|_{\Omega^0(U)}) = \{f \in C^\infty(U)\,;\, \mathrm{grad}(f) = 0\}.$$

But if $f \in C^\infty(U)$ satisfies $\mathrm{grad}(f) = 0$ in the connected set U, then f must be constant since

$$f(x) = f(0) + \int_0^1 \frac{d}{dt} f(tx)\,dt = f(0) + \int_0^1 \nabla f(tx) \cdot x\,dt = f(0).$$

Thus $H^0_{\mathrm{dR}}(U)$ is one-dimensional and thus isomorphic to $\mathbb{R}$.

We next show that $H^1_{\mathrm{dR}}(U) = \{0\}$, that is, for any $F \in C^\infty(U, \mathbb{R}^2)$ we have

$$\mathrm{curl}(F) = 0 \implies F = \mathrm{grad}(f) \text{ for some } f \in C^\infty(U).$$

Let $F = (F_1, F_2)$ satisfy $\partial_1 F_2 - \partial_2 F_1 = 0$. Then f should be some kind of integral of F, in fact we may just take

$$f(x) := \int_0^1 F_j(tx) x^j\,dt, \qquad x \in U.$$

Then, using that $\partial_1 F_2 = \partial_2 F_1$, we compute

$$\begin{aligned} \partial_1 f(x) &= \int_0^1 \left[\partial_1 F_j(tx) t x^j + F_1(tx)\right]\,dt \\ &= \int_0^1 \left[\partial_1 F_1(tx) t x^1 + \partial_2 F_1(tx) t x^2 + F_1(tx)\right]\,dt \\ &= \int_0^1 \frac{d}{dt}\left[t F_1(tx)\right]\,dt = F_1(x). \end{aligned}$$

Similarly $\partial_2 f(x) = F_2(x)$, showing that $F = \mathrm{grad}(f)$.

Finally we show that $H^2_{\mathrm{dR}}(U) = \{0\}$, which means that

$$f \in C^\infty(U) \implies f = \mathrm{curl}(F) \text{ for some } F \in C^\infty(U, \mathbb{R}^2).$$

Again F_j should be integrals of f. We may define

$$F_1(x) := -\int_0^1 f(tx)tx_2\,dt, \qquad F_2(x) := \int_0^1 f(tx)tx_1\,dt.$$

Then

$$\begin{aligned}\partial_1 F_2 - \partial_2 F_1 &= \int_0^1 \left[\partial_1 f(tx)t^2x_1 + \partial_2 f(tx)t^2x_2 + 2tf(tx)\right]\,dt \\ &= \int_0^1 \frac{d}{dt}\left[t^2 f(tx)\right]\,dt = f(x).\end{aligned}$$ □

Exercise 2.8. Prove Lemma 2.9 for general n when $k = 1$.

Exercise 2.9. Prove Lemma 2.9 for general n and k (you may use [**MT**] or some other reference).

In general, the cohomology groups for $k = 0$ are very simple.

Lemma 2.10. *If $U \subset \mathbb{R}^n$ is open, then* $\dim(H^0_{\mathrm{dR}}(U))$ *is the number of connected components of U (possibly ∞).*

Exercise 2.10. Prove Lemma 2.10.

For the case $k = 1$, a sufficient condition for $H^1_{\mathrm{dR}}(U) = \{0\}$ is given as follows.

Definition. Let $U \subset \mathbb{R}^n$ be open.

(a) Let $\gamma_0, \gamma_1 : [0,1] \to U$ be continuous closed curves (i.e. continuous maps with $\gamma_j(0) = \gamma_j(1)$). The curves γ_0 and γ_1 are said to be *homotopic* (resp. *smoothly homotopic*) within U if there is a continuous (resp. smooth) map $H : [0,1] \times [0,1] \to U$ such that $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$, and $H(s,0) = H(s,1)$ for all $s \in [0,1]$.

(b) We say that U is *simply connected* if U is connected and any continuous closed curve in U is null-homotopic, i.e. homotopic within U to some constant curve $\eta(t) \equiv x_0$ with $x_0 \in U$.

The map H in (a) is called a *homotopy* between γ_0 and γ_1. Writing $\gamma_s(t) = H(s,t)$, we can think of $(\gamma_s)_{s\in[0,1]}$ as a continuous deformation from γ_0 to γ_1.

Recall also that in complex analysis, a set $U \subset \mathbb{R}^2$ is called simply connected if both U and $\mathbb{R}^2 \setminus U$ are connected. This definition is equivalent to the one above for $n = 2$ but not for $n \geq 3$.

LEMMA 2.11. *If $U \subset \mathbb{R}^n$ is simply connected, then $H^1_{\mathrm{dR}}(U) = \{0\}$.*

The proof is contained in the following exercises.

EXERCISE 2.11. Let $U \subset \mathbb{R}^n$ be connected and $F \in \Omega^1(U)$. Fix $x_0 \in U$, and define

$$f(x) = \int_{\gamma_{x_0,x}} F := \int_0^1 F(\gamma_{x_0,x}(t)) \cdot \dot{\gamma}_{x_0,x}(t)\, dt$$

where $\gamma_{x_0,x} : [0,1] \to U$ is a smooth curve joining x_0 and x. *Assume* that the definition does not depend on the choice of the curve $\gamma_{x_0,x}$. Show that then $f \in \Omega^0(U)$ and $df = F$.

EXERCISE 2.12. If $U \subset \mathbb{R}^n$ is open, $F \in \Omega^1(U)$ is closed, and $\gamma : [0,1] \to U$ is a smooth closed curve that is smoothly homotopic to a constant curve, prove that

$$\int_\gamma F = 0.$$

(Hint: compute $\partial_s(\int_{\gamma_s} F)$.)

EXERCISE 2.13. Prove Lemma 2.11. (Hint: use Exercises 2.11–2.12 and smooth approximations to continuous maps.)

We conclude by mentioning some facts about the de Rham cohomology groups (for more details see [**MT**]):

- The de Rham cohomology groups are *topological invariants*: if U and V are homeomorphic open sets in Euclidean space, then $H^k_{\mathrm{dR}}(U)$ and $H^k_{\mathrm{dR}}(V)$ are isomorphic as vector spaces for each k. (In fact this holds more generally when U and V are *homotopy equivalent.*) This gives a potential way of showing that two sets U and V are not homeomorphic; it would be enough to check that some cohomology groups are not isomorphic.
- Note however that it is possible for non-homeomorphic spaces to have the same cohomology groups (e.g. the Whitehead manifold has the same cohomology groups as $\mathbb{R}^3$).
- In many cases (e.g. if $U \subset \mathbb{R}^n$ is a bounded open set with smooth boundary), the vector spaces $H^k_{\mathrm{dR}}(U)$ are finite dimensional. The dimension of $H^k_{\mathrm{dR}}(U)$ is an important topological invariant, namely the kth *Betti number* of U.

- Very loosely speaking, the cohomology groups may give some information about "holes" in a set. For instance, if $K_1, \ldots, K_N$ are disjoint closed balls in $\mathbb{R}^n$, then
$$H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \cup_{j=1}^N K_j) = \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ \mathbb{R}^N, & \text{if } k = n-1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Later in this course we will discuss Hodge theory, which studies the cohomology groups $H^k_{\mathrm{dR}}(M)$ where M is a compact manifold via the Laplace operator acting on differential forms on M.

2.4. Riemannian metrics

An open set $U \subset \mathbb{R}^n$ is often thought to be "homogeneous" (the set looks the same near every point) and "flat" (if U is considered as a subset of $\mathbb{R}^{n+1}$ lying in the hyperplane $\{x_{n+1} = 0\}$, then U has the geometry induced by the flat hypersurface $\{x_{n+1} = 0\}$). In this section we will introduce extra structure on U which makes it "inhomogeneous" (the properties of the set vary from point to point) and "curved" (U has some geometry that is different from the geometry induced by a flat hypersurface $\{x_{n+1} = 0\}$).

Motivation. An intuitive way of introducing this extra structure is to think of U as a medium where sound waves propagate. The properties of the medium are described by a function $c : U \to \mathbb{R}_+$, which is thought of as the sound speed of the medium. If U is homogeneous, the sound speed is constant ($c(x) = 1$ for each $x \in U$), but if U is inhomogeneous then the sound speed varies from point to point.

Consider now a C^1 curve $\gamma : [0,1] \to U$. The tangent vector $\dot\gamma(t)$ of this curve is thought to be a vector at the point $\gamma(t)$. If the sound speed is constant ($c \equiv 1$), the length of the tangent vector is just the Euclidean length:
$$|\dot\gamma(t)|_e := \left[\sum_{j=1}^n \dot\gamma^j(t)^2 \right]^{1/2}.$$
In the case of a general sound speed $c : U \to \mathbb{R}_+$, one can think that at points where c is large the curve moves very quickly and consequently has short length. Thus we may define the length of $\dot\gamma(t)$ with respect

to the sound speed c by

$$|\dot{\gamma}(t)|_c := \frac{1}{c(\gamma(t))} \left[\sum_{j=1}^{n} \dot{\gamma}^j(t)^2 \right]^{1/2}.$$

It is useful to generalize the above setup in two directions. First, in addition to measuring lengths of tangent vectors we would also like to measure angles between tangent vectors (in particular we want to know when two tangent vectors are orthogonal). Second, if the sound speed is a scalar function on U, then the length of a tangent vector is independent of its direction (the medium is *isotropic*). We wish to allow the medium to be *anisotropic*, which will mean that the sound speed may depend on direction and should be a matrix valued function.

In order to measures lengths and angles of tangent vectors, it is enough to introduce an inner product on the space of tangent vectors at each point. The tangent space is defined as follows:

Definition. If $U \subset \mathbb{R}^n$ is open and $x \in U$, the *tangent space at* x is defined as

$$T_x U := \{x\} \times \mathbb{R}^n.$$

The *tangent bundle* of U is the set

$$TU := \bigcup_{x \in U} T_x U.$$

Of course, each $T_x U$ can be identified with $\mathbb{R}^n$ (and we will often do so), and a vector $v \in T_x U$ is written in terms of its coordinates as $v = (v^1, \ldots, v^n)$. Now if $\langle \,\cdot\,,\,\cdot\, \rangle$ is any inner product on $\mathbb{R}^n$, there is some positive definite symmetric matrix $A = (a_{jk})_{j,k=1}^n$ such that

$$\langle v, w \rangle = Av \cdot w, \qquad v, w \in \mathbb{R}^n.$$

Exercise 2.14. Prove the above statement.

The next definition introduces an inner product on the space of tangent vectors at each point:

Definition. A *Riemannian metric* on U is a matrix-valued function $g = (g_{jk})_{j,k=1}^n$ such that each g_{jk} is in $C^\infty(U)$, and $(g_{jk}(x))$ is a positive definite symmetric matrix for each $x \in U$. The corresponding *inner product* on $T_x U$ is defined by

$$\langle v, w \rangle_g := g_{jk}(x) v^j w^k, \qquad v, w \in T_x U.$$

The *length* of a tangent vector is

$$|v|_g := \langle v, v\rangle_g^{1/2} = \left(g_{jk}(x)v^j v^k\right)^{1/2}, \qquad v \in T_xU.$$

The *angle* between two tangent vectors $v, w \in T_xU$ is the number $\theta_g(v, w) \in [0, \pi]$ defined by

$$\cos\,\theta_g(v, w) = \frac{\langle v, w\rangle_g}{|v|_g|w|_g}.$$

We will often drop the subscript and write $\langle\,\cdot\,,\,\cdot\,\rangle$ or $|\cdot|$ if the metric g is fixed. To connect the above definition to the discussion about sound speeds, a scalar sound speed $c(x)$ corresponds to the Riemannian metric

$$g_{jk}(x) = \frac{1}{c(x)^2}\delta_{jk}.$$

Example. (a) (Euclidean metric) If $U \subset \mathbb{R}^n$ is open, the most standard Riemannian metric that one can put on U is the *Euclidean metric* given by

$$g_{jk}(x) = \delta_{jk}, \qquad x \in U.$$

This is clearly a smooth positive definite symmetric matrix function on U. If $x \in U$ and $v, w \in T_xU$, the inner product $\langle v, w\rangle_g$ is the dot product $v \cdot w$ and the length $|v|_g$ is the Euclidean length of v. The set U equipped with this metric is homogeneous and flat (its Riemann curvature tensor is $\equiv 0$), and geodesics are straight lines.

(b) (Sphere) Let $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ be the n-sphere with north pole e_{n+1}. As we will see later, the Euclidean metric on $\mathbb{R}^{n+1}$ induces a standard Riemannian metric on the hypersurface S^n called the *round metric*. The sphere S^n with this metric has constant sectional curvature $+1$, and the geodesics are the great circles. Let Φ be the stereographic projection

$$\Phi : S^n \setminus \{e_{n+1}\} \to \mathbb{R}^n, \quad \Phi(\xi, x_{n+1}) = \frac{\xi}{1 - x_{n+1}}.$$

This is a bijective map. If we equip $\mathbb{R}^n$ with the metric

$$g_{jk}(x) = \frac{4}{(|x|^2 + 1)^2}\delta_{jk},$$

then the stereographic projection becomes an isometry (i.e. distance-preserving map) from $S^n \setminus \{e_{n+1}\}$ with the round metric to $(\mathbb{R}^n, g)$.

(c) (Hyperbolic space) Consider $U = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ with the metric
$$g_{jk}(x) = \frac{1}{x_n^2}\delta_{jk}.$$
The space (U, g) is one of the standard models of hyperbolic space (the model space with constant sectional curvature -1). Another equivalent model for hyperbolic space is (B, h) where $B \subset \mathbb{R}^n$ is the unit ball and
$$h_{jk}(x) = \frac{4}{(1-|x|^2)^2}\delta_{jk}.$$

Finally, we introduce some notation that will be very useful.

Notation. If $g = (g_{jk})$ is a Riemannian metric on U, we write
$$(g^{jk})_{j,k=1}^n = g^{-1}$$
for the inverse matrix of $(g_{jk})_{j,k=1}^n$, and
$$|g| = \det(g)$$
for the determinant of the matrix $(g_{jk})_{j,k=1}^n$.

In particular, we note that $g_{jk}g^{kl} = \delta_j^l$ for any $j, l = 1, \ldots, n$, where δ_j^l denotes the Kronecker symbol.

2.5. Geodesics

Lengths of curves. Consider an open set U that is equipped with a Riemannian metric g. As we saw above, one can measure lengths of tangent vectors with respect to g, and this makes it possible to measure lengths of curves as well.

Definition. A smooth map $\gamma : [a, b] \to U$ whose tangent vector $\dot{\gamma}(t)$ is always nonzero is called a *regular curve*. The *length* of γ is defined by
$$L(\gamma) := \int_a^b |\dot{\gamma}(t)|\,dt.$$
The length of a piecewise regular curve is defined as the sum of lengths of the regular parts. The *Riemannian distance* between two points $p, q \in U$ is defined by
$$d(p, q) := \inf\{L(\gamma)\,;\, \gamma : [a, b] \to U \text{ is a piecewise regular curve with } \gamma(a) = p \text{ and } \gamma(b) = q\}.$$

EXERCISE 2.15. Show that $L(\gamma)$ is independent of the way the curve γ is parametrized, and that we may always parametrize γ by *arc length* so that $|\dot\gamma(t)| = 1$ for all t.

The previous exercise shows that we can always reparametrize a piecewise regular curve by arc length, so that one will have $|\dot\gamma(t)| = 1$. A curve satisfying $|\dot\gamma(t)| \equiv 1$ is called a *unit speed curve* (similarly a curve with $|\dot\gamma(t)| \equiv \text{const}$ is called a *constant speed curve*).

EXERCISE 2.16. Show that d is a metric distance function on U, and that (U, d) is a metric space whose topology is the same as the Euclidean topology on U.

Geodesic equation. We now wish to show that any length minimizing curve satisfies a certain ordinary differential equation.

THEOREM 2.12. *(Length minimizing curves are geodesics) Suppose $U \subset \mathbb{R}^n$ is open, let g be a Riemannian metric on U, and let $\gamma : [a, b] \to U$ be a piecewise regular unit speed curve. Assume that γ minimizes the distance between its endpoints, in the sense that*

$L(\gamma) \leq L(\eta)$ for any piecewise regular curve η from $\gamma(a)$ to $\gamma(b)$.

Then γ is a regular curve, and it satisfies the geodesic equation

$$\ddot\gamma^l(t) + \Gamma^l_{jk}(\gamma(t))\dot\gamma^j(t)\dot\gamma^k(t) = 0, \quad 1 \leq l \leq n,$$

where Γ^l_{jk} are the Christoffel symbols *of the metric g:*

$$\Gamma^l_{jk} := \frac{1}{2} g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}), \quad 1 \leq j, k, l \leq n. \tag{2.6}$$

EXAMPLE. If g is the Euclidean metric on U, so that $g_{jk}(x) = \delta_{jk}$, then all the Christoffel symbols Γ^l_{jk} are zero. The geodesic equation becomes just

$$\ddot\gamma^l(t) = 0, \quad 1 \leq l \leq n.$$

Solving this equation shows that

$$\gamma(t) = tv + w$$

for some vectors $v, w \in \mathbb{R}^n$. Thus Theorem 2.12 recovers the classical fact that any length minimizing curve in Euclidean space is a line segment.

Any smooth curve that satisfies the geodesic equation is called a *geodesic*, and the conclusion of Theorem 2.12 can be rephrased so that any length minimizing curve is a geodesic. The fact that length minimizing curves satisfy the geodesic equation gives powerful tools for studying these curves. For instance, one can show that

- any geodesic has constant speed and is therefore regular
- given any $x \in U$ and $v \in T_x U$, there is a unique geodesic starting at point x in direction v
- any geodesic minimizes length at least locally (but not always globally)
- a set U with Riemannian metric g is geodesically complete, meaning that every geodesic is defined for all $t \in \mathbb{R}$, if and only if the metric space (U, d_g) is complete (this is the Hopf-Rinow theorem).

The rest of this section is occupied with the proof of Theorem 2.12. See [**Le1**, Chapter 6] for more details on these facts.

Variations of curves. Let $\gamma : [a, b] \to U$ be a piecewise regular length minimizing curve. We will prove Theorem 2.12 by considering families of curves (γ_s) where $s \in (-\varepsilon, \varepsilon)$ and $\gamma_0 = \gamma$, and all curves γ_s start at $\gamma(a)$ and end at $\gamma(b)$. Such a family is called a *variation* (or a *fixed-endpoint variation*) of γ. By the length minimizing property,

$$L(\gamma_0) \leq L(\gamma_s) \text{ for } s \in (-\varepsilon, \varepsilon),$$

so if the dependence on s is at least C^1 we obtain that $\frac{d}{ds} L(\gamma_s)\big|_{s=0} = 0$. This fact, applied to many different families (γ_s), will imply that γ is smooth and solves the geodesic equation.

If (γ_s) is a family or curves with $\gamma_0 = \gamma$, we think of $V(t) := \frac{\partial}{\partial s} \gamma_s(t)\big|_{s=0}$ as the "infinitesimal variation" of the curve γ that leads to the family (γ_s). The vector $V(t)$ should be thought of as an element of $T_{\gamma(t)} U$. The next result shows that one can reverse this process, and obtain a variation of γ from any given infinitesimal variation V.

In this result and below, we assume that the piecewise regular curve γ is fixed and that there is a subdivision of $[a, b]$,

$$a = t_0 < t_1 < \ldots < t_N < t_{N+1} = b,$$

such that $\gamma|_{(t_j, t_{j+1})}$ is regular for each j with $0 \leq j \leq N$.

LEMMA 2.13. *(Variations of curves) If* $V : [a, b] \to \mathbb{R}^n$ *is a continuous map such that* $V|_{(t_j,t_{j+1})}$ *is* C^∞ *for each* j *and* $V(a) = V(b) = 0$, *then there exists* $\varepsilon > 0$ *and a continuous map*

$$\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to U$$

such that the curves $\gamma_s : [a, b] \to U, \gamma_s(t) := \Gamma(s, t)$ *satisfy the following:*

(1) *each* γ_s *is a piecewise regular curve with endpoints* $\gamma(a)$ *and* $\gamma(b)$, *and* $\gamma_s|_{(t_j,t_{j+1})}$ *is regular for each* j,
(2) $\gamma_0 = \gamma$,
(3) $s \mapsto \gamma_s(t)$ *is* C^∞ *and* $\frac{\partial}{\partial s}\gamma_s(t)\big|_{s=0} = V(t)$ *for each* $t \in [a, b]$.

PROOF. Define

$$\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to U, \quad \Gamma(s, t) := \gamma(t) + sV(t)$$

where ε is so small that Γ takes values in U. The properties (1)–(3) follow immediately from the definition. □

We can now compute the derivative $\frac{d}{ds}L(\gamma_s)\big|_{s=0}$ that was mentioned above. In classical terminology, this is called the *first variation* of the length functional.

LEMMA 2.14. *(First variation formula) Let* γ *be a piecewise regular unit speed curve, and let* (γ_s) *be a variation of* γ *associated with* V *as in Lemma 2.13. Then*

$$\frac{d}{ds}L(\gamma_s)\big|_{s=0} = -\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \langle D_t\dot\gamma(t), V(t)\rangle\, dt - \sum_{j=1}^{N}\langle \Delta\dot\gamma(t_j), V(t_j)\rangle$$

where $D_t\dot\gamma(t)$ *is the element of* $T_{\gamma(t)}U$ *defined by*

$$(D_t\dot\gamma(t))^l := \ddot\gamma^l(t) + \Gamma^l_{jk}(\gamma(t))\dot\gamma^j(t)\dot\gamma^k(t), \quad 1 \le l \le n,$$

and $\Delta\dot\gamma(t_j) := \dot\gamma(t_j+) - \dot\gamma(t_j-)$ *is the jump of* $\dot\gamma(t)$ *at* t_j.

REMARK. We will later give an invariant meaning to $D_t\dot\gamma(t)$ and interpret is as the covariant derivative of $\dot\gamma(t)$ along the curve γ. However, at this point it is enough to think of $D_t\dot\gamma(t)$ just as some expression that comes out when we compute the derivative $\frac{d}{ds}L(\gamma_s)\big|_{s=0}$.

PROOF. Define

$$I(s) := L(\gamma_s) = \sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} [g_{pq}(\gamma_s(t))\dot\gamma_s^p(t)\dot\gamma_s^q(t)]^{1/2}\, dt.$$

To prepare for computing the derivative $I'(0)$, define two vector fields

$$T(t) := \partial_t \gamma_s(t)|_{s=0} = \dot{\gamma}(t), \quad V(t) := \partial_s \gamma_s(t)|_{s=0}.$$

Using that $|\dot{\gamma}_0(t)| = |T(t)| \equiv 1$ and (g_{jk}) is symmetric, we have

$$I'(0) = \frac{1}{2} \sum_{j=0}^{N} \int_{t_j}^{t_{j+1}} (\partial_r g_{pq}(\gamma(t)) V^r(t) T^p(t) T^q(t) + 2 g_{pq}(\gamma(t)) \dot{V}^p(t) T^q(t))\, dt.$$

Integrating by parts in the last term, this shows that

$$I'(0) = \sum_{j=0}^{N} \int_{t_j}^{t_{j+1}} \left[\frac{1}{2} \partial_r g_{pq}(\gamma) T^p T^q - \partial_m g_{rq}(\gamma) T^m T^q - g_{rq}(\gamma) \dot{T}^q \right] V^r\, dt$$

$$+ \sum_{j=0}^{N} \left[\langle V(t_{j+1}), T(t_{j+1}) \rangle - \langle V(t_j), T(t_j) \rangle \right].$$

Using that $V(t_0) = V(t_{N+1}) = 0$ and that V is continuous, the boundary term becomes $-\sum_{j=1}^{N} \langle \Delta\dot{\gamma}(t_j), V(t_j) \rangle$ as required. For the integrals, we use that

$$\partial_m g_{rq}(\gamma) T^m T^q = \frac{1}{2} (\partial_m g_{rq}(\gamma) + \partial_q g_{rm}(\gamma)) T^m T^q$$

which gives

$$\begin{aligned} - \langle D_t \dot{\gamma}(t), V(t) \rangle &= -g_{rq}(\gamma)(\dot{T}^q + \Gamma_{jk}^q T^j T^k) V^r \\ &= -g_{rq}(\gamma) \dot{T}^q - \frac{1}{2} \left[\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk} \right] T^j T^k) V^r \\ &= -g_{rq}(\gamma)(\dot{T}^q + \frac{1}{2} \partial_r g_{pq}(\gamma) T^p T^q - \partial_m g_{rq}(\gamma) T^m T^q) V^r. \end{aligned}$$

This proves the result. □

Proof of Theorem 2.12. Let $\gamma : [a, b] \to U$ be a piecewise regular unit speed curve that minimizes the length between its endpoints. If V is any vector field as in Lemma 2.13 and (γ_s) is the corresponding variation of γ, we must have

$$L(\gamma_0) \leq L(\gamma_s)$$

for $s \in (-\varepsilon, \varepsilon)$. Therefore $\frac{d}{ds} L(\gamma_s)\big|_{s=0} = 0$. But the first variation formula (Lemma 2.14) shows that

$$\sum_{j=0}^{N} \int_{t_j}^{t_{j+1}} \langle D_t \dot{\gamma}(t), V(t) \rangle\, dt + \sum_{j=1}^{N} \langle \Delta\dot{\gamma}(t_j), V(t_j) \rangle = 0 \tag{2.7}$$

for any such V.

We first show that γ solves the geodesic equation on each interval (t_j, t_{j+1}). Fix $j \in \{0, \ldots, N\}$ and choose V such that

$$V(t) := \varphi(t) D_t \dot{\gamma}(t)$$

where φ is any function in $C_c^\infty((t_j, t_{j+1}))$. This V is an admissible choice in Lemma 2.14, and then (2.7) implies that

$$\int_{t_j}^{t_{j+1}} |D_t \dot{\gamma}(t)|^2 \varphi(t)\, dt = 0$$

for any $\varphi \in C_c^\infty((t_j, t_{j+1}))$. Varying φ shows that we must have $D_t \dot{\gamma}(t)|_{(t_j, t_{j+1})} = 0$ for each j.

We next show that γ has no corners and is a C^1 curve in $[a, b]$. Going back to (2.7), we have

$$\sum_{j=1}^{N} \langle \Delta \dot{\gamma}(t_j), V(t_j) \rangle = 0$$

for any V with $V(a) = V(b) = 0$. Now, if $\Delta \dot{\gamma}(t_j) \neq 0$ for some j, we can choose V with $V(t_j) = \Delta \dot{\gamma}(t_j)$ and $V(t_k) = 0$ for $k \neq j$. This would imply that

$$|\Delta \dot{\gamma}(t_j)|^2 = 0$$

which contradicts the assumption $\Delta \dot{\gamma}(t_j) \neq 0$. This shows that we must have $\Delta \dot{\gamma}(t_j) = 0$ for each j, and it follows that γ is in fact a C^1 curve in $[a, b]$.

Finally, since $\gamma|_{(t_j, t_{j+1})}$ solves the geodesic equation for each j and since γ is C^1 near each t_j, the existence and uniqueness theorem for ODE implies that $\gamma|_{(t_j, t_{j+1})}$ is the unique smooth continuation of the solution $\gamma|_{(t_{j-1}, t_j)}$. Thus in fact γ solves the geodesic equation and is C^∞ near each t_j, and γ is a regular curve solving the geodesic equation on $[a, b]$. □

The previous proof shows actually more than stated in the theorem. We say that a piecewise regular curve γ is a critical point of the length functional L if $\frac{d}{ds} L(\gamma_s)\big|_{s=0} = 0$ for any fixed-endpoint variation of γ as in Lemma 2.13.

THEOREM 2.15. *The critical points of L are exactly the geodesic curves.*

PROOF. The proof of Theorem 2.12 shows that any critical point of L is a geodesic curve. To see the converse, let γ be a geodesic curve so that γ is C^∞ and $D_t\dot{\gamma}(t) = 0$ in $[a, b]$. By the first variation formula (Lemma 2.14) any such curve satisfies $\frac{d}{ds}L(\gamma_s)\big|_{s=0} = 0$, so any geodesic must be a critical point of L. □

Remark. Let us give a more geometric interpretation of the proof of Theorem 2.12. Suppose that γ is a piecewise regular curve which is smooth in (t_j, t_{j+1}) for $0 \leq j \leq N$. The preceding proof shows that

$$\frac{d}{ds}L(\gamma_s)\Big|_{s=0} = -\sum_{j=0}^{N} \int_{t_j}^{t_{j+1}} \langle D_t\dot{\gamma}(t), V(t)\rangle \, dt - \sum_{j=1}^{N} \langle \Delta\dot{\gamma}(t_j), V(t_j)\rangle$$

where (γ_s) is a variation of γ related to V as in Lemma 2.14. Choosing

$$V(t) := \varphi(t) D_t\dot{\gamma}(t)$$

where φ is a nonnegative function supported in (t_j, t_{j+1}) shows that

$$\frac{d}{ds}L(\gamma_s)\Big|_{s=0} = -\int_{t_j}^{t_{j+1}} \varphi(t)|D_t\dot{\gamma}(t)|^2 \, dt \leq 0.$$

Thus if $D_t\dot{\gamma}(t) \neq 0$ somewhere in (t_j, t_{j+1}), the derivative can be made strictly negative. This means we can always make the curve γ shorter by deforming it in the direction of $D_t\dot{\gamma}(t)$.

Assume now that γ solves the geodesic equation in each segment (t_j, t_{j+1}) where it is smooth. If one has $\Delta\dot{\gamma}(t_j) \neq 0$ and if we choose V so that $V(t_j) = \Delta\dot{\gamma}(t_j)$ and $V(t_k) = 0$ for $k \neq j$, then

$$\frac{d}{ds}L(\gamma_s)\Big|_{s=0} = -|\Delta\dot{\gamma}(t)|^2 < 0.$$

This shows that a "broken geodesic" with corner at t_j can always be made shorter by deforming it in the direction of $\Delta\dot{\gamma}(t_j)$. This argument of "rounding the corner" was the key point in showing that length minimizing curves are C^∞.

2.6. Integration and inner products

This section will largely consist of definitions. We explain a natural way of integrating functions with respect to a Riemannian metric g, given by the volume form dV_g. This leads to an L^2 inner product first for scalar functions and then for vector fields and tensor fields. Finally we discuss the codifferential operator δ, which is the adjoint of the exterior derivative of d with respect to the L^2 inner product on differential forms. On 1-forms δ can be interpreted as a Riemannian divergence operator. The operator δ will be used in the next section to define the Laplace operator.

Integration. Let U be an open set, and let g be a Riemannian metric on U. If f is a function in (say) $C_c(U)$, we wish to consider the integral of f over U with respect to the metric g. The idea is that the metric g gives a way of measuring infinitesimal volumes, in the same way that it allows to measure lengths and angles of tangent vectors.

Motivation. Since in this chapter we are restricting ourselves to using Cartesian coordinates, the integral of f over U should be approximately given by

$$\int_U f(x)\,d\mathrm{Vol}_g(x) \approx \sum_{j=1}^N f(x_j)\mathrm{Vol}_g(Q_j) \tag{2.8}$$

where $\{Q_1, \ldots, Q_N\}$ are very small congruent cubes whose sides are parallel to the Cartesian coordinate axes such that the cubes approximately tile U, and x_j is the center of Q_j. Now if Q_j has sidelength h, one should have

$$\mathrm{Vol}_g(Q_j) = \mathrm{Vol}_g(he_1|_{x_j}, \ldots, he_n|_{x_j})$$

where $\mathrm{Vol}_g(v_1, \ldots, v_n)$ is the Riemannian volume of the parallelepiped generated by the v_j (this is the set $\{\sum_{j=1}^n t_j v_j\,;\, t_j \in [0,1]\}$).

The volume should have the following properties if the v_j have very small (infinitesimal) length:

(a) If $v_1, \ldots, v_n$ are orthogonal with respect to g, one should have $\mathrm{Vol}_g(v_1, \ldots, v_n) \approx |v_1|_g \cdots |v_n|_g$.
(b) If A is a matrix with $Av_j = \lambda_j v_j$, $j = 1, \ldots, n$, one should have $\mathrm{Vol}_g(Av_1, \ldots, Av_n) \approx \lambda_1 \cdots \lambda_n \mathrm{Vol}_g(v_1, \ldots, v_n)$.
(c) More generally if A is any $n \times n$ matrix, then one should have $\mathrm{Vol}_g(Av_1, \ldots, Av_n) \approx \det(A)\mathrm{Vol}_g(v_1, \ldots, v_n)$.

Fix now a point $x \in U$, write $G = (g_{jk}(x))_{j,k=1}^n$, and note that the set $\{G^{-1/2}e_1, \ldots, G^{-1/2}e_n\}$ is an g-orthonormal basis of T_xU:

$$\begin{aligned}
&\langle G^{-1/2}e_j, G^{-1/2}e_k\rangle_g = g_{pq}(x)(G^{-1/2}e_j)^p(G^{-1/2}e_k)^q \\
&= G(G^{-1/2}e_j)\cdot(G^{-1/2}e_k) = G^{-1/2}GG^{-1/2}e_j \cdot e_k \\
&= e_j \cdot e_k = \delta_{jk}.
\end{aligned}$$

Thus the volume of an infinitesimal parallelepiped should be

$$\begin{aligned}\mathrm{Vol}_g(he_1|_x, \ldots, he_n|_x) &\approx h^n \mathrm{Vol}_g(G^{1/2}(G^{-1/2}e_1)|_x, \ldots, G^{1/2}(G^{-1/2}e_n)|_x) \\ &\approx h^n |g(x)|^{1/2}\end{aligned}$$

where $|g(x)| = \det(g_{jk}(x))$. Going back to (2.8), this would give

$$\int_U f(x)\, d\mathrm{Vol}_g(x) \approx \sum_{j=1}^N f(x_j)|g(x_j)|^{1/2} h^n \underset{h\to 0}{\to} \int_U f(x)|g(x)|^{1/2}\, dx.$$

The above discussion motivates the following definitions:

Definition. Let $U \subset \mathbb{R}^n$ be open, and let g be a Riemannian metric on U. If $f \in C_c(U)$, we define the integral of f by

$$\int_U f(x)\, dV_g(x) := \int_U f(x)|g(x)|^{1/2}\, dx.$$

The Riemannian volume of a measurable set $E \subset U$ is

$$\mathrm{Vol}_g(E) := \int_E |g(x)|^{1/2}\, dx.$$

If $1 \leq p < \infty$, the L^p norm of f is

$$\|f\|_{L^p(U,dV_g)} := \left(\int_U |f|^p \, dV_g \right)^{1/p}.$$

The space $L^p(U, dV_g)$ is the completion of $C_c(U)$ in the L^p norm.

It follows that $L^p(U, dV_g)$ is a Banach space whenever $1 \leq p < \infty$.

Remark. The quantity dV_g is usually called the *volume form* of the Riemannian manifold (U, g). To justify this terminology, one should interpret dV_g as the differential n-form (element of $\Omega^n(U)$) given by

$$dV_g = |g|^{1/2}\, dx^1 \wedge \ldots \wedge dx^n.$$

One can equivalently think of dV_g as a measure, i.e. (using the Riesz representation theorem for measures) as a linear operator acting on functions in $C_c(U)$ by

$$f \mapsto \int_U f\, dV_g.$$

In the present setting where $U \subset \mathbb{R}^n$, this measure is absolutely continuous with respect to Lebesgue measure ($dV_g(x) = |g(x)|^{1/2}\, dx$).

Inner products. The most important case of L^p spaces during this course is $p = 2$. In fact, $L^2(U, dV_g)$ is a Hilbert space with the following inner product.

DEFINITION. If $u, v \in L^2(U, dV_g)$ we define

$$(u, v)_{L^2} := \int_U uv \, dV_g.$$

We now wish to define an L^2 inner product for vector fields and tensor fields on U as well. The case of vector fields comes naturally: if $F, G \in C_c(U, \mathbb{R}^n)$ are two vector fields, so that $F(x), G(x) \in T_x U$ for each $x \in U$, the g-inner product of $F(x)$ and $G(x)$ is

$$\langle F(x), G(x) \rangle_g = g_{jk}(x) F^j(x) G^k(x). \tag{2.9}$$

The L^2 inner product of F and G is then defined by

$$\begin{aligned} (F, G)_{L^2} &:= \int_U \langle F(x), G(x) \rangle_g \, dV_g(x) \\ &= \int_U g_{jk}(x) F^j(x) G^k(x) |g(x)|^{1/2} \, dx. \end{aligned}$$

Next consider the case of 1-forms. Let α and β be two 1-forms in U whose coordinate functions are in $C_c(U)$, meaning that $\alpha = \alpha_j \, dx^j$ and $\beta = \beta_k \, dx^k$ where $\alpha_j, \beta_k \in C_c(U)$. If $\alpha(x)$ denotes the expression $\alpha_j(x) \, dx^j$, in analogy with (2.9) it seems natural to define the g-inner product

$$\langle \alpha(x), \beta(x) \rangle_g := g^{jk}(x) \alpha_j(x) \beta_k(x). \tag{2.10}$$

Recall that (g^{jk}) is the inverse matrix of (g_{jk}). The L^2 inner product of two compactly supported 1-forms α and β is defined by

$$\begin{aligned} (\alpha, \beta)_{L^2} &:= \int_U \langle \alpha(x), \beta(x) \rangle_g \, dV_g(x) \\ &= \int_U g^{jk}(x) \alpha_j(x) \beta_k(x) |g(x)|^{1/2} \, dx. \end{aligned} \tag{2.11}$$

Motivated by (2.10), one can define the L^2 inner product of two tensor fields with components in $C_c(U)$. In particular, this gives an L^2 inner product on differential forms since k-forms can be identified with certain (alternating) k-tensor fields by Theorem 2.8.

DEFINITION. Let $u = (u_{j_1\cdots j_m})_{j_1,\ldots,j_m=1}^n$ and $v = (v_{k_1\cdots k_m})_{k_1,\ldots,k_m=1}^n$ be two tensor fields such that each $u_{j_1\cdots j_m}$ and $v_{k_1\cdots k_m}$ is in $C_c(U)$. The L^2 inner product of u and v is

$$(u,v)_{L^2} := \int_U g^{j_1k_1}(x)\cdots g^{j_mk_m}(x)u_{j_1\cdots j_m}(x)v_{k_1\cdots k_m}(x)|g(x)|^{1/2}\,dx.$$

If α and β are differential k-forms whose component functions are in $C_c(U)$, we denote by

$$(\alpha,\beta)_{L^2} := (\tilde{\alpha},\tilde{\beta})_{L^2}$$

the inner product of the corresponding tensor fields as in Theorem 2.8.

EXERCISE 2.17. Show that the L^2 inner product of tensor fields defined above is indeed an inner product.

Recall that if $\alpha = \alpha_I\,dx^I$ is a k-form, Theorem 2.8 identifies α with the k-tensor $\tilde{\alpha}$ defined by

$$\tilde{\alpha}_{j_1\cdots j_k} := \begin{cases} 0, & (j_1,\ldots,j_k) \text{ contains a repeated index,} \\ \frac{1}{\sqrt{k!}}\varepsilon_{j_1\cdots j_k}\alpha_{R(j_1,\ldots,j_k)}, & (j_1,\ldots,j_k) \text{ contains no repeated index,} \end{cases}$$

where $R(j_1,\ldots,j_k) = (j_{\sigma(1)},\ldots,j_{\sigma(k)})$ where σ is the unique permutation of $\{1,\ldots,k\}$ such that $j_1 < j_2 < \ldots < j_k$ (thus R puts the indices in increasing order), and $\varepsilon_{j_1\cdots j_k} = (-1)^{\operatorname{sgn}(\sigma)}$.

Notice that if α and β are 1-forms, this inner product is equal to (2.11).

EXAMPLE. Let $U \subset \mathbb{R}^n$ be open and let g be the Euclidean metric, so $g_{jk} = \delta_{jk}$. Then $|g(x)| \equiv 1$ and $g^{jk} = \delta^{jk}$. If $\alpha = \alpha_j\,dx^j$ and $\beta = \beta_k\,dx^k$ are two 1-forms with $\alpha_j, \beta_k \in C_c(U)$, and if $\vec{\alpha} = (\alpha_1,\ldots,\alpha_n)$ and $\vec{\beta} = (\beta_1,\ldots,\beta_n)$ are the corresponding vector fields, then

$$(\alpha,\beta)_{L^2} = \int_U \sum_{j=1}^n \alpha_j\beta_j\,dx = \int_U \vec{\alpha}\cdot\vec{\beta}\,dx.$$

Moreover, if $u = (u_{j_1\cdots j_m})_{j_1,\ldots,j_m=1}^n$ and $v = (v_{k_1\cdots k_m})_{k_1,\ldots,k_m=1}^n$ are two tensor fields with components in $C_c(U)$, then

$$(u,v)_{L^2} = \int_U \sum_{j_1,\ldots,j_m=1}^n u_{j_1\cdots j_m}v_{j_1\cdots j_m}\,dx.$$

Codifferential. Our next purpose is to consider the exterior derivative $d : \Omega^k(U) \to \Omega^{k+1}(U)$ and to compute its formal adjoint operator in the L^2 inner product on forms. Below, we write $\Omega^k_c(U)$ for the set of compactly supported k-forms in U (thus $\alpha = \alpha_I \, dx^I$ is in $\Omega^k_c(U)$ if $\alpha_I \in C^\infty_c(U)$ for each I).

THEOREM 2.16. *(Codifferential) Let $U \subset \mathbb{R}^n$ be open and let g be a Riemannian metric on U. For each k with $0 \leq k \leq n$, there is a unique linear operator*

$$\delta : \Omega^k(U) \to \Omega^{k-1}(U)$$

having the property

$$(d\alpha, \beta)_{L^2} = (\alpha, \delta\beta)_{L^2}, \qquad \alpha \in \Omega^{k-1}_c(U), \quad \beta \in \Omega^k(U). \tag{2.12}$$

The operator δ satisfies $\delta \circ \delta = 0$ and $\delta|_{\Omega^0(U)} = 0$. It is a linear first order differential operator acting on component functions, and on 1-forms it is given by

$$\delta\beta := -|g|^{-1/2}\partial_j(|g|^{1/2} g^{jk}\beta_k), \quad \beta = \beta_k \, dx^k \in \Omega^1(U). \tag{2.13}$$

The proof is based on the integration by parts formula

$$\int_U u(\partial_j v) \, dx = -\int_U (\partial_j u) v \, dx, \quad u \in C(U), \quad v \in C_c(U).$$

PROOF. We begin with the case $k = 1$. Let $\beta = \beta \, dx^k \in \Omega^1(U)$. To compute $\delta\beta$ satisfying (2.12), we take $\alpha \in \Omega^0_c(U) = C^\infty_c(U)$ and compute

$$\begin{aligned}(d\alpha, \beta)_{L^2} &= \int_U \langle d\alpha, \beta \rangle_g \, dV_g = \int_U g^{jk} \partial_j \alpha \beta_k |g|^{1/2} \, dx \\ &= -\int_U \alpha |g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \beta_k) \, dV_g.\end{aligned}$$

Thus (2.12) will be satisfied for $k = 1$ if we define $\delta : \Omega^1(U) \to \Omega^0(U)$ by (2.13).

Let us now show that for any k, there is an operator $\delta : \Omega^k(U) \to \Omega^{k-1}(U)$ such that (2.12) holds. Let $\alpha \in \Omega^{k-1}_c(U)$ and $\beta \in \Omega^k(U)$.

Using the definitions and integration by parts, we obtain

$$\begin{aligned}(d\alpha, \beta)_{L^2} &= \int_U \langle \partial_i \alpha_I \, dx^i \wedge dx^I, \beta_J \, dx^J \rangle_g \, dV_g \\ &= \int_U (\partial_i \alpha_I) \beta_J \langle dx^i \wedge dx^I, dx^J \rangle_g |g|^{1/2} \, dx \\ &= -\int_U \alpha_I |g|^{-1/2} \partial_i \left[|g|^{1/2} \langle dx^i \wedge dx^I, dx^J \rangle_g \beta_J \right] \, dV_g.\end{aligned}$$

Write $\gamma^I := -|g|^{-1/2} \partial_i \left[|g|^{1/2} \langle dx^i \wedge dx^I, dx^J \rangle_g \beta_J \right]$. It follows that

$$(d\alpha, \beta)_{L^2} = \int_U \alpha_I \gamma^I \, dV_g.$$

We wish to find $\gamma = \gamma_L \, dx^L \in \Omega^{k-1}(U)$ such that $\alpha_I \gamma^I = \langle \alpha, \gamma \rangle_g$. This can be done by *lowering indices*. First let $\tilde{\alpha} = (\tilde{\alpha}_{i_1 \cdots i_{k-1}})$ and $\tilde{\gamma} = (\tilde{\gamma}^{i_1 \cdots i_{k-1}})$ be the alternating tensor fields corresponding to α_I and γ^I, so for instance $\tilde{\gamma}^{i_1 \cdots i_{k-1}} := \frac{1}{\sqrt{(k-1)!}} \varepsilon^{i_1 \cdots i_{k-1}} \gamma^{R(i_1, \ldots, i_{k-1})}$. Let

$$\tilde{\gamma}_{l_1 \cdots l_{k-1}} := g_{l_1 i_1} \cdots g_{l_{k-1} i_{k-1}} \tilde{\gamma}^{i_1 \cdots i_{k-1}}$$

and let $\gamma = \gamma_L \, dx^L$ be the $(k-1)$-form corresponding to $\tilde{\gamma}$. Then

$$\begin{aligned}\langle \alpha, \gamma \rangle_g &= \langle \tilde{\alpha}, \tilde{\gamma} \rangle_g \\ &= g^{i_1 l_1} \cdots g^{i_{k-1} l_{k-1}} \tilde{\alpha}_{i_1 \cdots i_{k-1}} \left[g_{l_1 p_1} \cdots g_{l_{k-1} p_{k-1}} \tilde{\gamma}^{p_1 \cdots p_{k-1}} \right] \\ &= \tilde{\alpha}_{i_1 \cdots i_{k-1}} \tilde{\gamma}^{i_1 \cdots i_{k-1}} = \frac{1}{(k-1)!} \alpha_{R(i_1 \cdots i_{k-1})} \gamma^{R(i_1 \cdots i_{k-1})} = \alpha_I \gamma^I.\end{aligned}$$

Combining the above arguments, we have proved that

$$(d\alpha, \beta)_{L^2} = (\alpha, \gamma)_{L^2}$$

for all $\alpha \in \Omega_c^{k-1}(U)$. Here $\gamma \in \Omega^{k-1}(U)$ is determined uniquely by this identity, thus setting $\delta\beta := \gamma$ satisfies (2.12). Inspecting the above argument shows that $\delta\beta = \gamma_L \, dx^L$ where for $L = (l_1, \ldots, l_{k-1})$

$$\begin{aligned}\gamma_L = -g_{l_1 i_1} \cdots g_{l_{k-1} i_{k-1}} |g|^{-1/2} \cdot \\ \partial_i \left[|g|^{1/2} \langle dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}, dx^J \rangle_g \beta_J \right].\end{aligned} \tag{2.14}$$

Thus δ is a first order operator acting on the component functions β_J.

It is clear that $\delta|_{\Omega^0(U)} = 0$, and the condition $\delta \circ \delta = 0$ follows from (2.12) and the fact that $d \circ d = 0$. □

EXERCISE 2.18. Let $U \subset \mathbb{R}^3$ and let g be the Euclidean metric. Use the property (2.12) directly to find a formula for δ acting on $\Omega^2(U)$ and on $\Omega^3(U)$.

EXERCISE 2.19. If $\beta \in \Omega^k(U)$, use (2.14) to show that $\delta\beta = \gamma_I \, dx^I$ where

$$\tilde{\gamma}_{i_1 \dots i_{k-1}}(x) = -g^{lr}(\partial_r \tilde{\beta}_{l i_1 \dots i_{k-1}} - \Gamma^j_{lr} \tilde{\beta}_{j i_1 \dots i_{k-1}})$$

where Γ^j_{lr} are the Christoffel symbols in (2.6).

If $U \subset \mathbb{R}^n$ is an open set, in Section 2.3 we studied the sequence

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U) \tag{2.15}$$

where $d \circ d = 0$. This sequence does not depend on any Riemannian metric on U. However, if we introduce a Riemannian metric g on U, then Theorem 2.16 shows that there is another sequence

$$\Omega^0(U) \xleftarrow{\delta} \Omega^1(U) \xleftarrow{\delta} \dots \xleftarrow{\delta} \Omega^{n-1}(U) \xleftarrow{\delta} \Omega^n(U) \tag{2.16}$$

where $\delta \circ \delta = 0$. As we will explain later, the sequences (2.15) and (2.16) and the corresponding cohomology groups turn out to be dual to each other: this is related to *Poincaré duality.*

2.7. Laplace-Beltrami operator

Definition. In this section we will see that on any open set equipped with a Riemannian metric there is a canonical second order elliptic operator, called the Laplace-Beltrami operator, which is an analogue of the usual Laplacian in $\mathbb{R}^n$.

MOTIVATION. Let first U be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and consider the Laplace operator

$$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}. \tag{2.17}$$

Solutions of the equation $\Delta u = 0$ are called harmonic functions, and by standard results for elliptic PDE [**Ev**, Chapter 6], for any $f \in H^1(U)$ there is a unique solution $u \in H^1(U)$ of the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } U, \\ u = f & \text{on } \partial U. \end{cases} \tag{2.18}$$

The last line means that $u - f \in H_0^1(U)$. Recall from [**Ev**, Chapter 5] that $H^1(U)$ and $H_0^1(U)$ denote the Sobolev spaces

$$\begin{aligned} H^1(U) &:= \{u \in L^2(U) \,:\, \partial_j u \in L^2(U) \text{ for } 1 \leq j \leq n\}, \\ H_0^1(U) &:= \text{closure of } C_c^\infty(U) \text{ in } H^1(U), \end{aligned}$$

where $\partial_j u$ denotes the weak derivative. Both $H^1(U)$ and $H_0^1(U)$ are Hilbert spaces, and $H_0^1(U)$ can be thought of as the set of those $u \in H^1(U)$ that vanish on ∂U (in the trace sense).

One way to produce the solution of (2.18) is based on variational methods and Dirichlet's principle [**Ev**, Chapter 2]. We define the Dirichlet energy

$$E(v) := \frac{1}{2} \int_U |\nabla v|^2 \, dx, \qquad v \in H^1(U).$$

If we define the admissible class

$$\mathcal{A}_f := \{v \in H^1(U)\,;\, v = f \text{ on } \partial U\},$$

then the solution of (2.18) is the unique function $u \in \mathcal{A}_f$ which minimizes the Dirichlet energy:

$$E(u) \leq E(v) \quad \text{for all } v \in \mathcal{A}_f.$$

The heuristic idea is that the solution of (2.18) represents a physical system in equilibrium, and therefore should minimize a suitable energy functional. The point is that one can start from the energy functional $E(\,\cdot\,)$ and conclude that any minimizer u must satisfy $\Delta u = 0$, which gives another way to define the Laplace operator.

From this point on, let $U \subset \mathbb{R}^n$ be open and let g be a Riemannian metric on U. Although there is no immediately obvious analogue of (2.17) that would take into account the metric g, there is a natural analogue of the Dirichlet energy. It is given by

$$E(v) := \frac{1}{2} \int_U |dv|^2 \, dV, \qquad v \in H^1(U).$$

Here $|dv| = |dv|_g$ is the Riemannian length of the 1-form dv, and $dV = dV_g$ is the volume form.

We wish to find a differential equation which is satisfied by minimizers of $E(\,\cdot\,)$. Suppose $u \in H^1(U)$ is a minimizer which satisfies

$E(u) \leq E(u + t\varphi)$ for all $t \in \mathbb{R}$ and all $\varphi \in C_c^\infty(U)$. We have

$$\begin{aligned} E(u + t\varphi) &= \frac{1}{2} \int_U \langle d(u + t\varphi), d(u + t\varphi) \rangle \, dV \\ &= E(u) + t \int_U \langle du, d\varphi \rangle \, dV + t^2 E(\varphi). \end{aligned}$$

Since $I_\varphi(t) := E(u+t\varphi)$ is a smooth function of t for fixed φ, and since $I_\varphi(0) \leq I_\varphi(t)$ for $|t|$ small, we must have $I_\varphi'(0) = 0$. This shows that if u is a minimizer, then

$$\int_U \langle du, d\varphi \rangle \, dV = 0 \tag{2.19}$$

for any choice of $\varphi \in C_c^\infty(U)$. Moreover, if one has $u \in C^\infty(\overline{U})$ (which is the case by elliptic regularity), then by the properties of the codifferential δ, one also has

$$\int_U (\delta du) \varphi \, dV = 0$$

for all $\varphi \in C_c^\infty(U)$. Thus any minimizer u has to satisfy the equation

$$\delta du = 0 \quad \text{in } U.$$

We have arrived at the definition of the Laplace-Beltrami operator.

Definition. The Laplace-Beltrami operator on (U, g) is defined by

$$\Delta_g u := -\delta du.$$

The next result implies, in particular, that in Euclidean space Δ_g is just the usual Laplacian.

Lemma 2.17. *The Laplace-Beltrami operator has the expression*

$$\Delta_g u = |g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \partial_k u)$$

where, as before, $|g| = \det(g_{jk})$ is the determinant of g.

Proof. Follows from the coordinate expression for δ. □

Remark. There are differing sign conventions for the Laplace-Beltrami operator. Honoring the title of this course ("Analysis on manifolds"), we have chosen the convention which is perhaps most common in analysis and makes the Laplace-Beltrami operator for Euclidean metric equal to $\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. However, it is very common in geometry define the Laplace-Beltrami operator with the opposite sign, which has the benefit that the operator becomes positive. Moreover, in probability theory a factor of $\frac{1}{2}$ is often included in the definition. In this course we will stick to the analysts' convention so that $\Delta_g = -\delta d$.

The existence of a canonical Laplace operator associated to a Riemannian metric implies that one has analogues of the classical linear PDE:

- $\Delta_g u = 0$ (Laplace)
- $\partial_t u - \Delta_g u = 0$ (heat)
- $\partial_t^2 u - \Delta_g u = 0$ (wave)
- $i\partial_t u + \Delta_g u = 0$ (Schrödinger)

Therefore in physical terms, any Riemannian manifold will support a theory for electrostatics, heat flow, acoustic wave propagation, and quantum mechanics. Note also that the theory of geodesics leads to a version of classical mechanics, and there are many relations between the classical and quantum picture (i.e. between the geodesic flow and the Laplace-Beltrami operator).

EXERCISE 2.20. Show that $\Delta_g(uv) = (\Delta_g u)v + 2\langle du, dv\rangle_g + u(\Delta_g v)$ for $u, v \in C^2(U)$.

EXERCISE 2.21. If $U \subset \mathbb{R}^n$ is a bounded open set with C^1 boundary, and if $u \in C^2(\overline{U})$ satisfies $\Delta_g u = 0$ in U and $u|_{\partial U} = 0$, show that $u = 0$.

EXERCISE 2.22. Let $U = \{x \in \mathbb{R}^n : x_n > 0\}$ and let $g_{jk}(x) = x_n^{-2}\delta_{jk}$ be the hyperbolic metric. Compute Δ_g, and show that

$$((-\Delta_g - \frac{1}{4}(n-1)^2)u, u)_{L^2(U,dV_g)} \geq 0 \text{ for any } u \in C_c^\infty(U).$$

(This implies that the spectrum of the hyperbolic Laplacian is in the interval $[\frac{1}{4}(n-1)^2, \infty)$.)

2.8. Solutions of the Laplace equation

In this section we make the standing assumption that $U \subset \mathbb{R}^n$ is a bounded open set with C^∞ boundary, and U is equipped with a Riemannian metric g which extends in a C^∞ way to $\overline{U}$. We will now give the basic properties related to solutions of the Laplace-Beltrami (or Laplace) equation

$$-\Delta_g u = 0 \text{ in } U.$$

This is a second order elliptic equation with C^∞ coefficients, and hence these basic properties follow from [**Ev**, Chapter 6]. However, we give a presentation that will easily generalize to the case of manifolds.

We will define a new Sobolev norm involving g:

$$\|u\|^2_{H^1(U,dV_g)} := \|u\|^2_{L^2(U,dV_g)} + \|du\|^2_{L^2(U,dV_g)}.$$

In the Riemannian setting it is more natural to work with $H^1(U, dV_g)$. However, by the next exercise, for $U \subset \mathbb{R}^n$ we can equivalently work with $H^1(U)$ at the expense of changing certain constants in the norm estimates.

EXERCISE 2.23. Show that there are $c, C > 0$ such that

$$c\|u\|_{L^2(U)} \leq \|u\|_{L^2(U,dV_g)} \leq C\|u\|_{L^2(U)},$$
$$c\|u\|_{H^1(U)} \leq \|u\|_{H^1(U,dV_g)} \leq C\|u\|_{H^1(U)}.$$

Use this to prove that $L^2(U) = L^2(U, dV_g)$ and $H^1(U) = H^1(U, dV_g)$ both as sets and as topological spaces.

First we show that the Dirichlet problem always has a unique weak solution. Motivated by (2.19), we say that $u \in H^1(U)$ is a *weak solution* of $-\Delta_g u = 0$ in U if

$$(du, d\varphi)_{L^2(U,dV_g)} = 0$$

for all $\varphi \in C^\infty_c(U)$, or equivalently for all $\varphi \in H^1_0(U)$. More generally, we will consider the equation $-\Delta_g u = F$ in U. Recall the Sobolev space

$$H^{-1}(U) := (H^1_0(U))^* = \{\text{bounded linear functionals } F : H^1_0(U) \to \mathbb{R}\}.$$

EXERCISE 2.24. State the natural definitions for $H^1_0(U, dV_g)$ and $H^{-1}(U, dV_g)$, and show that as sets one has $H^1_0(U, dV_g) = H^1_0(U)$ and $H^{-1}(U, dV_g) = H^{-1}(U)$.

THEOREM 2.18 (Dirichlet problem). *Given any $F \in H^{-1}(U)$ and $f \in H^1(U)$, there is a unique weak solution $u \in H^1(U)$ of the problem*

$$\left\{ \begin{array}{rll} -\Delta_g u = F & & \text{in } U, \\ u = f & & \text{on } \partial U. \end{array} \right. \tag{2.20}$$

Here we say that $u \in H^1(U)$ is a weak solution *of* (2.20) *iff*

$$(du, dv)_{L^2(U,dV_g)} = F(v) \text{ for all } v \in H^1_0(U), \qquad u - f \in H^1_0(U).$$

One has the norm estimate

$$\|u\|_{H^1(U)} \leq C(\|F\|_{H^{-1}(U)} + \|f\|_{H^1(U)})$$

where C is independent of F and f.

The following result states that if the data F and f are more regular, then also the solution u will be more regular. Here we use the Sobolev space

$$H^k(U) := \{u \in L^2(U) \,:\, \partial^\alpha u \in L^2(U) \text{ for all } |\alpha| \leq k\}.$$

THEOREM 2.19 (Higher regularity). *Assume the conditions in Theorem 2.18. If $F \in H^k(U)$ and $f \in H^{k+2}(U)$ for some $k \geq 0$, then $u \in H^{k+2}(U)$ and*

$$\|u\|_{H^{k+2}(U)} \leq C(\|F\|_{H^k(U)} + \|f\|_{H^{k+2}(U)})$$

where C is independent of F and f.

Finally, we state the weak maximum principle for solutions in $H^1(U)$. Here we say that $u \leq C$ in U (resp. $u \leq C$ on ∂U) if $(u - C)_+ = 0$ (resp. $(u - C)_+ \in H_0^1(U)$), where

$$u_+ := \max\{u, 0\}.$$

We also say that $u \geq c$ in U (resp. $u \geq c$ on ∂U) if $-u \leq -c$ in U (resp. $-u \leq -c$ on ∂U).

THEOREM 2.20 (Weak maximum principle). *Let $u \in H^1(U)$ solve*

$$-\Delta_g u = 0 \text{ in } U.$$

If $u \leq C$ on ∂U, then $u \leq C$ in U. Similarly, if $u \geq c$ on ∂U, then $u \geq c$ in U.

We proceed to the proofs of the above results.

Dirichlet problem. The solvability of the Dirichlet problem will follow from Hilbert space theory and the following simple inequality.

THEOREM 2.21 (Poincaré inequality). *There is $C > 0$ so that*

$$\|u\|_{L^2(U,dV_g)} \leq C\|du\|_{L^2(U,dV_g)}, \qquad u \in H_0^1(U).$$

PROOF. By density it is enough to prove the inequality for any $u \in C_c^\infty(U)$. Since $U \subset \mathbb{R}^n$ is bounded, one has $U \subset \{a \leq x_n \leq b\}$ for some $a < b$. Note that for any $x' \in \mathbb{R}^{n-1}$ and $x_n \in [a,b]$, one has

$$u(x', x_n) = u(x', x_n) - u(x', a) = \int_a^{x_n} \partial_n u(x', t)\, dt.$$

The Cauchy-Schwarz inequality gives that

$$|u(x', x_n)| \leq \int_a^{x_n} 1 \cdot |\partial_n u(x', t)|\, dt \leq (x_n - a)^{1/2} \left[\int_a^{x_n} |\partial_n u(x', t)|^2\, dt \right]^{1/2}.$$

This implies

$$|u(x', x_n)|^2 \leq (x_n - a) \int_a^b |\partial_n u(x', t)|^2\, dt.$$

Integrating over $\mathbb{R}^{n-1} \times (a, b)$, and using that $\mathrm{supp}(u) \subset U$, gives

$$\|u\|_{L^2(U)}^2 \leq \frac{1}{2}(b-a)^2 \|du\|_{L^2(U)}^2.$$

Finally, by Exercise 2.23 one can change the $L^2(U)$ norms to $L^2(U, dV_g)$ norms at the expense of increasing the constant. □

PROOF OF THEOREM 2.18. We first consider the case $f = 0$. Define $B(u, v) := (du, dv)_{L^2(U, dV_g)}$ for $u, v \in H_0^1(U)$. We need to show that there is $u \in H_0^1(U)$ satisfying

$$B(u, v) = F(v) \text{ for all } v \in H_0^1(U). \tag{2.21}$$

We claim that B is an inner product on $H_0^1(U)$. Clearly B is a symmetric bilinear form. By the Poincaré inequality in Theorem 2.21 we have

$$B(u, u) = \|du\|_{L^2(U, dV_g)}^2 \geq c\|u\|_{L^2(U, dV_g)}^2.$$

Thus $B(u, u) = 0$ implies $u = 0$, so B is positive definite and hence it is indeed an inner product on $H_0^1(U)$. Moreover, the norm induced by B is equivalent to the standard one on $H_0^1(U)$ in the sense that

$$\begin{aligned} B(u, u) &\leq \|u\|_{H^1(U, dV_g)}^2 \leq C\|u\|_{H^1(U)}^2, \\ B(u, u) &= \frac{1}{2}B(u, u) + \frac{1}{2}B(u, u) \\ &\geq (c + \frac{1}{2})(\|u\|_{L^2(U, dV_g)}^2 + \|du\|_{L^2(U, dV_g)}^2) \geq c'\|u\|_{H^1(U)}^2. \end{aligned}$$

This shows that $H_0^1(U)$ equipped with the inner product B has the same convergent sequences and Cauchy sequences as the standard $H_0^1(U)$. In particular, $(H_0^1(U), B) = H_0^1(U)$ as topological spaces, and $(H_0^1(U), B)$ is a Hilbert space since $H_0^1(U)$ is.

Now, since F is a bounded linear functional on $H_0^1(U) = (H_0^1(U), B)$, the Riesz representation theorem for Hilbert spaces shows that there is a unique $u \in H_0^1(U)$ satisfying (2.21). The same theorem also entails that u satisfies the norm estimate $\|u\|_{H^1(U)} \leq C\|F\|_{H^{-1}(U)}$. This proves the theorem for $f = 0$. The general case follows from this by the next exercise. □

EXERCISE 2.25. Prove the previous theorem for any $F \in H^{-1}(U)$ and $f \in H^1(U)$ by looking for a solution $u = f + \tilde{u}$ with suitable $\tilde{u} \in H_0^1(U)$.

Maximum principle. We will next prove the maximum principle. The proof hinges on the fact, proved in the following exercises, that $u_+ \in H^1(U)$ whenever $u \in H^1(U)$.

EXERCISE 2.26. Let $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$. If $u \in H^1(U)$, show that $f(u) \in H^1(U)$ and that the weak derivatives satisfy $\partial_j(f(u)) = f'(u)\partial_j u$.

EXERCISE 2.27. If $u \in H^1(U)$, use the previous exercise to show that $u_+ \in H^1(U)$ and that the weak derivatives satisfy

$$\partial_j u_+ = \begin{cases} \partial_j u & \text{when } u > 0, \\ 0 & \text{when } u \leq 0. \end{cases}$$

(Hint: consider $f(t) = (\varepsilon^2 + t^2)^{1/2} - \varepsilon$ when $t > 0$, $f(t) = 0$ when $t \leq 0$.)

PROOF OF THEOREM 2.20. Let $u \in H^1(U)$ solve $-\Delta_g u = 0$ in U. We will show that

$$u \leq 0 \text{ on } \partial U \implies u \leq 0 \text{ in } U.$$

The other statements follow easily from this. Now, since u is a weak solution, we have

$$(du, dv)_{L^2(U, dV_g)} = 0 \text{ for all } v \in H_0^1(U).$$

Since $u \leq 0$ on ∂U, by definition we have $u_+ \in H_0^1(U)$ and we may take $v = u_+$ above. We also write $u_- = (-u)_+$, so that $u = u_+ - u_-$. It follows that

$$(du_+, du_+)_{L^2(U, dV_g)} - (du_-, du_+)_{L^2(U, dV_g)} = 0.$$

However, by Exercise 2.27 we have $du_+ = 0$ on $\{u \leq 0\}$ and $du_- = 0$ on $\{u \geq 0\}$, which shows that the second term is zero. Thus

$$\|du_+\|^2_{L^2(U,dV_g)} = 0.$$

Since $u_+ \in H^1_0(U)$, the Poincaré inequality gives $u_+ = 0$. This shows that $u \leq 0$ in U. □

Higher regularity. We will now prove *interior regularity*, stating that if F is locally H^k in U, then the solution u is locally H^{k+2} in U. We refer to [**Ev**, Chapter 6] for the proof of the *boundary regularity* part of Theorem 2.19. Interior regularity is a purely local affair and it is enough to consider solutions in sufficiently small open balls in U. More generally, we consider operators of the form

$$Pu = a^{jk}\partial_{jk}u + b^j\partial_j u + cu \tag{2.22}$$

where $a^{jk}, b^j, c \in C^\infty(\overline{U})$ and a^{jk} satisfies for some $\lambda > 0$ the ellipticity condition

$$a^{jk}(x)\xi_j\xi_k \geq \lambda|\xi|^2, \qquad x \in \overline{U},\ \xi \in \mathbb{R}^n.$$

Consider the local Sobolev space

$$H^k_{\mathrm{loc}}(U) = \{u \,:\, u|_V \in H^k(V) \text{ for any } V \subset\subset U\},$$

where $V \subset\subset U$ means that V is open and $\overline{V}$ is a compact subset of U. We aim to prove the following result.

Proposition 2.22 (Interior regularity). *If $u \in H^1_{\mathrm{loc}}(U)$ is a weak solution of $Pu = F$ in U and $F \in H^k_{\mathrm{loc}}(U)$, then $u \in H^{k+2}_{\mathrm{loc}}(U)$. Moreover, if $U_1 \subset\subset U_2 \subset\subset U$, then there is $C = C_{P,k,U_1,U_2}$ such that*

$$\|u\|_{H^{k+2}(U_1)} \leq C(\|u\|_{H^1(U_2)} + \|F\|_{H^k(U_2)}). \tag{2.23}$$

Remark 2.23. Since $g_{jk} \in C^\infty(\overline{U})$, the operator Δ_g can be equivalently written in the form (2.22) (so called *nondivergence form*). Weak $H^1_{\mathrm{loc}}(U)$ solutions of $-\Delta_g u = F$ correspond to $H^1_{\mathrm{loc}}(U)$ solutions in the sense of distributions of the nondivergence form operator (2.22). The following arguments can be justified using either notion of the solutions (we will ignore the details).

We begin with an *a priori* estimate that is valid for test functions $u \in C^\infty_c(U)$.

LEMMA 2.24. *Let $K \subset \mathbb{R}^n$ be compact. There are $C, r_0 > 0$ such that whenever $x_0 \in K$ and $0 < r \leq r_0$, one has $B(x_0, r) \subset U$ and*

$$\|u\|_{H^2} \leq C(\|u\|_{H^1} + \|Pu\|_{L^2}), \qquad u \in C_c^\infty(B(x_0, r)).$$

PROOF. We do the proof under the simplifying assumptions that $a^{jk}(x_0) = \delta^{jk}$ and $b^j = c = 0$. Then $Pu = a^{jk}\partial_{jk}u$. First assume that $r_0 < \mathrm{dist}(K, \partial U)$. Then $B(x_0, r) \subset U$ for $r < r_0$, and for any $u \in C_c^\infty(B(x_0, r))$ we can perform the following integration by parts:

$$\begin{aligned} \sum_{j,k=1}^n (\partial_{jk}u, \partial_{jk}u) &= \sum_{j,k=1}^n (\partial_{jj}u, \partial_{kk}u) = \|a^{jk}(x_0)\partial_{jk}u\|_{L^2}^2 \\ &= \|Pu - (a^{jk} - a^{jk}(x_0))\partial_{jk}u\|_{L^2}^2. \end{aligned}$$

Since $a^{jk} \in C^\infty(\overline{U})$, we have $|a^{jk}(x) - a^{jk}(x_0)| \leq M|x - x_0|$ where $M = \max_{j,k}\|\nabla a^{jk}\|_{L^\infty(U)}$. Using that $\mathrm{supp}(u) \subset B(x_0, r)$, we have

$$\sum_{j,k=1}^n \|\partial_{jk}u\|_{L^2}^2 \leq 2\|Pu\|^2 + 2(Mr)^2 \sum_{j,k=1}^n \|\partial_{jk}u\|_{L^2}^2.$$

Choosing r_0 so that $2(Mr_0)^2 \leq 1/2$, we can absorb the last term of the right to the left hand side. This implies that

$$\sum_{j,k=1}^n \|\partial_{jk}u\|_{L^2}^2 \leq 4\|Pu\|^2.$$

The desired estimate follows by adding $\|u\|_{H^1(U)}^2$ on both sides. □

EXERCISE 2.28. Prove Lemma 2.24 in the general case.

Now, if $u \in H^1$ solves $Pu = F$ where $F \in L^2$, we would like to apply Lemma 2.24 to χu where $\chi \in C_c^\infty(\mathbb{R}^n)$ is a suitable cutoff function. This would at least morally show that u is H^2 near x_0. However, since u is only H^1 we cannot apply the a priori estimate directly to u. We will instead apply the estimate to convolution approximations u_ε of u, defined as

$$u_\varepsilon := u * \varphi_\varepsilon = \int_{\mathbb{R}^n} u(x - y)\varphi_\varepsilon(y)\, dy, \qquad \varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon),$$

where $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfies $0 \leq \varphi \leq 1$ and $\int_{\mathbb{R}^n} \varphi\, dx = 1$. The following exercise contains the basic properties of convolutions of L^2 functions.

EXERCISE 2.29. If $u \in L^2(\mathbb{R}^n)$, show that $u_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\partial^\alpha u_\varepsilon = u * \partial^\alpha \varphi_\varepsilon$ for any multi-index α, and $u_\varepsilon \to u$ in $L^2(\mathbb{R}^n)$. Moreover, if $u \in H^k(\mathbb{R}^n)$, show that $\partial^{\alpha+\beta} u_\varepsilon = \partial^\alpha u * \partial^\beta \varphi_\varepsilon$ for any multi-indices α, β with $|\alpha| \le k$, and that $\partial^\alpha u_\varepsilon \to \partial^\alpha u$ in $L^2(\mathbb{R}^n)$ whenever $|\alpha| \le k$.

The following result will allow us to take limits as $\varepsilon \to 0$.

LEMMA 2.25 (Friedrichs' lemma). *Let $u \in H^1(U)$ be compactly supported in U, and let $Pu \in L^2(U)$. Then*

$$P(u_\varepsilon) \to Pu \text{ in } L^2(U) \text{ as } \varepsilon \to 0.$$

PROOF. Let $K \subset U$ be compact and let $u \in H^1$ be supported in K. We have $b^j \partial_j u_\varepsilon + c u_\varepsilon \to 0$ in $L^2(U)$ by Exercise 2.29. It is thus enough to consider the case where $Pu = a^{jk}\partial_{jk} u$. We write $T_\varepsilon(u) = (Pu) * \varphi_\varepsilon - P(u * \varphi_\varepsilon)$ and use Exercise 2.29 together with the definition of weak derivatives to observe that

$$\begin{aligned}
T_\varepsilon(u)(x) &= \int (Pu)(x-y)\varphi_\varepsilon(y) - P(\int u(x-y)\varphi_\varepsilon(y)\,dy) \\
&= \int \partial_j u(x-y)\partial_{y_k}(a^{jk}(x-y)\varphi_\varepsilon(y))\,dy - \int a^{jk}(x)\partial_j u(x-y)\partial_k\varphi_\varepsilon(y))\,dy \\
&= \int (a^{jk}(x-y) - a^{jk}(x))\partial_j u(x-y)\partial_k\varphi_\varepsilon(y)) - \int (\partial_k a^{jk}(x-y))\partial_j u(x-y)\varphi_\varepsilon(y)\,dy.
\end{aligned}$$

Since a^{jk} is Lipschitz continuous, there is $M > 0$ such that

$$|T_\varepsilon u(x)| \le M \int |\nabla u(x-y)| g_\varepsilon(y)\,dy$$

where $g_\varepsilon(y) = |y||\nabla\varphi_\varepsilon(y)| + \varphi_\varepsilon(y)$. Since $g_\varepsilon(y) = \varepsilon^{-n} g_1(y/\varepsilon)$, this is another convolution approximation and it satisfies

$$\|T_\varepsilon u\|_{L^2} \le C\|\nabla u\|_{L^2} \tag{2.24}$$

where the constant C is independent of u (with support in K) and ε.

Finally, we note that if $v \in C_c^\infty(U)$, then $(Pv) * \varphi_\varepsilon \to Pv$ and $P(v * \varphi_\varepsilon) \to Pv$ in $L^2(U)$. Thus

$$T_\varepsilon(v) \to 0 \text{ in } L^2(U) \text{ for any } v \in C_c^\infty(U). \tag{2.25}$$

Now if $u \in H^1(U)$ is compactly supported and $v \in C_c^\infty(U)$, we use (2.24) to observe that

$$\|T_\varepsilon u\|_{L^2} \le \|T_\varepsilon v\|_{L^2} + \|T_\varepsilon(u-v)\|_{L^2} \le \|T_\varepsilon v\|_{L^2} + C\|u-v\|_{H^1}$$

where C depends on the fixed compact set containing the supports of u and v. We can make $\|u-v\|_{H^1}$ arbitrarily small by choosing

$v = u_\delta$ for δ small, and then by (2.25) we can make $\|T_\varepsilon v\|_{L^2}$ small by choosing ε small. This proves that $T_\varepsilon(u) \to 0$ in $L^2(U)$ when $\varepsilon \to 0$ as required. □

PROOF OF PROPOSITION 2.22. We first prove the case $k = 0$. Suppose that U_1, U_2 are as above, and let r_0 be as in Lemma 2.24 with $K = \overline{U}_1$. Let $\chi \in C_c^\infty(B(0, r_0))$ satisfy $\chi = 1$ for $|x| \leq r_0/2$, and write $\chi_{x_0}(x) = \chi(x - x_0)$. We also note that in the weak sense

$$P(\chi_{x_0} u) = \chi_{x_0} P u + [P, \chi_{x_0}]u$$

where $[P, \chi_{x_0}] = P \circ \chi_{x_0} - \chi_{x_0} \circ P$ is a first order operator with smooth coefficients. In particular, since $u \in H^1(U)$ and $Pu \in L^2(U)$, we have $P(\chi_{x_0} u) \in L^2(U)$.

Now given any $x_0 \in K$, we apply Lemma 2.24 to $(\chi_{x_0} u)_\varepsilon - (\chi_{x_0} u)_\delta$ with $\varepsilon, \delta > 0$ small:

$$\begin{aligned}
&\|(\chi_{x_0} u)_\varepsilon - (\chi_{x_0} u)_\delta\|_{H^2} \\
&\qquad \leq C(\|(\chi_{x_0} u)_\varepsilon - (\chi_{x_0} u)_\delta\|_{H^1} + \|P((\chi_{x_0} u)_\varepsilon) - P((\chi_{x_0} u)_\delta)\|_{L^2}).
\end{aligned}$$

Since $u \in H^1$ one has $(\chi_{x_0} u)_\varepsilon \to \chi_{x_0} u$ in H^1, and by Lemma 2.25 one has $P((\chi_{x_0} u)_\varepsilon) \to P(\chi_{x_0} u)$ in L^2 as $\varepsilon \to 0$. It follows that $(\chi_{x_0} u)_\varepsilon$ is a Cauchy sequence in $H^2(U)$ and hence converges to some $v \in H^2(U)$. Since $(\chi_{x_0} u)_\varepsilon \to \chi_{x_0} u$ in $H^1(U)$, the uniqueness of limits implies that $v = \chi_{x_0} u$. This proves that $\chi_{x_0} u \in H^2(U)$ for all $x_0 \in K$. By compactness one has $u \in H^2(U_1)$, and the estimate 2.23 for $k = 0$ also follows (exercise).

We have completed the proof for $k = 0$. Let now $F \in H^1_{\mathrm{loc}}(U)$ and let $u \in H^1_{\mathrm{loc}}(U)$ solve $Pu = F$ in U. By the case $k = 0$ we have $u \in H^2_{\mathrm{loc}}(U)$. We may now apply ∂_j to the equation $Pu = F$:

$$P(\partial_j u) = \partial_j F - [\partial_j, P]u.$$

Here $[\partial_j, P]$ is a second order operator with smooth coefficients, and hence the right hand side $\partial_j F - [\partial_j, P]u$ is a function in $L^2_{\mathrm{loc}}(U)$. Since $\partial_j u \in H^1_{\mathrm{loc}}(U)$ is a weak solution, using the case $k = 0$ again we obtain $\partial_j u \in H^2_{\mathrm{loc}}(U)$. This holds for any j, and thus one has $u \in H^3_{\mathrm{loc}}(U)$ and one also has the required estimate. The proof for arbitrary k follows by iterating this argument. □

EXERCISE 2.30. Complete the proof that when $F \in L^2_{\mathrm{loc}}(U)$, one has $u \in H^2(U_1)$ and (2.23) holds when $k = 0$.

CHAPTER 3

Calculus on Riemannian manifolds

In this chapter we will discuss the calculus concepts from Chapter 2 in the more general setting of smooth or Riemannian manifolds. Thus, instead of working on open sets $U \subset \mathbb{R}^n$, we wish to perform calculus operations on spaces such as

- surfaces in $\mathbb{R}^3$ (spheres, tori, double tori, etc)
- n-dimensional, possibly complicated hypersurfaces $S \subset \mathbb{R}^{n+k}$
- manifolds arising in dynamical systems for ODEs
- groups of transformations ($GL(n)$, $SO(n)$, $U(n)$ etc)
- phase spaces of dynamical systems on the above examples

Our aim is to present the material briefly, giving the definitions but omitting the proofs of their basic properties (for proofs see [**Le2**] and [**Le1**]). We hope that the readers will at this point have sufficient intuition from the $\mathbb{R}^n$ picture to appreciate what is going on.

3.1. Smooth manifolds

Manifolds. We recall some basic definitions from the theory of smooth manifolds. We will consistently also consider manifolds with boundary.

Definition. A *smooth n-dimensional manifold* is a topological space M, assumed to be Hausdorff and second countable, together with an open cover $\{U_\alpha\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to \tilde{U}_\alpha$ such that each $\tilde{U}_\alpha$ is an open set in $\mathbb{R}^n$, and $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map whenever $U_\alpha \cap U_\beta$ is nonempty.

Any family $\{(U_\alpha, \varphi_\alpha)\}$ as above is called an *atlas*. Any atlas gives rise to a maximal atlas, called a *smooth structure*, which is not strictly contained in any other atlas. We assume that we are always dealing with the maximal atlas. The pairs $(U_\alpha, \varphi_\alpha)$ are called *charts*, and the maps φ_α are called *local coordinate systems* (one usually writes $x = \varphi_\alpha$ and thus identifies points $p \in U_\alpha$ with points $x(p) \in \tilde{U}_\alpha$ in $\mathbb{R}^n$).

DEFINITION. A *smooth n-dimensional manifold with boundary* is a second countable Hausdorff topological space together with an open cover $\{U_\alpha\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to \tilde{U}_\alpha$ such that each $\tilde{U}_\alpha$ is an open set in $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n\,;\, x_n \geq 0\}$, and $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map whenever $U_\alpha \cap U_\beta$ is nonempty.

Here, if $A \subset \mathbb{R}^n$ we say that a map $F : A \to \mathbb{R}^n$ is smooth if it extends to a smooth map $\tilde{A} \to \mathbb{R}^n$ where $\tilde{A}$ is an open set in $\mathbb{R}^n$ containing A.

If M is a manifold with boundary we say that p is a boundary point if $\varphi(p) \in \partial\mathbb{R}^n_+$ for some chart φ, and an interior point if $\varphi(p) \in \mathrm{int}(\mathbb{R}^n_+)$ for some φ. We write ∂M for the set of boundary points and M^{int} for the set of interior points. Since M is not assumed to be embedded in any larger space, these definitions may differ from the usual ones in point set topology.

EXERCISE 3.1. If M is a manifold with boundary, show that the sets M^{int} and ∂M are always disjoint.

To clarify the relations between the definitions, note that a manifold is always a manifold with boundary (the boundary being empty), but a manifold with boundary is a manifold iff the boundary is empty (by the above exercise). However, we will loosely refer to manifolds both with and without boundary as 'manifolds'.

We have the following classes of manifolds:

- A *closed manifold* is compact, connected, and has no boundary
  - Examples: the sphere S^n, the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$
- An *open manifold* has no boundary and no component is compact
  - Examples: open subsets of $\mathbb{R}^n$, strict open subsets of a closed manifold
- A *compact manifold with boundary* is a manifold with boundary which is compact as a topological space
  - Examples: the closures of bounded open sets in $\mathbb{R}^n$ with smooth boundary, the closures of open sets with smooth boundary in closed manifolds

Smooth maps.

DEFINITION. Let $f : M \to N$ be a map between two manifolds. We say that f is *smooth* near a point p if $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth for some charts (U, φ) of M and (V, ψ) of N such that $p \in U$ and $f(U) \subset V$. We say that f is *smooth* in a set $A \subset M$ if it is smooth near any point of A. The set of all maps $f : M \to N$ which are smooth in A is denoted by $C^\infty(A, N)$. If $N = \mathbb{R}$ we write $C^\infty(A, N) = C^\infty(A)$.

EXERCISE 3.2. Verify that the definition of a smooth map does not depend on the particular choice of coordinate charts.

Tangent bundle. If $U \subset \mathbb{R}^n$ is open, we defined the tangent space $T_x U = \{x\} \times \mathbb{R}^n$ to be a copy of $\mathbb{R}^n$ sitting at x. Any $v \in T_x U$ can be thought of as an infinitesimal direction where one can move from x, and there is a corresponding directional derivative

$$\partial_v : C^\infty(U) \to \mathbb{R}, \quad \partial_v f(x) := v \cdot \nabla f(x).$$

Then ∂_v is a linear operator satisfying $\partial_v(fg) = (\partial_v f)g + f(\partial_v g)$. Such an object is called a derivation. It turns out that derivations can be identified with vectors in the tangent space, and this leads to a definition of tangent spaces on abstract manifolds.

DEFINITION. Let $p \in M$. A *derivation* at p is a linear map $v : C^\infty(M) \to \mathbb{R}$ which satisfies $v(fg) = (vf)g(p) + f(p)(vg)$. The *tangent space* $T_p M$ is the vector space consisting of all derivations at p. Its elements are called *tangent vectors*.

The tangent space $T_p M$ is an n-dimensional vector space when $\dim(M) = n$. If x is a local coordinate system in a neighborhood U of p, the *coordinate vector fields* ∂_j are defined for any $q \in U$ to be the derivations

$$\partial_j|_q f := \frac{\partial}{\partial x_j}(f \circ x^{-1})(x(q)), \quad j = 1, \ldots, n.$$

Then $\{\partial_j|_q\}$ is a basis of $T_q M$, and any $v \in T_q M$ may be written as $v = v^j \partial_j$.

EXERCISE 3.3. Prove that $T_p M$ is an n-dimensional vector space spanned by $\{\partial_j\}$ also when M is a manifold with boundary.

The *tangent bundle* is the disjoint union

$$TM := \bigvee_{p \in M} T_p M.$$

The tangent bundle has the structure of a $2n$-dimensional manifold defined as follows. For any chart (U, x) of M we represent elements of T_qM for $q \in U$ as $v = v^j(q)\partial_j|_q$, and define a map $\tilde{\varphi} : TU \to \mathbb{R}^{2n}, \tilde{\varphi}(q,v) = (x(q), v^1(q), \ldots, v^n(q))$. The charts $(TU, \tilde{\varphi})$ are called the *standard charts* of TM and they define a smooth structure on TM.

Since the tangent bundle is a smooth manifold, the following definition makes sense:

Definition. A *vector field* on M is a smooth map $X : M \to TM$ such that $X(p) \in T_pM$ for each $p \in M$.

Cotangent bundle. The dual space of a vector space V is

$$V^* := \{u : V \to \mathbb{R}\,;\, u \text{ linear}\}.$$

The dual space of T_pM is denoted by T_p^*M and is called the *cotangent space* of M at p. Let x be local coordinates in U, and let ∂_j be the coordinate vector fields that span T_qM for $q \in U$. We denote by dx^j the elements of the dual basis of T_q^*M, so that any $\xi \in T_q^*M$ can be written as $\xi = \xi_j\, dx^j$. The dual basis is characterized by

$$dx^j(\partial_k) = \delta_{jk}.$$

The *cotangent bundle* is the disjoint union

$$T^*M = \bigvee_{p \in M} T_p^*M.$$

This becomes a $2n$-dimensional manifold by defining for any chart (U,φ) of M a chart $(T^*U, \tilde{\varphi})$ of T^*M by $\tilde{\varphi}(q, \xi_j\, dx^j) = (\varphi(q), \xi_1, \ldots, \xi_n)$.

Definition. A *1-form* on M is a smooth map $\alpha : M \to T^*M$ such that $\alpha(p) \in T_p^*M$ for each $p \in M$.

Tensor bundles. If V is a finite dimensional vector space, the space of (covariant) k-tensors on V is

$$T^k(V) := \{u : \underbrace{V \times \ldots \times V}_{k \text{ copies}} \to \mathbb{R}\,;\, u \text{ linear in each variable}\}.$$

The *k-tensor bundle* on M is the disjoint union

$$T^kM = \bigvee_{p \in M} T^k(T_pM).$$

If x are local coordinates in U and dx^j is the basis for T_q^*M, then each $u \in T^k(T_qM)$ for $q \in U$ can be written as

$$u = u_{j_1 \cdots j_k} dx^{j_1} \otimes \ldots \otimes dx^{j_k}$$

Here $\otimes$ is the *tensor product*

$$T^k(V) \times T^{k'}(V) \to T^{k+k'}(V), \quad (u, u') \mapsto u \otimes u',$$

where for $v \in V^k, v' \in V^{k'}$ we have

$$(u \otimes u')(v, v') := u(v)u'(v').$$

It follows that the elements $dx^{j_1} \otimes \ldots \otimes dx^{j_k}$ span $T^k(T_qM)$. Similarly as above, T^kM has the structure of a smooth manifold (of dimension $n + n^k$).

Definition. A *k-tensor field* on M is a smooth map $u : M \to TM$ such that $u(p) \in T^k(T_pM)$ for each $p \in M$.

Exterior powers. The space of alternating k-tensors is

$$A^k(V) := \{u \in T^k(V)\,;\, u(v_1, \ldots, v_k) = 0 \text{ if } v_i = v_j \text{ for some } i \neq j\}.$$

This gives rise to the *exterior bundle*

$$\Lambda^k(M) := \bigvee_{p\in M} A^k(T_pM).$$

To describe a basis for $A^k(T_pM)$, we introduce the *wedge product*

$$A^k(V) \times A^{k'}(V) \to A^{k+k'}(V),\ (\omega, \omega') \mapsto \omega \wedge \omega' := \frac{(k + k')!}{k!(k')!}\mathrm{Alt}(\omega \otimes \omega'),$$

where $\mathrm{Alt} : T^k(V) \to A^k(V)$ is the projection to alternating tensors,

$$\mathrm{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

We have written S_k for the group of permutations of $\{1, \ldots, k\}$, and $\mathrm{sgn}(\sigma)$ for the signature of $\sigma \in S_k$.

The following properties of the wedge product can be checked from the definition:

Lemma 3.1. *The wedge product is associative, meaning that $\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$ for any alternating tensors. Moreover, if $\omega_1, \ldots, \omega_k$ are 1-tensors, then*

$$\omega_{\sigma(1)} \wedge \ldots \wedge \omega_{\sigma(k)} = (-1)^{\mathrm{sgn}(\sigma)}\omega_1 \wedge \ldots \wedge \omega_k, \quad \sigma \in S_k \tag{3.1}$$

and for any $v_1, \ldots, v_k \in V$ one has

$$(\omega_1 \wedge \ldots \wedge \omega_k)(v_1, \ldots, v_k) = \det \begin{bmatrix} \omega_1(v_1) & \ldots & \omega_1(v_k) \\ \vdots & \ddots & \vdots \\ \omega_k(v_1) & \ldots & \omega_k(v_k) \end{bmatrix}. \tag{3.2}$$

Exercise 3.4. Show that Alt maps $T^k(V)$ into $A^k(V)$ and that $(\mathrm{Alt})^2 = \mathrm{Alt}$.

Exercise 3.5. Prove Lemma 3.1.

If x is a local coordinate system in U, then a basis of $A^k(T_pM)$ is given by

$$\{dx^{j_1} \wedge \ldots \wedge dx^{j_k}\}_{1 \leq j_1 < j_2 < \ldots < j_k \leq n}.$$

Again, $\Lambda^k(M)$ is a smooth manifold (of dimension $n + \binom{n}{k}$).

Definition. A *k-form* on M is a smooth map $\omega : M \to \Lambda^k M$ such that $\omega(p) \in A^k(T_pM)$ for each $p \in M$.

Smooth sections. The above constructions of the tangent bundle, cotangent bundle, tensor bundles, and exterior powers are all examples of *vector bundles* with base manifold M. We will not need a precise definition here, but just note that in each case there is a natural vector space over any point $p \in M$ (called the *fiber over* p). A *smooth section* of a vector bundle E over M is a smooth map $s : M \to E$ such that for each $p \in M$, $s(p)$ belongs to the fiber over p. The space of smooth sections of E is denoted by $C^\infty(M, E)$.

We have the following terminology:

- $C^\infty(M, TM)$ is the set of *vector fields* on M,
- $C^\infty(M, T^kM)$ is the set of *k-tensor fields* on M,
- $\Omega^1(M) = C^\infty(M, T^*M)$ is the set of 1-*forms* on M,
- $\Omega^k(M) = C^\infty(M, \Lambda^k M)$ is the set of *(differential) k-forms* on M.

Let x be local coordinates in a set U, and let ∂_j and dx^j be the coordinate vector fields and 1-forms in U which span T_qM and T_q^*M, respectively, for $q \in U$. In these local coordinates,

- a vector field X has the expression $X = X^j \partial_j$,
- a 1-form α has expression $\alpha = \alpha_j \, dx^j$,
- a k-tensor field u can be written as

  $$u = u_{j_1 \cdots j_k} dx^{j_1} \otimes \ldots \otimes dx^{j_k},$$

- a k-form ω has the form

  $$\omega = \omega_I \, dx^I$$

  where $I = (i_1, \ldots, i_k)$ and $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, with the sum being over all I such that $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Here, the component functions $X^j, \alpha_j, u_{j_1 \cdots j_k}, \omega_I$ are all smooth real valued functions in U.

We mention briefly how the local coordinate formula for a k-tensor field u is obtained. If (U, x) is a local coordinate chart and $\{\partial_j\}$ are the associated coordinate vector fields, one can write any $v \in T_q M$ for $q \in U$ as $v = v^k \partial_k|_q$ for some $(v^1, \ldots, v^n) \in \mathbb{R}^n$. Thus by linearity

$$u_q(v_1, \ldots, v_k) = u_q(v_1^{j_1} \partial_{j_1}|_q, \ldots, v_1^{j_k} \partial_{j_k}|_q) = u_q(\partial_{j_1}|_q, \ldots, \partial_{j_k}|_q) v_1^{j_1} \cdots v_k^{j_k}.$$

If we define

$$u_{j_1 \cdots j_k}(q) := u_q(\partial_{j_1}|_q, \ldots, \partial_{j_k}|_q),$$

then the above computation and the definition of tensor product imply

$$u_q(v_1, \ldots, v_k) = (u_{j_1 \cdots j_k}(q) dx^{j_1}|_q \otimes \ldots \otimes dx^{j_k}|_q)(v_1, \ldots, v_k).$$

This proves that the local coordinate representation of a tensor field u is obtained just by evaluating u at coordinate vector fields.

Example. Some examples of the smooth sections that will be encountered in this text are:

- Vector fields: the gradient vector field $\mathrm{grad}(f)$ for $f \in C^\infty(M)$, coordinate vector fields ∂_j in a chart U
- One-forms: the exterior derivative df for $f \in C^\infty(M)$
- 2-tensor fields: Riemannian metrics g, Hessians $\mathrm{Hess}(f)$ for $f \in C^\infty(M)$, Ricci curvature R_{ab}
- 4-tensor fields: Riemann curvature tensor R_{abcd}
- k-forms: the volume form dV in Riemannian manifold (M, g) (then $k = n$)

Changes of coordinates. We next consider the transformation laws for vector and tensor fields under changes of coordinates. It is convenient to phrase these in terms of more general pullbacks or pushforwards by smooth maps between manifolds. We begin with pushforwards of tangent vectors.

Definition. Let $F : M \to N$ be a smooth map. The *pushforward* by F is the map acting on $T_p M$ for any $p \in M$ by

$$F_* : T_p M \to T_{F(p)} N, \quad F_* v(f) = v(f \circ F) \text{ for } f \in C^\infty(N).$$

The map F_* is also called the *derivative* or *tangent map* of F, and it is also denoted by DF.

We compute how F_* transforms vector fields in local coordinates.

LEMMA 3.2. *Let $F : M \to N$ be a smooth map and let X be a vector field in M. If (U, y) and (V, z) are coordinate charts near p in M and near $F(p)$ in N, respectively, and if Y and Z are corresponding coordinate representations of X and $F_* X$ so that*

$$X(q) = Y^j(y(q))\partial_{y^j}|_q, \quad F_* X(r) = Z^k(z(r))\partial_{z^k}|_r,$$

then

$$Z^k(z(F(q))) = \partial_{y^j}\tilde{F}^k(y(q))Y^j(y(q))$$

where $\tilde{F} = z \circ F \circ y^{-1}$.

PROOF. Given $q \in U$ with $F(q) \in V$, the tangent vector $F_* X|_{F(q)}$ is a derivation acting on $f \in C^\infty(N)$ and by the definitions

$$\begin{aligned} F_* X|_{F(q)} f &= X|_q(f \circ F) = Y^j(y(q))\partial_{y^j}|_q(f \circ z^{-1} \circ \tilde{F} \circ y) \\ &= Y^j(y(q))\partial_{y^j}((f \circ z^{-1}) \circ \tilde{F})(y(q)) \\ &= Y^j(y(q))\partial_{z^k}(f \circ z^{-1})(z(F(q)))\partial_{y^j}\tilde{F}^k(y(q)) \\ &= \partial_{y^j}\tilde{F}^k(y(q))Y^j(y(q))\partial_{z^k}|_{F(q)} f. \end{aligned}$$

□

REMARK. Applying Lemma 3.2 in the case where F is the identity map $F = i : M \to M$ shows that the representations Y and Z of a vector field X in two coordinate charts (U, y) and (V, z) with $U \cap V \neq \emptyset$ are related by

$$Z^k(z(q)) = \partial_{y^j}(z \circ y^{-1})^k(y(q))Y^j(y(q)), \quad q \in U \cap V. \tag{3.3}$$

This provides an alternative way to define vector fields on a manifold: if to each coordinate chart (U, y) on M one associates a vector field Y in $y(U) \subset \mathbb{R}^n$, and if the vector fields Y and Z for any two coordinate charts (U, y) and (V, z) with $U \cap V \neq \emptyset$ satisfy (3.3), then there is a vector field X in M whose coordinate representation in any chart (U, y) is Y. If (3.3) holds, we say that the coordinate representations Y *transform as a vector field in M*.

We now move to tensor fields. If $F : M \to N$ is any smooth map, we can associate to a tensor field $u \in C^\infty(N, T^k N)$ a corresponding tensor field $F^* u \in C^\infty(M, T^k M)$ in the following way.

DEFINITION. If $F : M \to N$ is a smooth map, the *pullback* by F acting on k-tensor fields is the map $F^* : C^\infty(N, T^k N) \to C^\infty(M, T^k M)$,

$$(F^* u)_p(v_1, \ldots, v_k) = u_{F(p)}(F_* v_1, \ldots, F_* v_k)$$

where $v_1, \ldots, v_k \in T_p N$.

It is easy to check that F^*u is indeed a tensor field on M, and that F^* has the following properties:

LEMMA 3.3. *(Properties of F^*) Let $F : M \to N$ be a smooth map, let $f \in C^\infty(N)$, let u and u' be tensor fields in N, and let ω and ω' be differential forms in N.*

- $F^*(fu) = (f \circ F)F^*u$
- $F^*(u \otimes u') = F^*u \otimes F^*u'$
- *F^* preserves alternating tensors and thus induces a map on differential forms,*

$$F^* : \Omega^k(N) \to \Omega^k(M), \quad 0 \le k \le n$$

- $F^*(\omega \wedge \omega') = F^*\omega \wedge F^*\omega'$

EXERCISE 3.6. Prove Lemma 3.3.

In terms of local coordinates, the pullback acts by

- $F^*f = f \circ F$ if f is a smooth function (=0-form)
- $F^*(\alpha_j \, dx^j) = (\alpha_j \circ F)\, d(x^j \circ F) = (\alpha_j \circ F)dF^j$ if α is a 1-form

and it has the following expression for higher order tensors:

LEMMA 3.4. *Let $F : M \to N$ be a smooth map and let u be a k-tensor field in N. If (U, y) and (V, z) are coordinate charts near p in M and near $F(p)$ in N, respectively, and if $(y_{i_1 \cdots i_k})$ and $(z_{j_1 \cdots j_k})$ are corresponding coordinate representations of F^*u and u so that*

$$\begin{aligned} F^*u(q) &= y_{i_1 \cdots i_k}(y(q))\, dy^{i_1} \otimes \cdots \otimes dy^{i_k}|_q, \\ u(r) &= z_{j_1 \cdots j_k}(z(r))\, dz^{j_1} \otimes \cdots \otimes dz^{j_k}|_r, \end{aligned}$$

then

$$y_{i_1 \cdots i_k}|_{y(q)} = (\partial_{y^{i_1}} \tilde{F}^{j_1}) \cdots (\partial_{y^{i_k}} \tilde{F}^{j_k})(z_{j_1 \cdots j_k} \circ \tilde{F})|_{y(q)}$$

where $\tilde{F} = z \circ F \circ y^{-1}$.

PROOF. Given $q \in U$ with $F(q) \in V$, we compute

$$\begin{aligned} y_{i_1 \cdots i_k}(y(q)) &= F^*u|_q(\partial_{y^{i_1}}, \ldots, \partial_{y^{i_k}}) \\ &= u|_{F(q)}(F_*\partial_{y^{i_1}}, \ldots, F_*\partial_{y^{i_k}}) \\ &= u|_{F(q)}(\partial_{y^{i_1}} \tilde{F}^{j_1}(y(q))\partial_{z^{j_1}}, \ldots, \partial_{y^{i_k}} \tilde{F}^{j_k}(y(q))\partial_{z^{j_k}}) \\ &= \partial_{y^{i_1}} \tilde{F}^{j_1}(y(q)) \cdots \partial_{y^{i_k}} \tilde{F}^{j_k}(y(q)) z_{j_1 \cdots j_k}(z(F(q)). \end{aligned}$$ ☐

Remark. We have defined F_* acting on vector fields and F^* acting on k-tensor fields. If $F : M \to N$ is a diffeomorphism, one can define in general $F_* = (F^{-1})^*$ and $F^* = (F^{-1})_*$, and thus for a diffeomorphism F the pushforward and pullback are defined both on vector and tensor fields.

Exterior derivative. The exterior derivative d is a first order differential operator mapping differential k-forms to $k+1$-forms. It can be defined first on 0-forms (that is, smooth functions f) by the local coordinate expression

$$df := \frac{\partial f}{\partial x_j} dx^j.$$

In general, if $\omega = \omega_I \, dx^I$ is a k-form we define

$$d\omega := d\omega_I \wedge dx^I.$$

Lemma 3.5. *The definition of d is independent of the choice of coordinates, and $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is a linear map for $0 \leq k \leq n$. The operator d has the properties*

- $d^2 = 0$
- $d|_{\Omega^n(M)} = 0$
- $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^k \omega \wedge d\omega'$ *for a k-form ω, k'-form ω'*
- $F^* d\omega = dF^* \omega$

Exercise 3.7. Prove Lemma 3.5.

Partition of unity. A major reason for including the condition of second countability in the definition of manifolds is to ensure the existence of partitions of unity. These make it possible to make constructions in local coordinates and then glue them together to obtain a global construction.

Lemma 3.6. *Let M be a manifold and let $\{U_\alpha\}$ be an open cover. There exists a family of C^∞ functions $\{\chi_\alpha\}$ on M such that $0 \leq \chi_\alpha \leq 1$,* $\mathrm{supp}(\chi_\alpha) \subset U_\alpha$, *any point of M has a neighborhood which intersects only finitely many of the sets* $\mathrm{supp}(\chi_\alpha)$, *and further*

$$\sum_\alpha \chi_\alpha = 1 \quad \text{in } M.$$

Integration on manifolds. The natural objects that can be integrated on an n-dimensional manifold are the differential n-forms. This is due to the transformation law for n-forms in $\mathbb{R}^n$ under smooth diffeomorphisms F in $\mathbb{R}^n$. If $f \in C^\infty(\mathbb{R}^n)$, one has

$$\begin{aligned} F^*(f\,dx^1 \wedge \cdots \wedge dx^n) &= (f \circ F) dF^1 \wedge \cdots \wedge dF^n \\ &= (f \circ F)(\partial_{j_1} F^1) \cdots (\partial_{j_n} F^n)\, dx^{j_1} \wedge \cdots \wedge dx^{j_n} \\ &= (f \circ F)(\det DF) dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

This is almost the same as the transformation law for integrals in $\mathbb{R}^n$ under changes of variables, the only difference being that in the latter the factor $|\det DF|$ instead $\det DF$ appears. To define an invariant integral, we therefore need to make sure that all changes of coordinates have positive Jacobian.

Definition. If M admits a smooth nowhere vanishing n-form we say that M is *orientable*. An *oriented manifold* is a manifold together with a given nowhere vanishing n-form.

If M is oriented with a given n-form Ω, a basis $\{v_1, \ldots, v_n\}$ of T_pM is called *positive* if $\Omega(v_1, \ldots, v_n) > 0$. There are many n-forms on an oriented manifold which give the same positive bases; we call any such n-form an *orientation form*. If (U, φ) is a connected coordinate chart, we say that this chart is *positive* if the coordinate vector fields $\{\partial_1, \ldots, \partial_n\}$ form a positive basis of T_qM for all $q \in M$.

A map $F : M \to N$ between two oriented manifolds is said to be *orientation preserving* if it pulls back an orientation form on N to an orientation form of M. In terms of local coordinates given by positive charts, one can see that a map is orientation preserving iff its Jacobian determinant is positive.

Example. The standard orientation of $\mathbb{R}^n$ is given by the n-form $dx^1 \wedge \cdots \wedge dx^n$, where x are the Cartesian coordinates.

If ω is a compactly supported n-form in $\mathbb{R}^n$, we may write $\omega = f\,dx^1 \wedge \cdots \wedge dx^n$ for some smooth compactly supported function f. Then the integral of ω is defined by

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f(x)\,dx^1 \cdots dx^n.$$

If ω is a smooth n-form in a manifold M whose support is compactly contained in U for some positive chart (U, φ), then the integral of ω

over M is defined by

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

Finally, if ω is a compactly supported n-form in a manifold M, the integral of ω over M is defined by

$$\int_M \omega := \sum_j \int_{U_j} \chi_j \omega.$$

where $\{U_j\}$ is some open cover of $\mathrm{supp}(\omega)$ by positive charts, and $\{\chi_j\}$ is a partition of unity subordinate to the cover $\{U_j\}$.

Exercise 3.8. Prove that the definition of the integral is independent of the choice of positive charts and the partition of unity.

The following result is a basic integration by parts formula which implies the usual theorems of Gauss and Green.

Theorem 3.7. *(Stokes theorem) If M is an oriented manifold with boundary and if ω is a compactly supported $(n-1)$-form on M, then*

$$\int_M d\omega = \int_{\partial M} i^* \omega$$

where $i : \partial M \to M$ is the natural inclusion.

Here, if M is an oriented manifold with boundary, then ∂M has a natural orientation defined as follows: for any point $p \in \partial M$, a basis $\{E_1, \ldots, E_{n-1}\}$ of $T_p(\partial M)$ is defined to be positive if $\{N_p, E_1, \ldots, E_{n-1}\}$ is a positive basis of $T_p M$ where N is some outward pointing vector field near ∂M (that is, there is a smooth curve $\gamma : [0, \varepsilon) \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = -N_p$).

Exercise 3.9. Prove that any manifold with boundary has an outward pointing vector field, and show that the above definition gives a valid orientation on ∂M.

3.2. Riemannian manifolds

Riemannian metrics. If u is a 2-tensor field on M, we say that u is *symmetric* if $u(v, w) = u(w, v)$ for any tangent vectors v, w, and that u is *positive definite* if $u(v, v) > 0$ unless $v = 0$.

Definition. Let M be a manifold. A *Riemannian metric* is a symmetric positive definite 2-tensor field g on M. The pair (M, g) is called a *Riemannian manifold*.

If g is a Riemannian metric on M, then $g_p : T_pM \times T_pM \to \mathbb{R}$ is an inner product on T_pM for any $p \in M$. We will write

$$\langle v, w \rangle := g(v, w), \qquad |v| := \langle v, v \rangle^{1/2}.$$

In local coordinates, a Riemannian metric is just a positive definite symmetric matrix. To see this, let (U, x) be a chart of M, and write $v, w \in T_qM$ for $q \in U$ in terms of the coordinate vector fields ∂_j as $v = v^j \partial_j$, $w = w^j \partial_j$. Then

$$g(v, w) = g(\partial_j, \partial_k) v^j w^k.$$

This shows that g has the local coordinate expression

$$g = g_{jk} dx^j \otimes dx^k$$

where $g_{jk} := g(\partial_j, \partial_k)$ and the matrix $(g_{jk})_{j,k=1}^n$ is symmetric and positive definite. We will also write $(g^{jk})_{j,k=1}^n$ for the inverse matrix of (g_{jk}), and $|g| := \det(g_{jk})$ for the determinant.

Example. Some examples of Riemannian manifolds:

1. (Euclidean space) If U is a bounded open set in $\mathbb{R}^n$, then (U, e) is a Riemannian manifold if e is the *Euclidean metric* for which $e(v, w) = v \cdot w$ is the Euclidean inner product of $v, w \in T_pU \approx \mathbb{R}^n$. In Cartesian coordinates, e is just the identity matrix.
2. If U is as above, then more generally (U, g) is a Riemannian manifold if $g(x) = (g_{jk}(x))_{j,k=1}^n$ is any family of positive definite symmetric matrices whose elements depend smoothly on $x \in U$.
3. If U is a bounded open set in $\mathbb{R}^n$ with smooth boundary, then $(\overline{U}, g)$ is a compact Riemannian manifold with boundary if $g(x)$ is a family of positive definite symmetric matrices depending smoothly on $x \in \overline{U}$.
4. (Hypersurfaces) Let S be a smooth hypersurface in $\mathbb{R}^n$ such that $S = f^{-1}(0)$ for some smooth function $f : \mathbb{R}^n \to \mathbb{R}$ which satisfies $\nabla f \neq 0$ when $f = 0$. Then S is a smooth manifold of dimension $n - 1$, and the tangent space T_pS for any $p \in S$ can be identified with $\{v \in \mathbb{R}^n \,;\, v \cdot \nabla f(p) = 0\}$. Using this identification, we define an inner product $g_p(v, w)$ on T_pS by taking the Euclidean inner product of v and w interpreted as vectors in $\mathbb{R}^n$. Then (S, g) is a Riemannian

manifold, and g is called the *induced Riemannian metric* on S (this metric being induced by the Euclidean metric in $\mathbb{R}^n$).

5. (Model spaces) The model spaces of Riemannian geometry are the Euclidean space $(\mathbb{R}^n, e)$, the sphere (S^n, g) where S^n is the unit sphere in $\mathbb{R}^{n+1}$ and g is the induced Riemannian metric, and the hyperbolic space (H^n, g) which may be realized by taking H^n to be the unit ball in $\mathbb{R}^n$ with metric $g_{jk}(x) = \frac{4}{(1-|x|^2)^2}\delta_{jk}$.

The Riemannian metric allows to measure lengths and angles of tangent vectors on a manifold, the *length* of a vector $v \in T_pM$ being $|v|$ and the *angle* between two vectors $v, w \in T_pM$ being the number $\theta(v, w) \in [0, \pi]$ which satisfies

$$\cos\theta(v, w) := \frac{\langle v, w\rangle}{|v||w|}. \tag{3.4}$$

Physically, one may think of a Riemannian metric g as the resistivity of a conducting medium (the conductivity matrix (γ^{jk}) of the medium corresponds formally to $(|g|^{1/2}g^{jk})$), or as the inverse of sound speed squared in a medium where acoustic waves propagate (if a medium $U \subset \mathbb{R}^n$ has scalar sound speed $c(x)$ then a natural Riemannian metric is $g_{jk}(x) = c(x)^{-2}\delta_{jk}$). In the latter case, regions where g is large (resp. small) correspond to low velocity regions (resp. high velocity regions). We will later define geodesics, which are length minimizing curves on a Riemannian manifold, and these tend to avoid low velocity regions as one would expect.

Exercise 3.10. Use a partition of unity to prove that any smooth manifold M admits a Riemannian metric.

Isometries and conformal maps. Let (M, g) and (N, h) be Riemannian manifolds. We say that a map $F : M \to N$ is a *local isometry* from (M, g) to (N, h) if $F^*h = g$, or more precisely

$$g_p(v, w) = h_{F(p)}(F_*v, F_*w), \qquad v, w \in T_pM.$$

We say that F is an *isometry* if it is additionally a diffeomorphism from M to N. Being isometric is an equivalence relation in the class of Riemannian manifolds, and one thinks of isometric manifolds as being identical in terms of their Riemannian structure.

In a similar way, we say that a map F from (M, g) to (N, h) is *conformal* if $F^*h = cg$ for some positive function $c \in C^\infty(M)$. In this case one has

$$c(p)g_p(v,w) = h_{F(p)}(F_*v, F_*w), \qquad v, w \in T_pM.$$

We say that F is an *conformal equivalence* if it is additionally a diffeomorphism from M to N.

A map F is a local isometry iff it preserves the lengths and angles of tangent vectors in the sense that $|F_*v|_h = |v|_g$ and $\cos\,\theta_h(F_*v, F_*w) = \cos\,\theta_g(v,w)$. Similarly, a conformal map preserves the angles, but then the lengths are scaled according to $|F_*v|_h = c^{1/2}|v|_g$.

Raising and lowering of indices. On a Riemannian manifold (M,g) there is a canonical way of converting tangent vectors into cotangent vectors and vice versa. We define a map

$$T_pM \to T_p^*M, \quad v \mapsto v^\flat$$

by requiring that $v^\flat(w) = \langle v, w\rangle$. This map (called the 'flat' operator) is an isomorphism, which is given in local coordinates by

$$(v^j\partial_j)^\flat = v_j\,dx^j, \quad \text{where } v_j := g_{jk}v^k.$$

We say that $v^\flat$ is the cotangent vector obtained from v by *lowering indices.* The inverse of this map is the 'sharp' operator

$$T_p^*M \to T_pM, \quad \xi \mapsto \xi^\sharp$$

given in local coordinates by

$$(\xi_j\,dx^j)^\sharp = \xi^j\partial_j, \quad \text{where } \xi^j := g^{jk}\xi_k.$$

We say that $\xi^\sharp$ is obtained from ξ by *raising indices* with respect to the metric g.

A standard example of this construction is the *metric gradient.* If $f \in C^\infty(M)$, the metric gradient of f is the vector field

$$\text{grad}(f) := (df)^\sharp.$$

In local coordinates, $\text{grad}(f) = g^{jk}(\partial_j f)\partial_k$.

Inner products of tensors. If (M,g) is a Riemannian manifold, we can use the Riemannian metric g to define inner products of tensors in a canonical way. The inner product of cotangent vectors is defined via the sharp operator by

$$\langle \alpha, \beta\rangle := \langle \alpha^\sharp, \beta^\sharp\rangle.$$

In local coordinates one has $\langle \alpha, \beta\rangle = g^{jk}\alpha_j\beta_k$ and $g^{jk} = \langle dx^j, dx^k\rangle$.

More generally, if u and v are k-tensors with local coordinate representations $u = u_{i_1 \cdots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}$, $v = v_{i_1 \cdots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}$, we define

$$\langle u, v \rangle := g^{i_1 j_1} \cdots g^{i_k j_k} u_{i_1 \cdots i_k} v_{j_1 \cdots j_k}. \tag{3.5}$$

This definition turns out to be independent of the choice of coordinates, and it gives a valid inner product on k-tensors. This inner product is natural in the sense that for any diffeomorphism F onto M,

$$F^*(\langle u, v \rangle_g) = \langle F^* u, F^* v \rangle_{F^* g}.$$

Orthonormal frames. If U is an open subset of M, we say that a set $\{E_1, \ldots, E_n\}$ of vector fields in U is a *local orthonormal frame* if $\{E_1(q), \ldots, E_n(q)\}$ forms an orthonormal basis of $T_q M$ for any $q \in U$.

Lemma 3.8. *(Local orthonormal frame) If (M, g) is a Riemannian manifold, then for any point $p \in M$ there is a local orthonormal frame in some neighborhood of p.*

If $\{E_j\}$ is a local orthonormal frame, the dual frame $\{\varepsilon^j\}$ which is characterized by $\varepsilon^j(E_k) = \delta_{jk}$ gives an orthonormal basis of $T_q^* M$ for any q near p. The inner product in (3.5) is the unique inner product on k-tensor fields such that $\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}\}$ gives an orthonormal basis of $T^k(T_q M)$ for q near p whenever $\{\varepsilon^j\}$ is a local orthonormal frame of 1-forms near p.

Exercise 3.11. Prove the lemma by applying the Gram-Schmidt orthonormalization procedure to a basis $\{\partial_j\}$ of coordinate vector fields, and prove the statements after the lemma.

Volume form, integration, and Sobolev spaces. From this point on, *all Riemannian manifolds will be assumed to be oriented* in order for the volume form to be defined. Clearly near any point p in (M, g) there is a positive local orthonormal frame (that is, a local orthonormal frame $\{E_j\}$ which gives a positive orthonormal basis of $T_q M$ for q near p).

Lemma 3.9. *(Volume form) Let (M, g) be a Riemannian manifold. There is a unique n-form on M, denoted by dV_g and called the* volume form*, such that $dV_g(E_1, \ldots, E_n) = 1$ for any positive local orthonormal frame $\{E_j\}$. In local coordinates*

$$dV_g = |g|^{1/2}\, dx^1 \wedge \ldots \wedge dx^n.$$

The volume form is natural in the sense that $F^*(dV_g) = dV_{F^*g}$ *for any orientation preserving diffeomorphism* F.

Exercise 3.12. Prove this lemma.

If f is a function on (M, g), we can use the volume form to obtain an n-form $f\,dV$. The integral of f over M is then defined to be the integral of the n-form $f\,dV$. Thus, on a Riemannian manifold there is a canonical way to integrate functions (instead of just n-forms).

If $u, v \in C^\infty(M)$ are real valued functions, we define the L^p norm for $1 < p < \infty$ and L^2 inner product by

$$\|u\|_{L^p} := \left(\int_M u^p\,dV\right)^{1/p},$$
$$(u, v)_{L^2} := \int_M uv\,dV.$$

The completion of $C^\infty(M)$ with respect to the L^p norm is a Banach space denoted by $L^p(M)$ or $L^p(M, dV)$. It consists of L^p-integrable functions defined almost everywhere on M with respect to the measure dV. The space $L^2(M)$ becomes a Hilbert space.

We now wish to define Sobolev spaces $W^{k,p}(M)$. This is possible on any oriented smooth manifold; we will assume compactness to avoid conditions at infinity.

Remark. One could ask whether Sobolev or even L^p spaces can be defined without assuming an orientation. If M is not oriented, there is an intrinsic $L^p_{\mathrm{loc}}(M)$ space but its elements are not functions but rather $1/p$-densities. However, if M is orientable and if one fixes an orientation, then the elements of the intrinsic L^p_{loc} space can be identified with functions on M.

Definition. Let M be a compact oriented smooth manifold, let $k \geq 0$ and $1 \leq p \leq \infty$. Let also $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ be an open cover of M by positive coordinate charts, and let (χ_α) be a subordinate partition of unity. We define the norm

$$\|u\|_{\tilde{W}^{k,p}} := \sum_{\alpha \in A} \|(\varphi_\alpha)_*(\chi_\alpha u)\|_{W^{k,p}(\mathbb{R}^n)}, \quad u \in C^\infty(M).$$

The space $W^{k,p}(M)$ is the completion of $C^\infty(M)$ under this norm. If $p = 2$ we write $H^k(M) := W^{k,2}(M)$.

EXERCISE 3.13. Show that $W^{k,p}(M)$ is a Banach space, and this space and its topology are independent of the choice of charts and of the partition of unity. (The crucial point is that $W^{k,p}$ spaces on open subsets of $\mathbb{R}^n$ behave well under changes of coordinates.)

EXERCISE 3.14. Show that $H^k(M)$ is a Hilbert space.

EXERCISE 3.15. If M is a compact oriented manifold and g is a Riemannian metric on M, show that $W^{0,p}(M) = L^p(M, dV_g)$.

Let now (M, g) be a compact oriented Riemannian manifold. We may define L^p spaces of k-forms or k-tensor fields, denoted by $L^p(M, \Lambda^k M)$ or $L^p(M, T^k M)$, by using the norm

$$\|u\|_{L^p} := \left(\int_M \langle u, u \rangle^{p/2} \, dV \right)^{1/p}.$$

If $p = 2$, we define the L^2 inner product of tensor fields

$$(u, v)_{L^2} := \int_M \langle u, v \rangle \, dV, \quad u, v \in L^2(M, T^k M).$$

Sobolev spaces $W^{k,p}(M, T^l M)$ and $W^{k,p}(M, \Lambda^l M)$ of l-tensor fields or l-forms can be defined via the norm

$$\|u\|_{\tilde{W}^{k,p}(M,T^l M)} := \sum_{\alpha \in A} \|(\varphi_\alpha)_*(\chi_\alpha u)\|_{W^{k,p}(\mathbb{R}^n, T^l \mathbb{R}^n)}$$

where $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is an open cover of M by positive coordinate charts, and (χ_α) is a subordinate partition of unity. If $p = 2$ we write $H^k = W^{k,2}$ as before.

Finally we observe that the $W^{1,p}(M)$ spaces of scalar functions on an oriented Riemannian manifold can be defined in an invariant way.

EXERCISE 3.16. Let (M, g) be a compact oriented Riemannian manifold. Show that

$$\|u\|_{W^{1,p}(M)} := (\|u\|^p_{L^p(M,dV_g)} + \|du\|^p_{L^p(M,dV_g)})^{1/p}$$

gives an equivalent norm on $W^{1,p}(M)$ $(p < \infty)$. Show also that the Hilbert structure of $H^1(M)$ is given by the inner product

$$(u, v)_{H^1} := (u, v)_{L^2} + (du, dv)_{L^2}.$$

REMARK. Also the $W^{k,p}(M)$ spaces can be defined invariantly in terms of the Riemannian metric, via the norm

$$\|u\|_{W^{k,p}(M)} := \left(\sum_{j=0}^{k} \|\nabla_g^j u\|_{L^p(M,T^jM)}^p \right)^{1/p}$$

where ∇_g is the total covariant derivative induced by the Levi-Civita connection on (M, g).

Codifferential. Using the inner product on k-forms, we can define the codifferential operator δ as the adjoint of the exterior derivative via the relation

$$(\delta u, v) = (u, dv)$$

where $u \in C^\infty(M, \Lambda^k)$ and $v \in C^\infty(M, \Lambda^{k-1})$ (with $v|_{\partial M} = 0$ if M has nonempty boundary). Applying Theorem 3.10 in coordinate neighborhoods covering M and using a partition of unity, we obtain the following:

THEOREM 3.10. *(Codifferential) Let (M, g) be an n-dimensional Riemannian manifold with or without boundary. For each k with $0 \leq k \leq n$, there is a unique linear operator*

$$\delta : \Omega^k(M) \to \Omega^{k-1}(M)$$

having the property

$$(du, v)_{L^2} = (u, \delta v)_{L^2}, \qquad u \in \Omega^{k-1}(M), \quad v \in \Omega^k(M), \tag{3.6}$$

where additionally $v|_{\partial M} = 0$ if M has nonempty boundary. The operator δ satisfies $\delta \circ \delta = 0$ and $\delta|_{\Omega^0(M)} = 0$. In any local coordinates (U, x) it is a linear first order differential operator acting on component functions, and on a 1-form $\beta = \beta_j \, dx^j$ it is given by

$$\delta\beta := -|g|^{-1/2}\partial_j(|g|^{1/2} g^{jk} \beta_k), \quad \beta = \beta_k \, dx^k \in \Omega^1(U). \tag{3.7}$$

It follows that $\delta\alpha$ is related to the divergence of vector fields by $\delta\alpha = -\mathrm{div}_g(\alpha^\sharp)$, where the divergence is defined in local coordinates by

$$\mathrm{div}_g(X) := |g|^{-1/2}\partial_j(|g|^{1/2} X^j).$$

EXERCISE 3.17. If X is a vector field with $\mathrm{div}_g(X) = 0$, show that for any $u, v \in C_c^\infty(M^{\mathrm{int}})$ one has

$$\int_M (Xu)v \, dV_g = -\int_M u(Xv) \, dV_g.$$

(If you wish, it is enough to do this exercise when M is an open subset of $\mathbb{R}^n$. Recall that $(Xu)(x) = X^j(x)\partial_j u(x)$ in this case.)

Laplace-Beltrami operator. On any Riemannian manifold there is a canonical second order elliptic operator, called the Laplace-Beltrami operator, which is an analogue of the usual Laplacian in $\mathbb{R}^n$. As in Section 2.7, we can start from the Dirichlet energy functional

$$E(v) = \frac{1}{2}\int_M |dv|^2 \, dV, \quad v \in H^1(M).$$

Since $E(v) = \frac{1}{2}(dv, dv)_{L^2}$, the same argument as in Section 2.7 shows that any minimizer u of the Dirichlet energy satisfies the equation

$$\delta du = 0.$$

We have arrived at the definition of the Laplace-Beltrami operator.

Definition. If (M, g) is a compact Riemannian manifold (with or without boundary), the Laplace-Beltrami operator is defined by

$$\Delta_g u := -\delta du.$$

The next result is clear from Section 2.7:

Lemma 3.11. *In local coordinates*

$$\Delta_g u = |g|^{-1/2}\partial_j(|g|^{1/2} g^{jk}\partial_k u)$$

where, as before, $|g| = \det(g_{jk})$ *is the determinant of* g.

Exercise 3.18. Show that $\Delta_g u = \mathrm{div}_g(\mathrm{grad}_g(u))$.

Exercise 3.19. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, and consider the product manifold (N, h) where $N = M_1 \times M_2$ and $h = g_1 \oplus g_2$ is the product metric that satisfies

$$h((v_1, v_2), (w_1, w_2)) = g_1(v_1, w_1) + g_2(v_2, w_2)$$

for $v_1, w_1 \in T_{x_1}M_1$ and $v_2, w_2 \in T_{x_2}M_2$. Show that h is indeed a Riemannian metric in N.

Exercise 3.20. Let (N, h) be as in the previous exercise. Show that if x_1 are local coordinates in M_1 and x_2 are local coordinates in M_2, then in the (x_1, x_2) coordinates the metric h looks like

$$h(x_1, x_2) = \begin{pmatrix} (g_{jk}(x_1)) & 0 \\ 0 & (h_{pq}(x_2)) \end{pmatrix}.$$

EXERCISE 3.21. Let (N, h) be as in the previous exercises. If $u = u(x_1, x_2) \in C^\infty(N)$, show that $\Delta_h u = \Delta_{g_1} u + \Delta_{g_2} u$ in the sense that

$$(\Delta_h u)(x_1, x_2) = \Delta_{g_1}(u(\,\cdot\,, x_2))(x_1) + \Delta_{g_2}(u(x_1, \,\cdot\,))(x_2).$$

(If you wish, it is enough to do this exercise when M_j are replaced by open sets in $\mathbb{R}^n$ and h has the matrix form in the previous exercise.)

3.3. Solutions of the Laplace equation

We now discuss properties of solutions of the Laplace-Beltrami equation $-\Delta_g u = F$ in M, where (M, g) is a compact connected oriented Riemannian manifold. The results will be different for manifolds with or without boundary, and they will involve the following Sobolev spaces.

DEFINITION. If (M, g) is a closed manifold (i.e. compact without boundary), we define

$$H^{-1}(M) := \{\text{continuous linear functionals on } H^1(M)\}.$$

If (M, g) is a compact manifold with smooth boundary, we also define

$$H_0^1(M) := \text{closure of } C_c^\infty(M^{\mathrm{int}}) \text{ in } H^1(M),$$
$$H^{-1}(M) := \{\text{continuous linear functionals on } H_0^1(M)\}.$$

Manifolds with boundary. Let (M, g) be a compact connected oriented Riemannian manifold with nonempty smooth boundary. We consider the Dirichlet problem for the Laplacian in (M, g). The results will be completely analogous to the case of bounded domains in $\mathbb{R}^n$ with smooth boundary discussed in Section 2.8.

THEOREM 3.12 (Dirichlet problem). *Given any $F \in H^{-1}(M)$ and $f \in H^1(M)$, there is a unique weak solution $u \in H^1(M)$ of the problem*

$$\left\{ \begin{array}{rll} -\Delta_g u = F & & \text{in } M, \\ u = f & & \text{on } \partial M. \end{array} \right. \tag{3.8}$$

Here we say that $u \in H^1(M)$ is a weak solution *of* (3.8) *iff*

$$(du, dv)_{L^2} = F(v) \text{ for all } v \in H_0^1(M), \qquad u - f \in H_0^1(M).$$

One has the norm estimate

$$\|u\|_{H^1(M)} \leq C(\|F\|_{H^{-1}(M)} + \|f\|_{H^1(M)})$$

where C is independent of F and f.

THEOREM 3.13 (Higher regularity). *Assume the conditions in Theorem 2.18. If $F \in H^k(M)$ and $f \in H^{k+2}(M)$ for some $k \geq 0$, then $u \in H^{k+2}(M)$ and*

$$\|u\|_{H^{k+2}(M)} \leq C(\|F\|_{H^k(M)} + \|f\|_{H^{k+2}(M)})$$

where C is independent of F and f.

Finally, we state the weak maximum principle for solutions in $H^1(M)$. Here we say that $u \leq C$ in M (resp. $u \leq C$ on ∂M) if $(u - C)_+ = 0$ (resp. $(u - C)_+ \in H_0^1(M)$), where

$$u_+ := \max\{u, 0\}.$$

We also say that $u \geq c$ in M (resp. $u \geq c$ on ∂M) if $-u \leq -c$ in M (resp. $-u \leq -c$ on ∂M).

THEOREM 3.14 (Weak maximum principle). *Let $u \in H^1(M)$ solve*

$$-\Delta_g u = 0 \text{ in } M.$$

If $u \leq C$ on ∂M, then $u \leq C$ in M. Similarly, if $u \geq c$ on ∂M, then $u \geq c$ in M.

The proof of Theorem 3.12 is the same as that of Theorem 2.18, except that we need to use a version of the Poincaré inequality that is valid on compact manifolds with boundary.

THEOREM 3.15 (Poincaré inequality). *Let (M, g) be compact with smooth boundary. There is $C > 0$ so that*

$$\|u\|_{L^2(M)} \leq C\|du\|_{L^2(M)}, \qquad u \in H_0^1(M).$$

There are different ways of proving this inequality. In the case where $\Omega \subset \mathbb{R}^n$ is bounded, we used the fact that $\Omega \subset \{a < x_n < b\}$ for some a, b and that any $u \in C_c^\infty(\Omega)$ satisfies

$$u(x', x_n) = u(x', x_n) - u(x', a) = \int_a^{x_n} \partial_n u(x', t)\, dt.$$

The Cauchy-Schwarz inequality, together with the fact that integration over $\{a < x_n < b\}$ can be divided into integration over $x' \in \mathbb{R}^{n-1}$ and $x_n \in (a, b)$, then implied the result.

One can follow a similar strategy for a compact manifold with boundary, if one considers unit speed geodesics $\gamma_z(t)$ in (M, g) starting at points $z \in \partial M$ and going in the inward normal direction (then the set ∂M replaces the set $\{x_n = a\}$ above). See [**KKL**, Section 2.1.16] for the following facts on normal geodesics. One can define the boundary exponential map

$$\exp_{\partial M} : \{(z,t) \,:\, z \in \partial M \text{ and } t \in [0, \tau_{\partial M}(z))\} \to M, \quad (z,t) \mapsto \gamma_z(t)$$

where $\tau_{\partial M}(z)$ is the unique time for which $\gamma_z([0,t])$ is the shortest curve between ∂M and $\gamma_z(t)$ for $t < \tau_{\partial M}(z)$, but it is no longer the shortest when $t > \tau_{\partial M}(z)$. Then the map $\exp_{\partial M}$ is a diffemorphism onto $M \setminus \omega$, where ω is the *boundary cut locus*

$$\omega = \{\gamma_z(\tau_{\partial M}(z)) \,:\, z \in \partial M\}.$$

The set ω has zero measure, and hence integrals of smooth functions over M may be evaluated by using the (z,t) coordinates. This leads to a proof of Theorem 3.15.

EXERCISE 3.22. Let B be the unit ball in $\mathbb{R}^n$. Use polar coordinates and the approach outlined above to prove the Poincaré inequality

$$\int_B |u|^2 \, dx \leq C \int_B |\nabla u|^2 \, dx$$

for any $u \in C^1(\overline{B})$ with $u|_{\partial B} = 0$.

Another proof can be given based on a version of the compact Sobolev embedding theorem (also known as the Rellich-Kondrachov theorem). This is an extremely important result in its own right. For a proof see [**Ta**, Sections 4.3 and 4.4].

THEOREM 3.16. *Let (M,g) be a compact Riemannian manifold with or without boundary. Then the natural inclusion map $i : H^1(M) \to L^2(M)$ is a compact linear operator.*

PROOF OF THEOREM 3.15. We argue by contradiction and assume that the result is not true. Then for any $k \geq 1$ there is $u_k \in H_0^1(M)$ satisfying

$$\|u_k\|_{L^2(M)} > k \|du_k\|_{L^2(M)}.$$

After replacing u_k by $u_k/\|u_k\|_{L^2}$, we may assume that $\|u_k\|_{L^2} = 1$. Then it follows that

$$\|du_k\|_{L^2(M)} < \frac{1}{k}.$$

In particular, (u_k) is a bounded sequence in $H_0^1(M)$. By Theorem 3.16 it has a subsequence, still denoted by (u_k), which converges in $L^2(M)$

to some $u \in L^2(M)$. Then we have

$$\|u_k\|_{L^2} = 1, \qquad \|du_k\|_{L^2} < \frac{1}{k}.$$

Taking $k \to \infty$, we see that (u_k) converges in $H_0^1(M)$ to a function with $\|u\|_{L^2} = 1$ and $du = 0$ in M. Thus u is constant in the connected set M, and one must have $u = 0$ since $u \in H_0^1(M)$. This is a contradiction with $\|u\|_{L^2} = 1$. □

Given the Poincaré inequality, the proof of Theorem 3.15 proceeds by using the same Hilbert space argument as in the case of domains in $\mathbb{R}^n$. The higher regularity result, Theorem 3.13, is of a local nature and reduces to the corresponding result in $\mathbb{R}^n$. The proof of the maximum principle in Theorem 3.14 is also essentially the same as that in $\mathbb{R}^n$.

Manifolds without boundary. Let (M, g) now be a closed manifold, i.e. a compact connected oriented Riemannian manifold with no boundary. We consider solutions of

$$-\Delta_g u = F \text{ in } M.$$

Now there are no boundary conditions, and there are two immediate differences to the boundary case.

- Solutions are not unique: if u is a solution, then $u + C$ is a solution for any constant C.
- Solutions do not exist for all F: indeed, if u is a smooth enough solution, then necessarily

$$(F, 1)_{L^2} = (\delta du, 1)_{L^2} = (du, d(1))_{L^2} = 0.$$

One also observes that the only global harmonic functions in a closed manifold are the constants:

Lemma 3.17. *If (M, g) is a closed manifold and $u \in C^2(M)$ satisfies $\Delta_g u = 0$ in M, then $u = $ const.*

Proof. One has $0 = (-\Delta_g u, u)_{L^2} = (\delta du, u)_{L^2} = (du, du)_{L^2}$, showing that $du = 0$ and hence $u = $ const. □

It turns out that these are the only obstructions for having unique solutions to the Laplace equation in a closed manifold. Below we write $(u)_M$ for the average

$$(u)_M := \frac{1}{\int_M dV_g} \int_M u \, dV_g.$$

THEOREM 3.18 (Dirichlet problem). *Given any $F \in H^{-1}(M)$ satisfying $F(1) = 0$, there is a weak solution $u \in H^1(M)$, unique up to adding a constant, of the problem*

$$-\Delta_g u = F \quad \text{in } M. \tag{3.9}$$

Here we say that $u \in H^1(M)$ is a weak solution *of (3.9) iff*

$$(du, dv)_{L^2} = F(v) \text{ for all } v \in H^1(M).$$

One has the norm estimate

$$\|u - (u)_M\|_{H^1(M)} \leq C\|F\|_{H^{-1}(M)}$$

where C is independent of F.

THEOREM 3.19 (Higher regularity). *Assume the conditions in Theorem 3.18. If $F \in H^k(M)$ for some $k \geq 0$, then $u \in H^{k+2}(M)$ and*

$$\|u - (u)_M\|_{H^{k+2}(M)} \leq C\|F\|_{H^k(M)}$$

where C is independent of F.

If (M, g) is a closed manifold, the maximum principle in M does not make sense (since there is no boundary, and the only global harmonic functions are constants). However, the maximum principle in Theorem 3.14 can still be applied to solutions of $\Delta_g u = 0$ in U whenever U is any open set in M such that $\overline{U}$ has smooth boundary.

As in the case of manifolds with boundary, the only real difference in the proofs of the above results as compared to the proofs in $\mathbb{R}^n$ is the fact that one needs to use a different Poincaré inequality. This inequality now applies to any function $u \in H^1(M)$, but one has $u-(u)_M$ on the left hand side.

THEOREM 3.20 (Poincaré inequality). *Let (M, g) be a closed manifold. There is $C > 0$ so that*

$$\|u - (u)_M\|_{L^2(M)} \leq C\|du\|_{L^2(M)}, \qquad u \in H^1(M).$$

This theorem can be proved as in the boundary case, by using a contradiction argument based on the compact Sobolev embedding.

EXERCISE 3.23. Give a proof of Theorem 3.20.

The proof of Theorem 3.18 is then completed by the standard Hilbert space argument but now considering solutions in the space

$$H^1_\diamond(M) := \{u \in H^1(M) : (u)_M = 0\}.$$

The higher regularity result, Theorem 3.19, follows again from the corresponding Euclidean result.

EXERCISE 3.24. Let (M, g) be a closed manifold. We say that a number $\lambda \in \mathbb{R}$ is an *eigenvalue* of $-\Delta_g$ if there is $\varphi \in H^1(M)$, $\varphi \not\equiv 0$, satisfying $-\Delta_g \varphi = \lambda\varphi$. The function φ is then called an *eigenfunction* corresponding to eigenvalue λ.

(a) Show that 0 is always an eigenvalue.
(b) Show that any other eigenvalue besides 0 is positive.
(c) Show that any eigenfunction is in $C^\infty(M)$.
(d) Show that if φ and ψ are eigenfunctions corresponding to two distinct eigenvalues, then $(\varphi, \psi)_{L^2(M)} = 0$.

CHAPTER 4

Riemann surfaces

In this section we discuss the fact that on an oriented 2-manifold M, a Riemannian metric g induces a complex structure and thus (M, g) becomes a *Riemann surface* (that is, a complex manifold with complex dimension one). We will also discuss the fundamental *uniformization theorem* which classifies the possible geometries, or conformal structures, on a given Riemann surface.

4.1. Generalities

We begin with some generalities.

Definition (Complex manifold). An N*-dimensional complex manifold* is a $2N$-dimensional smooth (real) manifold with an open cover U_α and charts $\varphi_\alpha : U_\alpha \to \mathbb{C}^N$ such that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic $\varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{C}^N$. The charts φ_α are called *complex* or *holomorphic coordinates*. The atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is called a *complex atlas*. Two complex atlases are called *equivalent* if their union is a complex atlas. A *complex structure* is an equivalence class of complex atlases.

Definition (Surface). A one-dimensional complex manifold is called a *Riemann surface* (or just *surface*).

Definition (Almost complex structure). If M is a differentiable manifold, an *almost complex structure* on M is a $(1, 1)$ tensor field J such that the restriction $J_p : T_pM \to T_pM$ satisfies $J_p^2 = -\mathrm{Id}$ for any p in M. If g is a Riemannian metric on M, we say that J is *compatible* with g if $g(Jv, Jw) = g(v, w)$ for all $v, w \in T_pM$.

If M is a complex manifold, let $z = (z_1, \ldots, z_N)$ be a holomorphic chart $U_\alpha \to \mathbb{C}^N$, and write $z_j = x_j + iy_j$ with x_j and y_j real. There is a canonical almost complex structure J on M, defined for holomorphic charts by

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

Conversely, if M is a differentiable manifold equipped with an almost complex structure J (so it is necessarily even dimensional and orientable), then by the Newlander-Nirenberg theorem M has the structure of a complex manifold, having J as its canonical almost complex structure, if J satisfies an additional integrability condition.

Definition (Holomorphic functions). If M is a complex manifold with complex charts $\varphi_\alpha : U_\alpha \to \mathbb{C}^N$, a C^1 function $f : M \to \mathbb{C}$ is called *holomorphic* (resp. *antiholomorphic*) if $f \circ \varphi_\alpha^{-1}$ is holomorphic (resp. antiholomorphic) from $\varphi_\alpha(U_\alpha) \subset \mathbb{C}^N$ to $\mathbb{C}$ for any α.

It is clear that all local properties of holomorphic functions in domains of $\mathbb{C}^N$ are valid also for holomorphic functions on complex manifolds.

4.2. Isothermal coordinates

Let now (M, g) be a two-dimensional oriented (real) manifold with Riemannian metric g. In this case everything becomes very simple. In particular, the almost complex structures correspond to rotation by $90°$.

Definition (Rotation by $90°$). For any $v \in T_xM$, let $v^\perp \in T_xM$ be the unique vector (the rotation of v by $90°$ counterclockwise) such that

$$|v^\perp|_g = |v|_g, \qquad \langle v, v^\perp \rangle = 0,$$

and $(v, v^\perp)$ is a positively oriented basis of T_xM when $v \neq 0$.

Exercise 4.1. Show that in local coordinates, if $g(x) = (g_{jk}(x))$ and $v = (v_1, v_2)^t$, the vector $v^\perp$ is given by

$$v^\perp = g(x)^{-1/2}(-(g(x)^{1/2}v)_2, (g(x)^{1/2}v)_1)^t$$

where ^{t} denotes transpose and $A^{1/2}$ is the square root of a positive definite matrix A.

Lemma 4.1 (Almost complex structures). *If (M, g) is an oriented two-dimensional manifold, then J is an almost complex structure compatible with g iff*

$$J(v) = \pm v^\perp, \qquad v \in TM.$$

PROOF. Let J be an almost complex structure compatible with g. Given $p \in M$ and $v \in T_pM$, the fact that J is compatible with g implies that $|Jv| = |v|$. Moreover, one has

$$\langle Jv, v\rangle = -\langle Jv, J^2 v\rangle = -\langle v, Jv\rangle$$

which implies that $\langle Jv, v\rangle = 0$. Thus Jv is orthogonal to v and has the same length as v. Since T_pM is two-dimensional, one must have $Jv = \pm v^\perp$. Conversely, $Jv = \pm v^\perp$ clearly satisfies $J^2 = -\mathrm{Id}$ and $\langle Jv, Jw\rangle = \langle v, w\rangle$. $\square$

We wish to find a complex structure on M associated with $J(v) = v^\perp$. The following fundamental result, proved by Gauss in 1822 in the real-analytic case, will yield complex local coordinates that are compatible with J. The uniformization theorem that will be proved later can be viewed as a *global* version of this result.

THEOREM 4.2 (Isothermal coordinates). *Let (M, g) be an oriented two-dimensional manifold. Near any point of M there are positively oriented local coordinates $x = (x_1, x_2)$, called* isothermal coordinates*, so that the metric has the form*

$$g_{jk}(x) = e^{2\lambda(x)}\delta_{jk}$$

where λ is a smooth real-valued function.

Given the existence of isothermal coordinates, it is easy to show that any 2D Riemannian manifold has a complex structure. The proof uses the basic complex analysis fact proved in the next exercise that a smooth bijective map φ between open subsets of $\mathbb{R}^2$ is holomorphic iff it is conformal and orientation preserving. Recall that φ being conformal means that

$$\varphi^* h = ch$$

for some smooth positive function c where h is the Euclidean metric on $\mathbb{R}^2$.

EXERCISE 4.2. Let $f : \Omega \to \mathbb{R}^2$ be a smooth map, where $\Omega \subset \mathbb{R}^2$ is open, and let $Df(z) = (\partial_k f_j(z))_{j,k=1}^2$ be the Jacobi matrix of f.

(a) Show that f is conformal, i.e. $f^*h = ch$ where h is the Euclidean metric, if and only if $(Df(z))^t Df(z) = c(z)\mathrm{Id}$.
(b) Show that f is holomorphic in Ω if and only if

$$(Df(z))^t Df(z) = c(z)\mathrm{Id}, \quad \det Df(z) > 0 \text{ in } \Omega,$$

for some smooth positive function $c(z)$ in Ω. (Hint: identify vectors in $\mathbb{R}^2$ with complex numbers and use complex notation to write $v \cdot w = \mathrm{Re}(v\bar{w})$, $Df(z)v = (\partial f)v + (\overline{\partial} f)\bar{v}$, $\det Df = |\partial f|^2 - |\overline{\partial} f|^2$, where $\overline{\partial} f = \frac{1}{2}(\partial_1 f + i\partial_2 f)$ and $\partial f = \frac{1}{2}(\partial_1 f - i\partial_2 f)$. Also use that f satisfies the Cauchy-Riemann equations if and only if $\overline{\partial} f = 0$.)

Theorem 4.3 (Complex structure induced by g). *Let (M, g) be an oriented 2D manifold, and let (U_α) be an open cover of M so that there are isothermal coordinate charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$. Then $\varphi_\beta^{-1} \circ \varphi_\alpha$ is holomorphic $\varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^2$ whenever $U_\alpha \cap U_\beta \neq \emptyset$. Thus the charts $(U_\alpha, \varphi_\alpha)$ induce a complex structure on M corresponding to $J(v) = v^\perp$. This complex structure is independent of the choice of the isothermal coordinate charts, and hence it is uniquely determined by g.*

Proof. The fact that $g_{jk}(x) = e^{2\lambda(x)}\delta_{jk}$ in isothermal coordinates can be rewritten as

$$(\varphi_\alpha^{-1})^* g = e^{2\lambda_\alpha} h$$

where h is the Euclidean metric in $\mathbb{R}^2$. Suppose that $U_\alpha \cap U_\beta \neq \emptyset$ and let $\Phi = \varphi_\beta \circ \varphi_\alpha^{-1}$. Then Φ is a smooth map from an open set of $\mathbb{R}^2$ to $\mathbb{R}^2$, and one has

$$\Phi^* h = (\varphi_\alpha^{-1})^* \varphi_\beta^* h = (\varphi_\alpha^{-1})^* (e^{-2\varphi_\beta^* \lambda_\beta} g) = e^{2(\lambda_\alpha - \Phi^* \lambda_\beta)} h.$$

Since h is the Euclidean metric, the identity $\Phi^* h = ch$, where $c = e^{2(\lambda_\alpha - \Phi^* \lambda_\beta)}$ is a positive smooth function, means that Φ is a conformal bijective map between open sets in $\mathbb{R}^2$. Since isothermal coordinate charts are positively oriented, Φ is orientation preserving. Thus Φ must be holomorphic. This proves that any atlas consisting of isothermal coordinate charts is a complex atlas. It is also clear from this argument that if one uses different isothermal coordinate charts, then one obtains an equivalent atlas.

It remains to show that the almost complex structure J given by isothermal coordinates satisfies $J(v) = v^\perp$. But in isothermal coordinates $J(\partial_{x_1}) = \partial_{x_2} = (\partial_{x_1})^\perp$ and $J(\partial_{x_2}) = -\partial_{x_1} = (\partial_{x_2})^\perp$, so one must have $J(v) = v^\perp$. $\square$

If (M, g) is a two-dimensional oriented Riemannian manifold, we will always use the complex structure induced by g on M. In fact the complex structure only depends on the conformal class

$$[g] = \{cg \,;\, c \in C^\infty(M) \text{ positive}\},$$

and conversely any complex structure on M arises from some conformal class.

THEOREM 4.4 (Complex structures vs conformal classes). *Let M be an oriented two-dimensional manifold. There is a $1-1$ correspondence between conformal classes of Riemannian metrics on M and complex structures on M.*

PROOF. Isothermal coordinates for a metric g are also isothermal for cg: if $(\varphi^{-1})^* g = e^{2\lambda} h$ with h the Euclidean metric, then $(\varphi^{-1})^*(cg) = e^{2\mu} h$ for $\mu = \lambda + \frac{1}{2}\log((\varphi^{-1})^* c)$. Thus the complex structure on M obtained in Theorem 4.3 is the same for g and cg.

Conversely, suppose that M is equipped with a complex structure. We wish to produce a metric g which induces this structure. Such a metric can be defined locally: if $p \in M$ and if (U, φ) is a complex coordinate chart near p, we can define $g = \varphi^* h$ in U where h is the Euclidean metric in $\varphi(U) \subset \mathbb{R}^2$. More generally, if M is covered by complex coordinate charts $(U_\alpha, \varphi_\alpha)$ and if (χ_α) is a locally finite partition of unity subordinate to the cover (U_α), we can define

$$g = \sum \chi_\alpha \varphi_\alpha^* h.$$

Then g is a Riemannian metric on M. The complex coordinate charts $(U_\alpha, \varphi_\alpha)$ above are isothermal for g, since

$$(\varphi_\alpha^{-1})^* g = \sum_\beta ((\varphi_\alpha^{-1})^* \chi_\beta)(\varphi_\beta \circ \varphi_\alpha^{-1})^* h = \sum_\beta ((\varphi_\alpha^{-1})^* \chi_\beta) c_{\alpha\beta} h = ch$$

for some positive smooth functions $c_{\alpha\beta}$ and c. Here we used that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic, hence conformal, and thus satisfies $(\varphi_\beta \circ \varphi_\alpha^{-1})^* h = c_{\alpha\beta} h$. This shows that the complex structure on M induced by g is the same as the original one. □

It remains to prove Theorem 4.2. It is convenient to consider rotations on T^*M instead of TM.

DEFINITION (Hodge star). For any $\xi \in T_x^* M$ let $*\xi \in T_x^* M$ be the rotation of ξ by $90°$ counterclockwise, i.e.

$$*\xi := ((\xi^\sharp)^\perp)^\flat,$$

where $\sharp, \flat$ are the musical isomorphisms associated with g.

EXERCISE 4.3. If $x = (x^1, x^2)$ are local coordinates and $\xi = \xi_1 \, dx^1 + \xi_2 \, dx^2$, show that $*\xi = \eta_1 \, dx^1 + \eta_2 \, dx^2$ where

$$\vec{\eta} = g^{1/2}(-(g^{-1/2}\vec{\xi})_2, (g^{-1/2}\vec{\xi})_1)$$

with the notation $\vec{\xi} = (\xi_1, \xi_2)^t$, $\vec{\eta} = (\eta_1, \eta_2)^t$, $g = (g_{jk})$, and where $A^{1/2}$ denotes the square root of a positive definite symmetric matrix A.

Clearly $*\xi$ is the unique covector so that $|*\xi|_g = |\xi|_g$, $\langle \xi, *\xi \rangle = 0$, and $(\xi, *\xi)$ is a positively oriented basis of $T_x^* M$ when $\xi \neq 0$. The operator $*$ is just the Hodge star operator specialized to 1-forms on a two-dimensional manifold. We can identify the almost complex structure $J(v) = v^\perp$ with the operator $*$.

PROOF OF THEOREM 4.2. Let $p \in M$. We wish to show that there are smooth functions u and v near p so that

$$|du|_g = |dv|_g > 0, \qquad \langle du, dv \rangle = 0 \qquad \text{near } p. \tag{4.1}$$

Since du and dv are linearly independent at p, the inverse function theorem shows that choosing $x_1 = u$, $x_2 = v$ and $\lambda = -\log |du|_g$ yields the required coordinate system near p.

The equations (4.1) state that du and dv should be orthogonal and have the same (positive) length. Since M is two-dimensional, it follows that dv must be the rotation of du by 90° (either clockwise or counterclockwise). Thus, given u with $du|_p \neq 0$, it would be enough to find v such that

$$dv = *du \tag{4.2}$$

where $*$ is the Hodge star operator in Definition 4.2.

Now if the metric were Euclidean, the equations (4.2) would read

$$\partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v.$$

These are exactly the Cauchy-Riemann equations for an analytic function $f = u + iv$ in the complex plane. In particular, u and v would necessarily be harmonic. The same is true in the general case: by Exercise 4.5 below, on a two-dimensional oriented manifold one has

$$d * du = (\Delta_g u) \, dV_g.$$

Since $d^2 = 0$, it follows from (4.2) that u and v have to be harmonic.

We use Lemma 4.6 below which shows that there is a harmonic function u near p with $du|_p \neq 0$. Then $*du$ is a closed 1-form (since

$d(*du) = *\Delta_g u = 0$), and the Poincaré lemma shows that in any small ball near p one can find a smooth function v satisfying (4.2). Since $du|_p \neq 0$, one has (4.1) in some neighborhood of p which proves the theorem. □

EXERCISE 4.4. Prove the formula $d * du = (\Delta_g u)\, dV_g$ used in the proof of Theorem 4.2 in the case where $M \subset \mathbb{R}^n$ and g is the Euclidean metric.

EXERCISE 4.5. Prove the formula $d * du = (\Delta_g u)\, dV_g$ on a general Riemannian manifold (M, g).

We formulate part of the above proof as a lemma:

LEMMA 4.5 (Harmonic conjugate). *Let (M, g) be a simply connected oriented 2-manifold. Given any $u \in C^\infty(M)$ satisfying $\Delta_g u = 0$ in M, there is $v \in C^\infty(M)$ satisfying*

$$dv = *du \text{ in } M.$$

The function v, called a harmonic conjugate *of u, is harmonic and unique up to an additive constant. The function $f = u + iv$ is holomorphic in the complex structure induced by g. Conversely, the real and imaginary parts of any holomorphic function are harmonic.*

LEMMA 4.6. *Let (M, g) be a Riemannian n-manifold and let $p \in M$. There is a harmonic function u near p with $du|_p \neq 0$.*

PROOF. We will work in normal coordinates at p. Writing out the local coordinate formula for Δ_g, it follows that

$$\Delta_g u = \Delta_e u + Qu, \qquad Qu = a^{jk}\partial_{jk} u + b^k \partial_k u,$$

where Δ_e is the Euclidean Laplacian and a^{jk}, b^k are smooth functions near 0. Since in normal coordinates one has $g_{jk}(0) = \delta_{jk}$ and $\partial_j g_{kl}(0) = 0$, it follows that

$$a^{jk}(0) = b^k(0) = 0.$$

We will look for u in the ball $B_r = B_r(0)$, where $r > 0$ is small, in the form

$$u(x) := x_1 + w(x).$$

The idea is that if r is small, then $\Delta_g x_1 \approx 0$ in B_r (since Δ_g is close to Δ_e and $\Delta_e x_1 = 0$), so there should be a solution of $\Delta_g u = 0$ close to x_1. We choose w as the solution of

$$\Delta_g w = -\Delta_g x_1 \text{ in } B_r, \qquad w|_{\partial B_r} = 0.$$

Clearly $\Delta_g u = 0$ in B_r. In order to estimate w, note that w solves

$$\Delta_e w = -Qu \text{ in } B_r, \qquad w|_{\partial B_r} = 0.$$

Writing $w_r(x) = w(rx)$ etc, we can rescale the previous equation to the unit ball:

$$\Delta_e w_r = -r^2 (Qu)_r \text{ in } B_1, \qquad w_r|_{\partial B_1} = 0.$$

For any $m \geq 0$, we may use elliptic regularity for the Dirichlet problem to get that

$$\|w_r\|_{H^{m+2}(B_1)} \lesssim r^2 \|(Qu)_r\|_{H^m(B_1)}$$

with the implied constant independent of r. Now $a^{jk}(0) = b^k(0) = 0$ and $u = x_1 + w$, so a short computation gives that

$$r^2 \|(Qu)_r\|_{H^m(B_1)} \lesssim r^3 + r\|w_r\|_{H^{m+2}(B_1)}.$$

If r is small enough, combining the last two equations gives

$$\|w_r\|_{H^{m+2}(B_1)} \lesssim r^3.$$

Choosing $m+2 > n/2+1$, the Sobolev embedding gives $\|\nabla w_r\|_{L^\infty(B_1)} \lesssim r^3$, which yields

$$\|\nabla w\|_{L^\infty(B_r)} \lesssim r^2.$$

If we choose r small enough, it follows that $du|_0 = dx_1|_0 + dw|_0 \neq 0$. □

4.3. The uniformization theorem

We will next prove the existence of global isothermal coordinates on compact simply connected surfaces with boundary. This is part of the uniformization theorem for Riemann surfaces, and reduces to the following result. (Recall that $\mathbb{D}$ denotes the unit disk in $\mathbb{R}^2$.)

THEOREM 4.7 (Uniformization theorem in the boundary case). *Let (M, g) be a compact oriented simply connected 2-manifold with smooth boundary and let $p \in M^{\text{int}}$. There is a bijective holomorphic map*

$$\Phi : M^{\text{int}} \to \mathbb{D}$$

with $\Phi(p) = 0$ which extends smoothly as a diffeomorphism $M \to \overline{\mathbb{D}}$.

The result can be reformulated as follows:

Theorem 4.8 (Global isothermal coordinates). *If (M, g) is a compact oriented simply connected 2-manifold with smooth boundary, then there are global coordinates (x_1, x_2) in M so that in these coordinates*

$$g_{jk}(x) = e^{2\lambda(x)}\delta_{jk}$$

where λ is a smooth real-valued function.

We will begin by giving a proof for a version of the standard Riemann mapping theorem. This proof is based on constructing a Green function for the Laplace operator. We will then indicate how to extend this argument to prove Theorem 4.7.

Theorem 4.9 (Riemann mapping theorem). *Let $\Omega \subset \mathbb{C}^2$ be a bounded simply connected domain with smooth boundary and let $z_0 \in \Omega$. There is a bijective holomorphic map*

$$\Phi : \Omega \to \mathbb{D}$$

with $\Phi(z_0) = 0$ which extends smoothly as a diffeomorphism $\overline{\Omega} \to \overline{\mathbb{D}}$.

Proof. After a translation, we may assume that $z_0 = 0$. We look for Φ in the form $\Phi = e^{\Psi}$ where Ψ is holomorphic in $\Omega \setminus \{0\}$. Write $\Psi = u + iv$ where $u = \mathrm{Re}(\Psi)$ and $v = \mathrm{Im}(\Psi)$. The condition $\Phi(0) = 0$ means that Ψ should behave roughly like $\log z$ near 0, and the condition $\Phi(\partial\Omega) = \partial\mathbb{D}$ means that u should vanish on $\partial\Omega$. The condition that Ψ is holomorphic implies that u and v should be harmonic in $\Omega \setminus \{0\}$.

Noting that $\mathrm{Re}(\log z) = \log|z|$ is harmonic in $\Omega \setminus \{0\}$, we look for u in the form

$$u = \log|z| + h$$

where h solves the equation

$$\Delta h = 0 \text{ in } \Omega, \qquad h|_{\partial\Omega} = -(\log|z|)|_{\partial\Omega}.$$

The Dirichlet data is in $C^\infty(\partial\Omega)$ since $0 \notin \partial\Omega$, and hence there is a unique solution $h \in C^\infty(\overline{\Omega})$. We mention that the function u thus obtained is a multiple of the Green function for the Laplacian in Ω (it satisfies $-\Delta u = 2\pi\delta_0$ in Ω with $u|_{\partial\Omega} = 0$).

We turn to finding the imaginary part v. Since Ψ should be holomorphic, v should be a harmonic conjugate of u. It is enough to choose some harmonic conjugate $h_1 \in C^\infty(\overline{\Omega})$ of h in Ω (this is possible by

Lemma 4.5 using the assumption that Ω is simply connected). We then define

$$v = \operatorname{Im}(\log z) + h_1 \text{ in } \Omega.$$

The function $\operatorname{Im}(\log z)$ is multivalued (it is well defined modulo $2\pi\mathbb{Z}$), but e^{iv} is well defined in $\Omega \setminus \{0\}$ since $e^{2\pi i k} = 1$ for any $k \in \mathbb{Z}$. We then define in $\Omega \setminus \{0\}$

$$\Phi := e^{u+iv} = e^{\log z + h + ih_1} = z e^{h+ih_1}. \tag{4.3}$$

Since $h + ih_1 \in C^\infty(\overline{\Omega})$ is holomorphic in Ω, it follows that Φ extends as a holomorphic function to Ω and satisfies

$$\Phi \in C^\infty(\overline{\Omega}), \qquad \Phi(0) = 0, \qquad |\Phi(z)| = 1 \text{ for } z \in \partial\mathbb{D}.$$

The maximum modulus principle then implies that $|\Phi| \leq 1$ in $\mathbb{D}$, i.e. that Φ maps Ω into $\mathbb{D}$.

It remains to show that Φ is bijective $\overline{\Omega} \to \overline{\mathbb{D}}$. Given any $q \in \mathbb{D}$, the argument principle in complex analysis states that the number of points $z \in \Omega$ satisfying $\Phi(z) = q$, counting multiplicity, is equal to the winding number of the curve $\Phi \circ \gamma$ around q, where γ is the boundary curve of Ω. Using the formula (4.3) for Φ, we see that there is precisely one $z \in \Omega$ with $\Phi(z) = 0$ and this point is $z = 0$. Thus the winding number of $\Phi \circ \gamma$ around 0 is 1. Since the winding number of $\Phi \circ \gamma$ is constant in $\mathbb{D}$, it follows that the winding number around any $q \in \mathbb{D}$ is 1. This proves that

$$\Phi : \Omega \to \mathbb{D} \text{ is bijective.}$$

The fact that Φ is bijective $\partial\Omega \to \partial\mathbb{D}$ follows since $\Phi \circ \gamma$ has winding number 1 and since $\Phi|_{\partial\Omega}$ has nowhere vanishing derivative (see [**Ta**, Section 8.4] for the details). □

We next indicate how the proof above can be modified in order to prove Theorem 4.7.

Proof of Theorem 4.7. We look for the desired function Φ with $\Phi(p) = 0$ in the form $\Phi = e^\Psi$, where $\Psi = u + iv$ is holomorphic in $M^{\mathrm{int}} \setminus \{p\}$ and satisfies $u|_{\partial M} = 0$. Choose a complex coordinate chart $x = (x^1, x^2)$ in a neighborhood U of p and write $z = x^1 + ix^2$. We look for u in the form

$$u = \chi \log|z| + h$$

where $\chi \in C_c^\infty(U)$ satisfies $0 \leq \chi \leq 1$ and $\chi = 1$ near p, and where h solves

$$\Delta_g h = -\Delta_g(\chi \log |z|) \text{ in } M, \qquad h|_{\partial M} = 0.$$

The right hand side function looks like it might have a singularity at $z = 0$ (i.e. at p), but since $\log |z|$ is harmonic in $U \setminus \{p\}$ the right hand side actually vanishes in a neigborhood of p and can be continued smoothly by zero across p. It follows that there is a unique real valued solution $h \in C^\infty(M)$. This completes the construction of the Green function u.

To construct the (multivalued) harmonic conjugate v, we start with the function

$$v_0 = \chi \operatorname{Im}(\log\, z) + h_1$$

where h_1 is the solution of

$$\Delta_g h_1 = -\Delta_g(\chi \operatorname{Im}(\log z)) \text{ in } M, \qquad h_1|_{\partial M} = 0.$$

As before, the right hand side is a function in $C^\infty(M)$ vanishing near p, and hence $h_1 \in C^\infty(M)$. Then both u and v_0 are harmonic in $M \setminus \{p\}$, but they are not necessarily harmonic conjugates. To rectify this we will look for v in the form $v = v_0 + v_1$ where v should satisfy $dv = *du$ (the equation for harmonic conjugates). In other words, we wish to find v_1 satisfying

$$dv_1 = *du - dv_0.$$

One can check that the right hand side is a closed smooth 1-form in M that vanishes near p. Then by the Poincaré lemma (i.e. the fact that $H^1_{\mathrm{dR}}(M) = \{0\}$) there is indeed a function $v_1 \in C^\infty(M)$ with this property. Then $\Phi = e^{u+iv}$ is the required bijective holomorphic map by an argument similar to that in the end of proof of Theorem 4.9. □

We now proceed to the full uniformization theorem. There are several possible ways to state this theorem, and one of them is the following.

Theorem 4.10 (Uniformization theorem, first version). *Let (X, g) be a Riemann surface with no boundary. Assume that X is simply connected, or that $H^1_{\mathrm{dR}}(X) = \{0\}$. Then (X, g) is conformally equivalent either with the sphere S^2, the plane $\mathbb{C}$, or the open unit disk $\mathbb{D}$.*

We sketch one possible proof following [**Hu**]. Suppose first that X is noncompact and $p \in X$. Then one can construct an exhaustion of X via a sequence of open subsets $(X_j)_{j=1}^{\infty}$, i.e.

$$X = \bigcup_{j=1}^{\infty} X_j, \qquad \overline{X}_j \subset X_{j+1},$$

where each $\overline{X}_j$ is compact, simply connected (or at least $H^1_{\mathrm{dR}}(X) = \{0\}$), has smooth boundary, and $p \in X_j$. By Theorem 4.7 there are conformal bijective maps $\Phi_j : X_j \to \mathbb{D}$ with $\Phi_j(p) = 0$. If we fix $v \in T_pX$ and apply scalings and rotations, we obtain new conformal bijective maps $\phi_j : X_j \to B(0, r_j)$ that satisfy $\phi_j(p) = 0$ and $(\phi_j)_*(v) = e_1$. These maps form a *normal family* in complex analysis, and hence some subsequence of (φ_j) converges uniformly on compact subsets to a conformal bijective map $\phi : X \to B(0, R)$ where $R = \sup r_j \in (0, \infty]$. This map ϕ is the required conformal equivalence with $\mathbb{C}$ if $R = \infty$, and with $\mathbb{D}$ otherwise.

Now suppose that X is compact and $p \in X$. One can show that then $H^1_{\mathrm{dR}}(X \setminus \{p\}) = \{0\}$, and the argument above shows that $X \setminus \{p\}$ is conformally equivalent with $\mathbb{C}$ or $\mathbb{D}$. But it is not hard to see that $X \setminus \{p\}$ cannot be conformally equivalent with $\mathbb{D}$. Hence $X \setminus \{p\}$ must be conformally equivalent with $\mathbb{C}$, and X must be conformally equivalent with S^2.

We now state a version of the uniformization theorem that applies to closed manifolds.

THEOREM 4.11 (Uniformization theorem, second version). *Let (M, g) be a closed connected oriented 2-manifold. Then there is a conformal metric on M that has constant Gaussian curvature $+1$, 0, or -1.*

The proof is based on the fact that the universal cover X of such a manifold M is a simply connected Riemann surface with no boundary. By Theorem 4.10, X is conformally equivalent to S^2, $\mathbb{C}$ or $\mathbb{D}$ and each of these has a constant curvature metric. Since M is the quotient of X by its fundamental group, also M has a constant curvature metric.

CHAPTER 5

Hodge theory

Let (M, g) be a compact oriented Riemannian manifold with no boundary, having dimension $\dim(M) = n$. In this section we introduce a Laplace operator acting on differential forms in M, prove the Hodge decomposition for differential forms that generalizes the Helmholtz decomposition for vector fields, and study the topology of M by identifying the de Rham cohomology groups with spaces of harmonic differential forms.

Motivation. Recall that we defined the Laplace-Beltrami operator Δ_g acting on scalar functions in M by looking at minimisers of the Dirichlet energy functional

$$E(u) = \int_M |du|^2 \, dV = (du, du)_{L^2}, \quad u \in H^1(M).$$

One has the trivial inequality

$$\|u\|_{H^1(M)}^2 \leq E(u) + \|u\|_{L^2}^2, \quad u \in H^1(M).$$

This shows that $E(u)$ "controls all derivatives of u", which leads to the fact that Δ_g is an elliptic operator.

Now if u is a k-form in M with $k \geq 1$, we have seen two types of derivatives of u: the exterior derivative $du \in \Omega^{k+1}(M)$ and also the codifferential $\delta u \in \Omega^{k-1}(M)$. We could introduce an energy functional

$$E^{(k)}(u) = (du, du)_{L^2} + (\delta u, \delta u)_{L^2}, \quad u \in H^1(M, \Lambda^k M).$$

The following result shows that this energy functional controls all first order derivatives of the k-form u. We refer to [**Ta**, Proposition 8.1] for a proof.

Theorem 5.1. *(Gaffney's inequality) There is $C > 0$ such that*

$$\|u\|_{H^1} \leq C(\|u\|_{L^2} + \|du\|_{L^2} + \|\delta u\|_{L^2})$$

whenever $u \in H^1(M, \Lambda^k M)$ and $0 \leq k \leq n$.

Example. Let us look at this inequality in a simple case. If u is a compactly supported 1-form in $\mathbb{R}^3$, so that $u = F_j\, dx^j$ where $F = (F_1, F_2, F_3) \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$, then an analogue of Gaffney's inequality would be

$$\sum_{j=1}^{3} \|F_j\|_{H^1} \leq C(\|F\|_{L^2} + \|\nabla \times F\|_{L^2} + \|\nabla \cdot F\|_{L^2}).$$

Integration by parts gives

$$\|\nabla \times F\|_{L^2}^2 + \|\nabla \cdot F\|_{L^2}^2 = (\nabla \times (\nabla \times F) - \nabla(\nabla \cdot F), F)_{L^2}.$$

But $\nabla \times (\nabla \times F) - \nabla(\nabla \cdot F) = (-\Delta F_1, -\Delta F_2, -\Delta F_3)$ (this is quickly seen on the Fourier side), so another integration by parts gives

$$\sum_{j=1}^{3} \|\nabla F_j\|_{L^2}^2 = \|\nabla \times F\|_{L^2}^2 + \|\nabla \cdot F\|_{L^2}^2.$$

This implies the required inequality.

Now, if u is a minimiser of $E^{(k)}$ in $H^1(M, \Lambda^k M)$, then for any $\varphi \in H^1(M, \Lambda^k M)$ we have

$$\begin{aligned} 0 &= \frac{d}{dt} E^{(k)}(u + t\varphi)\Big|_{t=0} \\ &= \frac{d}{dt}(E^{(k)}(u) + 2t\,[(du, d\varphi) + (\delta u, \delta\varphi)] + t^2 E^{(k)}(\varphi)) \\ &= ((d\delta + \delta d)u, \varphi). \end{aligned}$$

This is true for any φ, so a minimizer u must satisfy $(d\delta + \delta d)u = 0$.

Definition. If $0 \leq k \leq n$, we define the *Hodge Laplacian* to be the map $\Delta : \Omega^k(M) \to \Omega^k(M)$ satisfying

$$-\Delta = d\delta + \delta d.$$

Example. If $U \subset \mathbb{R}^3$ is an open set and $u = u_j\, dx^j$ is a 1-form in U, the computation in the previous example implies that

$$(d\delta + \delta d)u = \nabla \times (\nabla \times \vec{u}) - \nabla(\nabla \cdot \vec{u}) = (-\Delta u_j)\, dx^j.$$

A similar (but much longer) computation shows that if $U \subset \mathbb{R}^n$ is open and if $u = u_I\, dx^I$ is a k-form in U, then

$$\Delta u = (\Delta u_I)\, dx^I$$

where Δu_I is the Euclidean Laplacian of $u_I \in C^\infty(U)$.

Next we study the solvability of the equation $-\Delta u = f$ on k-forms.

DEFINITION. Let $H^{-1}(M, \Lambda^k M)$ be the dual space of $H^1(M, \Lambda^k M)$ (i.e. the space of bounded linear functionals on $H^1(M, \Lambda^k M)$). Given $f \in H^{-1}(M, \Lambda^k M)$, we say that $u \in H^1(M, \Lambda^k M)$ is a *weak solution* of

$$-\Delta u = f \text{ in } M$$

if

$$(du, dv)_{L^2} + (\delta u, \delta v)_{L^2} = f(v) \text{ for all } v \in H^1(M, \Lambda^k M).$$

The next theorem gives a detailed account of the existence, uniqueness and regularity of weak solutions to $-\Delta u = f$; we postpone the proof until the end of the section.

THEOREM 5.2. *Fix k with $0 \leq k \leq n$.*

1. *(Weak solutions) There is a countable set $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{R}$ with*
$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$$
such that whenever $\lambda \in \mathbb{C} \setminus \{\lambda_1, \lambda_2, \ldots\}$, the equation
$$(-\Delta - \lambda)u = f$$
has a unique weak solution $u \in H^1(M, \Lambda^k M)$ for any $f \in H^{-1}(M, \Lambda^k M)$.
2. *(Kernel of $-\Delta$) The space*
$$\mathcal{H}_k := \mathrm{Ker}(\Delta|_{H^1(M, \Lambda^k M)}) = \{u \in H^1(M, \Lambda^k M)\,;\, \Delta u = 0\}$$
is finite dimensional and its elements are C^∞.
3. *(Elliptic regularity) There is a bounded linear map*
$$G : L^2(M, \Lambda^k M) \to H^2(M, \Lambda^k M)$$
such that
$$-\Delta G u = (I - P_k)u, \quad u \in L^2(M, \Lambda^k M)$$
where P_k is the orthogonal projection from $L^2(M, \Lambda^k M)$ onto $\mathcal{H}_k$. For $j \geq 0$, G is a bounded map $H^j(M, \Lambda^k M) \to H^{j+2}(M, \Lambda^k M)$.

The finite dimensional space $\mathcal{H}_k$ is called the *space of harmonic k-forms*, and it has the following characterization:

THEOREM 5.3. *One has*

$$\mathcal{H}_k = \{u \in \Omega^k(M)\,;\, du = \delta u = 0\}.$$

One has

$$\mathcal{H}_0 = \{u \in C^\infty(M)\,;\, u \text{ is constant on each component of } M\}$$

and thus $\dim(\mathcal{H}_0)$ *is the number of connected components of* M*.*

Proof. If $u \in \mathcal{H}_k$, so that $(d\delta + \delta d)u = 0$, then using u as a test function gives

$$0 = ((d\delta + \delta d)u, u)_{L^2} = (du, du)_{L^2} + (\delta u, \delta u)_{L^2} = \|du\|_{L^2}^2 + \|\delta u\|_{L^2}^2$$

which implies $du = \delta u = 0$. Conversely, if $u \in \Omega^k(M)$ satisfies $du = \delta u = 0$, then clearly $(d\delta + \delta d)u = 0$ so $u \in \mathcal{H}_k$.

If $k = 0$ one has

$$\mathcal{H}_0 = \{u \in C^\infty(M)\,;\, du = 0\}$$

and clearly this consists of the functions that are constant on each connected component. □

The next result is a powerful generalization of the Helmholtz decomposition, which allows to decompose a vector field F in $\mathbb{R}^n$ into curl-free and divergence-free components, i.e.

$$F = \nabla p + W$$

where p is a scalar function and $\nabla \cdot W = 0$. The Helmholtz decomposition corresponds to the next theorem in the case of 1-forms.

Theorem 5.4. *(Hodge decomposition) Any* $u \in L^2(M, \Lambda^k M)$ *has the decomposition*

$$u = d\delta G u + \delta d G u + P_k u$$

where the three components are L^2*-orthogonal.*

Remark. The Hodge decomposition of $u \in L^2(M, \Lambda^k M)$ can also be written as

$$u = d\alpha + \delta\beta + \gamma$$

where $\alpha = \delta G u \in H^1(M, \Lambda^{k-1} M)$ and $\beta = dGu \in H^1(M, \Lambda^{k+1} M)$, and where $\gamma = P_k u \in \mathcal{H}_k$ is a harmonic k-form (and hence C^∞).

Proof of Theorem 5.4. Let $u \in L^2(M, \Lambda^k M)$. By Theorem 5.2 we have

$$-\Delta(Gu) = (I - P_k)u.$$

The decomposition follows by using that $-\Delta = d\delta + \delta d$. The orthogonality follows since

$$(d\alpha, \delta\beta)_{L^2} = (d^2\alpha, \beta)_{L^2} = 0$$

and since any harmonic form γ is L^2-orthogonal to any $d\alpha$ or $\delta\beta$ using that $d\gamma = \delta\gamma = 0$. $\square$

Let now M be a compact smooth manifold. We define the *de Rham cohomology groups* for $0 \leq k \leq n$ by

$$H^k_{\mathrm{dR}}(M) := \mathrm{Ker}(d|_{\Omega^k(M)})/\mathrm{Im}(d|_{\Omega^{k-1}(M)}).$$

These are actually vector spaces. If $F : M \to N$ is a diffeomorphism between two compact smooth manifolds, the property $dF^* = F^*d$ immediately implies that F^* induces an isomorphism between the vector spaces $H^k_{\mathrm{dR}}(N)$ and $H^k_{\mathrm{dR}}(M)$. Thus the de Rham cohomology groups are diffeomorphism invariants; it is not too hard to show that they are actually topological and even homotopy invariants (and thus do not depend on the particular smooth structure that M has).

The next theorem due to Hodge shows that if one assigns a Riemannian metric g on M, then $H^k_{\mathrm{dR}}(M)$ can be identified with the space of harmonic k-forms. This shows, in particular, that the dimension of $\mathcal{H}_k$ is independent of g and in fact is a topological invariant.

THEOREM 5.5. *(Hodge isomorphism) If $0 \leq k \leq n$, then any equivalence class in $H^k_{\mathrm{dR}}(M)$ has a unique harmonic representative. The map*

$$J_k : \mathcal{H}_k \to H^k_{\mathrm{dR}}(M), \quad u \mapsto [u]$$

is an isomorphism.

PROOF. Let $w \in \Omega^k(M)$ satisfy $dw = 0$, and let $[w] \in H^k_{\mathrm{dR}}(M)$ be the corresponding equivalence class. We need to show that $[w] = [u]$ for a unique $u \in \mathcal{H}_k$. To show existence, write the Hodge decomposition for w:

$$w = d\delta Gw + \delta dGw + P_k w.$$

But since $dw = 0$, we have $(w, \delta\alpha) = 0$ for all α, and in particular

$$0 = (w, \delta dGw) = (d\delta Gw + \delta dGw + P_k w, \delta dGw) = \|\delta dGw\|^2.$$

Thus $\delta dGw = 0$, which implies that

$$w = u + d\delta Gw$$

where $u = P_k w$ is harmonic. This shows that $[w] = [u]$ for some harmonic u. To show uniqueness we note that if $[u_1] = [u_2]$ with u_j harmonic then $u_1 - u_2 = d\alpha$ for some α, but then

$$\|u_1 - u_2\|^2 = (u_1 - u_2, d\alpha) = (\delta(u_1 - u_2), \alpha) = 0$$

showing that $u_1 = u_2$. The fact that J_k is an isomorphism follows immediately from the above facts. □

We record an immediate consequence:

COROLLARY 5.6. *(Betti numbers) Let M be a compact oriented smooth manifold. The de Rham cohomology groups of M are finite dimensional vector spaces, and their dimensions are given by*

$$b_k(M) = \dim(H^k_{\mathrm{dR}}(M)) = \dim(\mathrm{Ker}(\Delta_g|_{\Omega^k(M)}))$$

where g is any Riemannian metric on M.

Next we discuss Poincaré duality, which states that there is a natural isomorphism between $H^k_{\mathrm{dR}}(M)$ and $H^{n-k}_{\mathrm{dR}}(M)$ whenever $0 \leq k \leq n$. In terms of Betti numbers, this implies that $b_k(M) = b_{n-k}(M)$. The isomorphism is given by the following operator.

THEOREM 5.7. *(Hodge star operator) Let (M, g) be an oriented Riemannian manifold of dimension n. There is a unique linear operator (called the* Hodge star operator*)*

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

which satisfies the following identity for $u, v \in \Omega^k(M)$:

$$u \wedge *v = \langle u, v \rangle \, dV. \tag{5.1}$$

It has the following properties:

(1) $** = (-1)^{k(n-k)}$ *on k-forms*
(2) $*1 = dV$
(3) $*(\varepsilon^1 \wedge \ldots \wedge \varepsilon^k) = \varepsilon^{k+1} \wedge \ldots \wedge \varepsilon^n$ *whenever $(\varepsilon^1, \ldots, \varepsilon^n)$ is a positive local orthonormal frame on T^*M*
(4) *The codifferential has the expression*

$$\delta = (-1)^{(k-1)(n-k)-1} * d * \quad \text{on } k\text{-forms.}$$

Before the proof, we give two examples.

Example. Let $\dim(M) = 2$ and $u \in \Omega^1(M)$. If $(\varepsilon^1, \varepsilon^2)$ is a local orthonormal frame of 1-forms, we may write $u = u_1\varepsilon^1 + u_2\varepsilon^2$. Then the property (3) in the theorem implies that

$$*(u_1\varepsilon^1 + u_2\varepsilon^2) = -u_2\varepsilon^1 + u_1\varepsilon^2.$$

Consequently

$$|{*u}| = |u|, \quad \langle u, *u \rangle = 0.$$

Thus on 2D manifolds the Hodge star on 1-forms corresponds to rotation by 90° counterclockwise.

Example. Let $\dim(M) = 3$ and $u \in \Omega^1(M)$, so we may write $u = u_j\varepsilon^j$ if $(\varepsilon^1, \varepsilon^2, \varepsilon^3)$ is a local orthonormal frame of 1-forms. Property (3) in the theorem implies that

$$*(u_j\varepsilon^j) = u_j\varepsilon^{\hat{j}}$$

where $\varepsilon^{\hat{1}} = \varepsilon^2 \wedge \varepsilon^3$, $\varepsilon^{\hat{2}} = \varepsilon^3 \wedge \varepsilon^1$, $\varepsilon^{\hat{3}} = \varepsilon^1 \wedge \varepsilon^2$.

Proof of Theorem 5.7. Let us first show that if two linear operators $*$ and $\tilde{*}$ satisfy (5.1), then $* = \tilde{*}$. In fact, in this case one has

$$u \wedge (*v - \tilde{*}v) = 0$$

for any $u, v \in \Omega^k(M)$. If U is a coordinate neighborhood and if $\{\varepsilon^1, \ldots, \varepsilon^n\}$ is an orthonormal frame of T^*U, we may write $*v - \tilde{*}v|_U = w_J\varepsilon^J$ where the sum is over $J \in \mathcal{I}_{n-k}$. Choosing $u = \chi\varepsilon^I$ above where $\chi \in C_c^\infty(U)$ and $I \in \mathcal{I}_k$, and varying χ and I imply that $w_J = 0$ in U for all J. Thus $*v = \tilde{*}v$ in U, and varying U shows that $* \equiv \tilde{*}$.

Let us next construct a linear operator $*$ satisfying (5.1). It is enough to define $* : \Lambda^k(T_qM) \to \Lambda^{n-k}(T_qM)$ for q in a coordinate neighborhood U. If $(\varepsilon^1, \ldots, \varepsilon^n)$ is a positive orthonormal frame of T^*U, then $\Lambda^k(T_qM)$ has an orthonormal basis $\{\varepsilon^I\}_{I\in\mathcal{I}_k}$. We define

$$*(\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k}) := \varepsilon^{j_{k+1}} \wedge \ldots \wedge \varepsilon^{j_n}$$

where the indices are chosen so that $(\varepsilon^{j_1}, \ldots, \varepsilon^{j_n})$ is a positive orthonormal frame. This gives a well-defined operator acting on basis elements, and we extend it as a linear operator acting on $\Lambda^k(T_qM)$. It is easy to check that for any $I, J \in \mathcal{I}_k$,

$$\varepsilon^I \wedge *\varepsilon^J = \begin{cases} dV, & I = J, \\ 0, & I \neq J. \end{cases}$$

If $u, v \in \Omega^k(M)$ have local expressions $u = u_I \varepsilon^I, v = v_J \varepsilon^J$, then

$$u \wedge *v = u_I v_J \varepsilon^I \wedge *\varepsilon^J = \sum_I u_I v_I \, dV$$

and

$$\langle u, v \rangle \, dV = u_I v_J \langle \varepsilon^I, \varepsilon^J \rangle \, dV = \sum_I u_I v_I \, dV$$

since the ε^I are orthonormal. Thus our operator satisfies (5.1). We have seen that this defines $*$ uniquely, so we have an invariantly defined operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ satisfying (5.1).

The properties (1)–(3) follow from the definition of $*$ in terms of the ε^j. To prove (4), we let $u \in \Omega^{k-1}(M)$, $v \in \Omega^k(M)$ and compute

$$\begin{aligned}
(du, v)_{L^2} &= \int_M \langle du, v \rangle \, dV = \int_M du \wedge *v \\
&= \int_M \big[d(u \wedge *v) - (-1)^{k-1} u \wedge (d * v) \big] \\
&= (-1)^k (-1)^{(n-k+1)(k-1)} \int_M u \wedge (* * d * v) \\
&= \int_M \langle u, (-1)^{(k-1)(n-k)-1} * d * v \rangle \, dV \\
&= (u, (-1)^{(k-1)(n-k)-1} * d * v)_{L^2}.
\end{aligned}$$

We used the definitions, the formula for $d(u \wedge *v)$, and the Stokes theorem. This shows (4). □

THEOREM 5.8. *(Poincaré duality) If (M, g) is a compact oriented Riemannian manifold and $0 \le k \le n$, there is an isomorphism*

$$H^k_{\mathrm{dR}}(M) \approx H^{n-k}_{\mathrm{dR}}(M).$$

PROOF. Consider the Hodge star operator acting on harmonic k-forms,

$$* : \mathcal{H}_k \to \Omega^{n-k}(M).$$

If $u \in \mathcal{H}_k$, the formulas $** = \pm 1$ and $\delta = \pm * d *$ (the precise sign does not matter here) imply that $d(*u) = \pm * * d(*u) = \pm * \delta u = 0$ and $\delta(*u) = \pm * d(* * u) = \pm * du = 0$. Thus

$$* : \mathcal{H}_k \to \mathcal{H}_{n-k}.$$

But since $** = \pm 1$, the above map is invertible and hence is a vector space isomorphism. Now $H^k_{\mathrm{dR}}(M)$ is isomorphic to $\mathcal{H}_k$ by Theorem 5.5, so the result follows. □

Since $H^0_{\mathrm{dR}(M)}$ is the number of connected components of M, we have the following corollary.

Corollary 5.9. *If (M,g) is a compact connected oriented Riemannian manifold without boundary, then*

$$\dim(H^0_{\mathrm{dR}}(M)) = \dim(H^n_{\mathrm{dR}}(M)) = 1.$$

We remark that the Hodge star operator also explains the duality between the sequences (2.15) and (2.16) in Section 2.6. If

$$H^k_d(M) := \mathrm{Ker}(d|_{\Omega^k(M)})/\mathrm{Im}(d|_{\Omega^{k-1}(M)}),$$
$$H^k_\delta(M) := \mathrm{Ker}(\delta|_{\Omega^k(M)})/\mathrm{Im}(\delta|_{\Omega^{k+1}(M)}),$$

it is easy to check that $* : \Omega^k(M) \to \Omega^{n-k}(M)$ induces an isomorphism between $H^k_d(M)$ and $H^{n-k}_\delta(M)$.

To end this section we will sketch the proof of Theorem 5.2. We assume throughout that (M,g) is a compact oriented n-dimensional Riemannian manifold with no boundary, and $0 \leq k \leq n$. We begin with a simple result that only uses Gaffney's inequality and elementary Hilbert space methods.

Lemma 5.10. *For any positive real number μ, the equation*

$$(-\Delta + \mu)u = f$$

has a unique solution $u \in H^1(M, \Lambda^k M)$ for any $f \in H^{-1}(M, \Lambda^k M)$.

Proof. Define the bilinear form

$$B_\mu(u,v) = (du,dv)_{L^2} + (\delta u, \delta v)_{L^2} + \mu(u,v)_{L^2}, \quad u,v \in H^1(M,\Lambda^k M).$$

This is a symmetric bilinear form, and Gaffney's inequality implies that it satisfies for some $c_\mu > 0$

$$B_\mu(u,u) \geq c_\mu \|u\|^2_{H^1}, \quad u \in H^1(M, \Lambda^k M).$$

We also have $|B_\mu(u,v)| \leq C\|u\|_{H^1}\|v\|_{H^1}$. Consequently $B_\mu(\,\cdot\,,\,\cdot\,)$ is an inner product on $H^1(M,\Lambda^k M)$ that induces a norm equivalent to the usual norm on H^1 (hence also the usual topology). Then for any $f \in H^{-1}(M, \Lambda^k M)$, the Riesz representation theorem shows that there is a unique $u \in H^1(M,\Lambda^k M)$ satisfying

$$B_\mu(u,v) = f(v), \quad v \in H^1(M, \Lambda^k M).$$

This proves the theorem. □

The previous result can be considerable improved if one observes that the inverse of $-\Delta + \mu$ is a compact operator and applies the spectral theorem for compact operators. The basic underlying result is the compact Sobolev embedding theorem [**Ta**, Proposition 4.3.4].

THEOREM 5.11. *(Rellich-Kondrachov compact embedding theorem) The inclusion $H^1(M, \Lambda^k M) \to L^2(M, \Lambda^k M)$ is compact, meaning that any bounded sequence in $H^1(M, \Lambda^k M)$ has a convergent subsequence in $L^2(M, \Lambda^k M)$.*

For the proof of Theorem 5.2 we also need the following elliptic regularity result [**Ta**, Theorem 5.1.3].

THEOREM 5.12. *(Elliptic regularity) If $u \in H^1(M, \Lambda^k M)$ is a weak solution of $-\Delta u = f$ where $f \in H^j(M, \Lambda^k M)$ for some $j \geq 0$, then $u \in H^{j+2}(M, \Lambda^k M)$ and*

$$\|u\|_{H^{j+2}} \leq C(\|f\|_{H^j} + \|u\|_{H^{j+1}})$$

where C is independent of u and f.

PROOF OF THEOREM 5.2 PART 1. We fix $\mu > 0$ and let

$$T = (-\Delta + \mu)^{-1} : H^{-1}(M, \Lambda^k M) \to H^1(M, \Lambda^k M)$$

be the solution operator from Lemma 5.10. By compact embedding, we have that $T : L^2(M, \Lambda^k M) \to L^2(M, \Lambda^k M)$ is compact. It is also self-adjoint and positive semidefinite, since for $f, h \in L^2$ (with $u = Tf$)

$$\begin{aligned}
(Tf, h) &= (u, (-\Delta + \mu)Th) = (du, dTh) + (\delta u, \delta Th) + \mu(u, Th) \\
&= ((-\Delta + \mu)u, Th) = (f, Th), \\
(Tf, f) &= (Tf, (-\Delta + \mu)Tf) = (dTf, dTf) + (\delta Tf, \delta Tf) + \mu(Tf, Tf) \\
&\geq 0.
\end{aligned}$$

By the spectral theorem for compact operators [**Ta**], there exist $\mu_1 \geq \mu_2 \geq \ldots$ with $\mu_j \to 0$ and $\phi_l \in L^2(M, \Lambda^k M)$ with $T\phi_l = \mu_l \phi_l$ such that $\{\phi_l\}_{l=1}^{\infty}$ is an orthonormal basis of $L^2(M, \Lambda^k M)$ and

$$\text{Ker}(T - \mu_l) \text{ is finite dimensional for each } l. \tag{5.2}$$

Note that 0 is not in the spectrum of T, since $Tf = 0$ implies $f = 0$. Defining

$$\lambda_l = \frac{1}{\mu_l} - \mu$$

gives that

$\{\phi_l\}_{l=1}^{\infty}$ is an orthonormal basis of $L^2(M, \Lambda^k M)$ and $-\Delta\phi_l = \lambda_l \phi_l$.

If $\lambda \neq \lambda_l$ for all l then for $u \in H^1(M, \Lambda^k M)$ and $f \in H^{-1}(M, \Lambda^k M)$,

$$(-\Delta - \lambda)u = f \Leftrightarrow u = T(f + (\lambda+\mu)u) \Leftrightarrow (\frac{1}{\lambda+\mu} - T)u = \frac{1}{\lambda+\mu} Tf.$$

Since $\frac{1}{\lambda+\mu} \neq \mu_l$ for all l, $\frac{1}{\lambda+\mu}\mathrm{Id} - T$ is invertible and we see that $-\Delta - \lambda$ is bijective and bounded $H^1 \to H^{-1}$, therefore an isomorphism. □

Proof of Theorem 5.2 part 2. If 0 is an eigenvalue of $-\Delta$ (i.e. $\lambda_1 = 0$), then $1/\mu$ is an eigenvalue of T. The equivalence

$$-\Delta u = 0 \Leftrightarrow Tu = \frac{1}{\mu} u$$

and (5.2) show that $\mathrm{Ker}(-\Delta)$ is finite-dimensional. On the other hand, if 0 is not an eigenvalue of $-\Delta$, then $\mathrm{Ker}(-\Delta) = \{0\}$. By elliptic regularity, elements of $\mathrm{Ker}(-\Delta)$ are C^∞. □

Proof of Theorem 5.2 part 3. Let $\dim(\mathrm{Ker}(-\Delta)) = m \geq 0$, so $\lambda_1 = \ldots = \lambda_m = 0$ and $\lambda_{m+1} > 0$. Using the notation above, we define

$$Gu := \sum_{l=m+1}^{\infty} \frac{1}{\lambda_l} (u, \phi_l)_{L^2} \phi_l, \quad u \in L^2(M, \Lambda^k M).$$

The sum converges in L^2 by orthogonality and G becomes a bounded operator on L^2. Since $-\Delta\phi_l = \lambda_l \phi_l$, it is not hard to check that

$$-\Delta G u = \sum_{l=m+1}^{\infty} (u, \phi_l)_{L^2} \phi_l = (I - P_k)u.$$

A short argument using elliptic regularity shows that G is a bounded operator from L^2 to H^2, and also from H^j to H^{j+2} for $j \geq 0$. □

CHAPTER 6

Curvature

6.1. Background

Curvature of curves. Let $\gamma : (a,b) \to \mathbb{R}^3$ be a regular smooth curve, so $\dot\gamma(t) \neq 0$ for $t \in (a,b)$. We can always reparametrize γ by arc length, and in the new parametrisation one has $|\dot\gamma(t)| \equiv 1$ (such a curve is called a *unit speed* curve). Differentiating the identity $|\dot\gamma(t)|^2 \equiv 1$ shows that

$$\ddot\gamma(t) \cdot \dot\gamma(t) = 0.$$

Thus for unit speed curves the acceleration vector $\ddot\gamma(t)$ is orthogonal to the tangent vector. The vector $\ddot\gamma(t)$ measures how quickly the curve deviates from its tangent line at $\gamma(t)$, and leads to the notions of curvature and the osculating circle which is a good second order approximation of γ.

Lemma 6.1. *(Osculating circle) Let $\gamma : (a,b) \to \mathbb{R}^3$ be a smooth unit speed curve. Given $t_0 \in (a,b)$, there is a unique unit speed circle $\eta : [t_0 - \pi R, t_0 + \pi R] \to \mathbb{R}^3$ (called the* osculating circle *for γ at $\gamma(t_0)$) that satisfies*

$$\eta(t_0) = \gamma(t_0), \quad \dot\eta(t_0) = \dot\gamma(t_0), \quad \ddot\eta(t_0) = \ddot\gamma(t_0).$$

If $\ddot\gamma(t_0) = 0$ then η is a tangent line of γ, and if $\ddot\gamma(t_0) \neq 0$ then η is a circle with radius $R = \frac{1}{|\ddot\gamma(t_0)|}$ lying in the two-plane spanned by $\dot\gamma(t_0)$ and $\ddot\gamma(t_0)$ (called the osculating plane*).*

Proof. We normalise matters so that $t_0 = 0$. If $\ddot\gamma(0) = 0$, then η is given by $\eta(t) = \gamma(0) + t\dot\gamma(0)$. Assume now that $\ddot\gamma(0) \neq 0$. We look for η in the form

$$\eta(t) = x_0 + R(\cos(t/R)q_1 + \sin(t/R)q_2)$$

where $x_0 \in \mathbb{R}^3$, $R > 0$, and the unit vectors $q_1, q_2 \in \mathbb{R}^3$ are to be determined. The equations for η and γ at $t = 0$ imply that

$$x_0 + Rq_1 = \gamma(0), \quad q_2 = \dot\gamma(0), \quad -\frac{1}{R}q_1 = \ddot\gamma(0).$$

Taking absolute values in the last equation gives $R = \frac{1}{|\ddot\gamma(0)|}$, and then the last two equations give $q_1 = -\frac{\ddot\gamma(0)}{|\ddot\gamma(0)|}$ and $q_2 = \dot\gamma(0)$. The first equation implies $x_0 = \gamma(0) - Rq_1$, and this determines η uniquely. $\square$

The number $R = R(s)$ above is called the *radius of curvature* of γ at $\gamma(s)$, and its reciprocal

$$\kappa(s) = \frac{1}{R(s)}$$

is called the *curvature* at $\gamma(s)$. If $\ddot\gamma(s) \neq 0$ and if we choose a normal vector $N(s) = \pm\frac{\ddot\gamma(s)}{|\ddot\gamma(s)|}$ to γ in the osculating plane, we can also define the *signed curvature*

$$\kappa_N(t) := \ddot\gamma(t) \cdot N(t).$$

Curvature of surfaces in $\mathbb{R}^3$. Let now M be a smooth hypersurface in $\mathbb{R}^3$, equipped with the Riemannian metric induced by the Euclidean metric in $\mathbb{R}^3$. We assume for simplicity that $M = f^{-1}(0)$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function with $\nabla f \neq 0$ on M.

For a fixed point $p \in M$, we can study the curvature of M at p by computing the signed curvatures $\kappa(v)$ with respect to a normal $N(p)$ of curves γ_v, where $v \in T_pM$ is a unit tangent vector and γ_v is a unit speed curve on M with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. We will use the choice $N := -\nabla f/|\nabla f|$, so N is a smooth unit normal vector field on M (to see this, observe that if $v \in T_pM$ and if γ_v is as above, then $0 = \frac{d}{dt} f(\gamma_v(t))\big|_{t=0} = \nabla f(p) \cdot v$).

The curvatures $\kappa(v)$ depend on the point p and on the direction v (but not on the particular choice of γ_v, as shown by the next proof). The curvatures $\kappa(v)$ are conveniently described by the shape operator.

Lemma 6.2. *(Shape operator) There is a smooth map $S : TM \to TM$, called the* shape operator *of S, such that $S|_{T_pM}$ is a linear map on T_pM for each p and $\kappa(v) = \langle S(v), v\rangle$ for any unit tangent vector $v \in TM$. The map S is characterised by*

$$\langle S(v), w\rangle = \langle \frac{f''(p)}{|\nabla f(p)|} v, w\rangle, \quad p \in M, \quad v, w \in T_pM.$$

Proof. The last identity defines a symmetric linear map S on T_pM. It is enough to check that $\kappa(v) = \langle S(v), v\rangle$ for any unit tangent vector $v \in T_pM$. If γ is a unit speed curve on M with $\gamma(0) = p$ and

$\dot\gamma(0) = v$, it follows that $f(\gamma(t)) \equiv 0$. Thus

$$0 = \frac{d^2}{dt^2} f(\gamma(t))\Big|_{t=0} = f''(p)v \cdot v + \nabla f(p) \cdot \ddot\gamma(0).$$

It follows that

$$\kappa(v) = N(p) \cdot \ddot\gamma(0) = -\frac{1}{|\nabla f(p)|} \nabla f(p) \cdot \ddot\gamma(0) = \frac{f''(p)}{|\nabla f(p)|} v \cdot v$$

which proves the result. □

Definition. The *principal curvatures* of M at p are the eigenvalues κ_1 and κ_2 (with $\kappa_1 \leq \kappa_2$) of the linear map $S|_{T_pM}$ considered as a symmetric 2×2 matrix. The *Gaussian curvature* (or *total curvature*) of M is

$$K := \kappa_1 \kappa_2 = \det(S|_{T_pM})$$

and the *mean curvature* of M is

$$H := \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\mathrm{tr}(S|_{T_pM}).$$

Example. Consider two hypersurfaces M and $\tilde M$ in $\mathbb{R}^3$, defined by

$$M = \{x \in \mathbb{R}^3\,;\, x_3 = 0, \quad 0 < x_2 < \pi\},$$
$$\tilde M = \{x \in \mathbb{R}^3\,;\, x_2^2 + x_3^2 = 1, \quad x_3 > 0\}.$$

The map $F : M \to \tilde M$, $(x_1, x_2, 0) \mapsto (x_1, \cos x_2, \sin x_2)$ is an isometry between M and $\tilde M$ (equipped with the metric induced by the Euclidean metric in $\mathbb{R}^3$), since the vectors

$$F_*\partial_1|_{F(x_1,x_2,0)} = (1, 0, 0),$$
$$F_*\partial_2|_{F(x_1,x_2,0)} = (0, -\sin x_2, \cos x_2)$$

give an orthonormal basis at each point and consequently

$$F_* v \cdot F_* w = v \cdot w, \qquad v, w \in T_pM.$$

The principal curvatures κ_j and $\tilde\kappa_j$ of M and $\tilde M$ are

$$\kappa_1 = \kappa_2 = 0, \qquad \tilde\kappa_1 = 0, \quad \tilde\kappa_2 = 1.$$

(We use the normal vector on M pointing downward, and the corresponding vector on $\tilde M$.)

The previous example shows that the principal curvatures and mean curvature are not invariant under isometries. However, both M and $\tilde M$ have the same Gaussian curvature. The Gaussian curvature turns out to be invariant under isometries in general:

THEOREM 6.3. *(Gauss's Theorema Egregium, 1827) The Gaussian curvature is* intrinsic, *in the sense that it only depends on the structure of (M, g) as a Riemannian manifold (and not in its embedding in $\mathbb{R}^3$) and is invariant under isometries.*

The above theorem means that the Gaussian curvature of a 2D hypersurface M can be measured by inhabitants of M, whereas measuring the principal or mean curvatures would require information about the particular embedding in $\mathbb{R}^3$. Also in this direction, the Gaussian curvature is uniquely determined by the perimeters of small geodesic balls:

THEOREM 6.4. *(Bertrand, Puiseux) If (M, g) is a 2D hypersurface in $\mathbb{R}^3$ and if $B(p, \varepsilon) = \{q \in M\,;\, d_g(p, q) < \varepsilon\}$, then $\partial B(p, \varepsilon)$ is a smooth curve for $\varepsilon > 0$ small and*

$$L_g(\partial B(p, \varepsilon)) = 2\pi\varepsilon - \frac{\pi}{3}K(p)\varepsilon^3 + o(\varepsilon^3) \quad \text{as } \varepsilon \to 0.$$

Since the Gaussian curvature K is invariant under isometries, any hypersurface that is isometric to a piece of the flat plane $\{x_3 = 0\}$ satisfies $K \equiv 0$. The converse also holds: if (M, g) is a hypersurface and if $K = 0$ near p, then some neighborhood of p is isometric to a piece of $\{x_3 = 0\}$. This shows that the Gaussian curvature is a sufficiently powerful invariant to characterize local flatness.

The arguments above suggest that the Gaussian curvature can be defined for any 2D Riemannian manifold, not just for hypersurfaces. An important (and nontrivial) related theorem is the Gauss-Bonnet theorem:

THEOREM 6.5. *(Gauss-Bonnet) Let (M, g) be a compact oriented smooth Riemannian manifold with* $\dim(M) = 2$. *Then*

$$\int_M K\, dV_g = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M.

Since $\chi(M)$ is a topological invariant, the theorem implies among other things that the topology of M puts strong constraints on the kinds of Riemannian metrics that M admits. In particular, if M admits a metric with $\begin{cases} K > 0 \\ K = 0 \\ K < 0 \end{cases}$ everywhere, then $\begin{cases} \chi(M) > 0 \\ \chi(M) = 0 \\ \chi(M) < 0. \end{cases}$

Curvature in higher dimensions. The Habilitation lecture of Riemann in 1854 is a landmark in geometry. In this lecture, Riemann

- considered (not so rigorously) the notion of an abstract smooth manifold
- suggested that the geometry of such a space could be described by a length element (i.e. a Riemannian metric)
- introduced a higher dimensional generalization of Gaussian curvature.

There are many different approaches to understanding curvature. We describe some of these informally.

1. **Riemann's approach.** Riemann's idea for measuring curvature in higher dimensions was to look at certain second order coefficients R_{ijkl} in the Taylor expansion of the Riemannian metric in normal coordinates (i.e. coordinates obtained by following geodesics starting at a fixed point).
2. **Sectional curvature approach.** If (M, g) is a Riemannian manifold and $p \in M$, consider a 2-plane Π in T_pM. Following geodesics in M starting at p with initial direction in Π, one obtains a 2-dimensional Riemannian manifold M_Π. By the Theorema Egregium, the total curvature $K(\Pi)$ of M_Π only depends on the metric structure. The numbers $K(\Pi)$, called the *sectional curvatures* of M at p, for different 2-planes $\Pi \subset T_pM$ can be used to measure the curvature of (M, g). Knowing $K(\Pi)$ for each Π is equivalent to knowing the numbers R_{ijkl}.
3. **Parallel transport approach.** On any Riemannian manifold, if γ is a smooth regular curve from p to q and if $v \in T_pM$, there is a unique way of transporting v along γ to a vector $P_\gamma v \in T_qM$. Let X and Y be two vector fields near p that commute ($[X, Y] = 0$), and let $P_X(t)$ be the parallel transport for time t along the flow of X. Given $v \in T_pM$, let
$$Q_{X,Y}(s,t)v = P_Y(-t)P_X(-s)P_Y(t)P_X(s)v.$$
Since X and Y commute, $Q_{X,Y}(t)$ is a linear map $T_pM \to T_pM$ that corresponds to parallel translating v along a small quadrilateral with sidelength t determined by X and Y. It turns out that
$$R_{ijkl} = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\langle Q_{\partial_i,\partial_j}(s,t)\partial_k, \partial_l\rangle\Big|_{s,t=0}.$$

Thus, curvature measures how tangent vectors are changed under parallel translation along small loops.

4. **Connection approach.** On any Riemannian manifold, there is a natural way of differentiating a vector field Y in the direction of another vector field X to produce a new vector field $\nabla_X Y$. The operator ∇ is called the Levi-Civita connection, Riemannian connection, or covariant derivative. Curvature measures the extent to which second order covariant derivatives commute:

$$R_{ijkl} = \langle (\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_i}) \partial_k, \partial_l \rangle.$$

These constructions are equivalent and they give a complete set $\{R_{ijkl}\}$ of isometry invariants, in the sense that the vanishing of all these invariants near p is equivalent with local flatness (i.e. a neighbourhood of p being isometric to a piece of $\mathbb{R}^n$). The functions R_{ijkl} are the component functions of the coordinate representation a certain 4-tensor field on (M, g), called the Riemann curvature tensor.

At this point it is convenient to pause the geometric discussion, in order to develop some abstract machinery that could be used to

- compute curvatures
- prove some basic properties of curvature.

We will begin by discussing geodesics.

6.2. Geodesics and the Riemannian connection

Lengths of curves and the distance function d_g can be defined on any Riemannian manifold (M, g) in the same way as we did in open sets in $\mathbb{R}^n$. In that setting, recall that if γ is a curve that minimizes length between its endpoints, we showed that γ satisfies the geodesic equation by computing

$$\frac{d}{ds} L(\gamma_s) = \int_a^b \frac{\partial}{\partial s} \langle \dot{\gamma}_s(t), \dot{\gamma}_s(t) \rangle^{1/2} \, dt$$

where (γ_s) was a variation of γ. This computation involved inserting the local coordinate expression of $\langle \dot{\gamma}_s(t), \dot{\gamma}_s(t) \rangle$ and taking its derivative. The geodesic equation and Christoffel symbols then came out from this local coordinate computation.

It will be very useful to be able to do computations like this in an invariant way, without resorting to local coordinates. For this purpose we want to be able to take derivatives of vector fields in a way which is compatible with the Riemannian inner product $\langle \cdot\,, \cdot \rangle$.

We first recall the commutator of vector fields. Any vector field $X \in C^\infty(M, TM)$ gives rise to a first order differential operator $X : C^\infty(M) \to C^\infty(M)$ by

$$Xf(p) = X(p)f.$$

If X and Y are vector fields, their commutator $[X, Y]$ is the differential operator acting on smooth functions by

$$[X, Y]f := X(Yf) - Y(Xf).$$

The commutator of two vector fields is itself a vector field, and any coordinate vector fields satisfy $[\partial_i, \partial_j] = 0$ (both results follow by the equality of mixed partial derivatives in $\mathbb{R}^n$).

The next result is sometimes called the fundamental lemma of Riemannian geometry.

THEOREM 6.6. *(Riemannian connection) On any Riemannian manifold (M, g) there is a unique $\mathbb{R}$-bilinear map*

$$\begin{aligned} \nabla : C^\infty(M, TM) \times C^\infty(M, TM) &\to C^\infty(M, TM), \\ (X, Y) &\mapsto \nabla_X Y, \end{aligned}$$

which satisfies

(1) $\nabla_{fX} Y = f \nabla_X Y$ *(linearity)*
(2) $\nabla_X (fY) = f \nabla_X Y + (Xf)Y$ *(Leibniz rule)*
(3) $\nabla_X Y - \nabla_Y X = [X, Y]$ *(symmetry)*
(4) $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$ *(metric connection).*

Here X, Y, Z are vector fields and f is a smooth function on M.

PROOF. If ∇ satisfies (1)–(4), it is possible to derive the following identity known as Koszul's formula:

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle = {} & X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle \\ & + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle. \end{aligned} \tag{6.1}$$

It turns out that this identity defines a unique bilinear map satisfying (1)–(4). See [**Le1**] for the details. ☐

The map ∇ is called the *Riemannian connection* or *Levi-Civita connection* of (M, g). The vector field $\nabla_X Y$ is called the *covariant derivative* of the vector field Y in direction X.

EXAMPLE. In $(\mathbb{R}^n, e)$ the Levi-Civita connection is given by

$$\nabla_X Y = X^j(\partial_j Y^k)\partial_k.$$

This is just the natural derivative of Y in direction X.

EXAMPLE. On a general Riemannian manifold (M, g), applying Koszul's formula (6.1) to coordinate vector fields gives that

$$\begin{aligned} 2\langle \nabla_{\partial_j}\partial_k, \partial_l\rangle &= \partial_j\langle \partial_k, \partial_l\rangle + \partial_k\langle \partial_j, \partial_l\rangle - \partial_l\langle \partial_j, \partial_k\rangle \\ &= \partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}. \end{aligned}$$

It follows that

$$\nabla_{\partial_j}\partial_k = \Gamma_{jk}^l \partial_l$$

where Γ_{jk}^l are the Christoffel symbols

$$\Gamma_{jk}^l = \frac{1}{2} g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}).$$

For any two vector fields $X = X^j \partial_j$ and $Y = Y^k \partial_k$, one has

$$\nabla_X Y = X^j(\partial_j Y^k)\partial_k + X^j Y^k \Gamma_{jk}^l \partial_l.$$

Covariant derivative of tensors. At this point we will define the connection and covariant derivatives also for other tensor fields. Let X be a vector field on M. The covariant derivative of 0-tensor fields is given by

$$\nabla_X f := Xf.$$

For k-tensor fields u, the covariant derivative is defined by

$$\nabla_X u(Y_1, \ldots, Y_k) := X(u(Y_1, \ldots, Y_k)) - \sum_{j=1}^{k} u(Y_1, \ldots, \nabla_X Y_j, \ldots, Y_k).$$

EXERCISE 6.1. Show that these formulas give a well defined covariant derivative

$$\nabla_X : C^\infty(M, T^k M) \to C^\infty(M, T^k M).$$

EXAMPLE. An example of the above construction is the covariant derivative of 1-forms, which is uniquely specified by the identity

$$\nabla_{\partial_j} dx^k = -\Gamma_{jl}^k dx^l.$$

By using ∇_X on tensors, it is possible to define the *total covariant derivative* as the map

$$\nabla : C^\infty(M, T^k M) \to C^\infty(M, T^{k+1} M),$$
$$\nabla u(X, Y_1, \ldots, Y_k) := \nabla_X u(Y_1, \ldots, Y_k).$$

EXAMPLE. On 0-forms $\nabla f = df$.

EXAMPLE. If f is a smooth function, then the *covariant Hessian* of f is

$$\mathrm{Hess}(f) := \nabla^2 f.$$

In local coordinates it is given by

$$\nabla^2 f = (\partial_{jk} f - \Gamma_{jk}^l \partial_l f)\, dx^j \otimes dx^k.$$

Finally, we mention that the total covariant derivative can be used to define higher order Sobolev spaces invariantly on a Riemannian manifold.

DEFINITION. If $k \geq 0$, consider the inner product on $C^\infty(M)$ given by

$$(u, v)_{H^k(M)} := \sum_{j=0}^{k} (\nabla^j u, \nabla^j v)_{L^2(M)}.$$

Here the L^2 norm is the natural one using the inner product on tensors. The Sobolev space $H^k(M)$ is defined to be the completion of $C^\infty(M)$ with respect to this inner product. This coincides with the earlier definition which was based on local coordinates.

The next result says that the Riemannian connection is invariant under isometries. In particular, this will imply that the curvature tensors constructed via ∇ will also be invariant under isometries.

LEMMA 6.7. *If F is a diffeomorphism, then if T is any tensor or vector field one has*

$$F^*(\nabla_g T) = \nabla_{F^* g} F^* T.$$

Geodesics. Let us return to length minimizing curves. If $\gamma : [a, b] \to M$ is a curve and $X : [a, b] \to TM$ is a smooth vector field along γ (meaning that $X(t) \in T_{\gamma(t)} M$), we define the derivative of X along γ by

$$\nabla_{\dot\gamma} X := \nabla_{\dot\gamma} \tilde{X}$$

where $\tilde{X}$ is any vector field defined in a neighborhood of $\gamma([a,b])$ such that $\tilde{X}_{\gamma(t)} = X_{\gamma(t)}$. It is easy to see that this does not depend on the choice of $\tilde{X}$. The relation to geodesics now comes from the fact that in local coordinates, if $\gamma(t)$ corresponds to $x(t)$,

$$\begin{aligned}\nabla_{\dot\gamma}\dot\gamma &= \nabla_{\dot x^j\partial_j}(\dot x^k\partial_k)\\ &= (\ddot x^l + \Gamma^l_{jk}(x)\dot x^j\dot x^k)\partial_l.\end{aligned}$$

Thus the geodesic equation is satisfied iff $\nabla_{\dot\gamma}\dot\gamma = 0$. We now give the precise definition of a geodesic.

Definition. A regular curve γ is called a geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$.

The arguments above give evidence to the following result. The first statement follows from Theorem 2.12 and the second statement is proved for instance in [**Le1**].

Theorem 6.8. *(Length minimizing curves) If γ is a piecewise regular length minimizing curve from p to q, then γ is regular and $\nabla_{\dot\gamma}\dot\gamma = 0$. Conversely, if γ is a regular curve and $\nabla_{\dot\gamma}\dot\gamma = 0$, then γ minimizes length at least locally.*

We next list some basic properties of geodesics.

Lemma 6.9. *(Properties of geodesics) Let (M, g) be a Riemannian manifold without boundary. Then*

1. *for any $p \in M$ and $v \in T_pM$, there is an open interval I containing 0 and a geodesic $\gamma_v : I \to M$ with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$,*
2. *any two geodesics with $\gamma_1(0) = \gamma_2(0)$ and $\dot\gamma_1(0) = \dot\gamma_2(0)$ agree in their common domain,*
3. *any geodesic satisfies $|\dot\gamma(t)| = const$,*
4. *if M is compact then any geodesic γ can be uniquely extended as a geodesic defined on all of $\mathbb{R}$.*

Exercise 6.2. Prove this theorem by using the existence and uniqueness of solutions to ordinary differential equations.

By (3) in the theorem, we may (and will) always assume that geodesics are parametrized by arc length and satisfy $|\dot\gamma| = 1$. Part (4) says that the maximal domain of any geodesic on a closed manifold is $\mathbb{R}$, where the maximal domain is the largest interval to which the geodesic can be extended. We will always assume that the geodesics are defined on their maximal domain.

Normal coordinates. The following important concept enables us to parametrize a manifold locally by its tangent space.

DEFINITION. If $p \in M$ let $\mathcal{E}_p := \{v \in T_pM\,;\, \gamma_v \text{ is defined on } [0,1]\}$, and define the *exponential map*

$$\exp_p : \mathcal{E}_p \to M, \quad \exp_p(v) = \gamma_v(1).$$

This is a smooth map and satisfies $\exp_p(tv) = \gamma_v(t)$. Thus, the exponential map is obtained by following radial geodesics starting from the point p. This parametrization also gives rise to a very important system of coordinates on Riemannian manifolds.

THEOREM 6.10. *(Normal coordinates) For any $p \in M$, $\exp_p$ is a diffeomorphism from some neighborhood V of 0 in T_pM onto a neighborhood of p in M. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM and we identify T_pM with $\mathbb{R}^n$ via $v^j e_j \leftrightarrow (v^1, \ldots, v^n)$, then there is a coordinate chart (U, φ) such that $\varphi = \exp_p^{-1} : U \to \mathbb{R}^n$ and*

(1) $\varphi(p) = 0$,
(2) *if $v \in T_pM$ then $\varphi(\gamma_v(t)) = (tv^1, \ldots, tv^n)$,*
(3) *one has*

$$g_{jk}(0) = \delta_{jk}, \quad \partial_l g_{jk}(0) = 0, \quad \Gamma^l_{jk}(0) = 0.$$

PROOF. The smoothness of the exponential map follows by expressing the geodesics starting near p in terms of the flow of a certain vector field, called the geodesic vector field, on TM. Then the fact that $\exp_p$ is smooth near 0 follows from the existence and uniqueness theorem for ODEs. It is a diffeomorphism near 0 since a short computation shows that the derivative $(\exp_p)_* : T_0(T_pM) \to T_pM$ is just the identity map under the identification $T_0(T_pM) = T_pM$. For details see [**Le1**]. □

The local coordinates in the theorem are called *normal coordinates* at p. In these coordinates geodesics through p correspond to rays through the origin, and thus these geodesics are called *radial geodesics.* Further, by (3) the metric and its first derivatives have a simple form at 0. This fact is often exploited when proving an identity where both sides are invariantly defined, and thus it is enough to verify the identity in some suitable coordinate system. The properties given in (3) sometimes simplify these local coordinate computations dramatically.

Finally, we will need the fact that when switching to polar coordinates in a normal coordinate system, the metric has special form in a full neighborhood of 0 instead of just at the origin.

THEOREM 6.11. *(Polar normal coordinates) Let* (U, φ) *be normal coordinates at* p. *If* (r, θ) *are the corresponding polar coordinates (thus* $r(q) = |\varphi(q)| > 0$ *and* $\theta(q)$ *is the corresponding direction in* S^{n-1}*), then the metric has the form*

$$(g_{jk}(r,\theta)) = \begin{pmatrix} 1 & 0 \\ 0 & g_{\alpha\beta}(r,\theta) \end{pmatrix}.$$

This implies that $\mathrm{grad}_g(r) = \partial/\partial r$, $|\partial/\partial r| = 1$, $\langle \partial/\partial r, \partial/\partial\theta \rangle = 0$, *and* $r(q) = d(p,q)$.

PROOF. This is essentially Theorem 6.10 combined with the Gauss lemma, which states that $\langle \partial/\partial r, \partial/\partial\theta \rangle = 0$. To prove the last statement, one shows that it holds at the origin and the inner product in question is constant along radial geodesics (this uses the symmetry of the Riemannian connection and the fact that geodesics have unit speed). For details see [**Le1**]. □

6.3. Curvature tensors

It is now possible to give a precise definition of the Riemann curvature tensor described earlier.

DEFINITION. If X, Y, Z, W are vector fields in some open set in M, the *Riemann curvature tensor* is defined by

$$Rm(X,Y,Z,W) := \langle (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z, W \rangle.$$

If $p \in M$ and the vector fields are defined near p, one can check that $Rm(X,Y,Z,W)|_p$ only depends on the values of the vector fields at p. Thus Rm is in fact a smooth 4-tensor field on M. If x are local coordinates and $\{\partial_i\}$ are corresponding coordinate vector fields, the tensor Rm has the coordinate representation

$$Rm = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

where

$$R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l) = \langle (\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_i})\partial_k, \partial_l \rangle.$$

We also define the Ricci tensor and scalar curvature, which are obtained from the Riemann tensor by taking traces with respect to certain indices.

DEFINITION. If $v, w \in T_pM$, we define the *Ricci tensor*

$$Ric(v,w) = \sum_{j=1}^{n} R(e_j, v, w, e_j)$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of T_pM. We also define the *scalar curvature*

$$S = \sum_{j=1}^{n} Ric(e_j, e_j).$$

It follows that Ric is a smooth 2-tensor field and S is a smooth function on M. The Ricci tensor has coordinate representation

$$Ric = R_{jk}\, dx^j \otimes dx^k, \quad R_{jk} = g^{il} R_{ijkl},$$

and the scalar curvature has coordinate representation

$$S = g^{jk} R_{jk}.$$

The Riemann curvature tensor has the following basic symmetries [**Le1**, Proposition 7.4]:

(a) $Rm(X,Y,Z,W) = -Rm(Y,X,Z,W)$
(b) $Rm(X,Y,Z,W) = -Rm(X,Y,W,Z)$
(c) $Rm(X,Y,Z,W) = Rm(Z,W,X,Y)$
(d) $Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0.$

Here (a) is trivial, (b) follows since ∇ is compatible with the metric, (d) follows since ∇ is symmetric, and (c) follows by combining the other symmetries. The identity in (d) is called the first Bianchi identity. These are all the algebraic symmetries of the curvature tensor, since any 4-tensor satisfying (a)–(d) at a point p can be realised as the curvature tensor at p of some Riemannian metric. There is an additional differential symmetry called the second Bianchi identity.

The following related notion allows to connect the above abstract definitions to geometry:

DEFINITION. Let $p \in M$. We define the *sectional curvature* at p for any 2-plane $\Pi \subset T_pM$ by

$$K(\Pi) := \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - \langle X, Y \rangle}$$

where $X, Y \in T_pM$ are any vectors with $\Pi = \mathrm{span}\{X, Y\}$ (the definition is independent of the choice of X and Y).

To illustrate the above notions, we give a list of facts (without proofs) related to curvature tensors.

1. The Riemann curvature tensor at p and the sectional curvatures $\{K(\Pi)\,;\,\Pi \subset T_pM \text{ 2-plane}\}$ are equivalent information. The proof is a simple algebraic argument using the symmetries (a)–(d) of the curvature tensor [**Le1**, Lemma 8.9].
2. If (M, g) is a 2-dimensional manifold, then any T_pM is 2-dimensional. The *Gaussian curvature* of (M, g) is defined to be the function

$$K(p) := K(T_pM), \quad p \in M.$$

   On 2D manifolds, the Gaussian curvature completely determines the Riemann, Ricci and scalar curvatures [**Le1**, Lemma 8.7]:

$$\begin{aligned} R_{ijkl} &= K(g_{il}g_{jk} - g_{ik}g_{jl}), \\ R_{jk} &= Kg_{jk}, \\ S &= 2K. \end{aligned}$$

3. On 3D manifolds, the Ricci tensor completely determines the Riemann curvature tensor:

$$Rm = (Ric - \frac{S}{3}g) \circ g + \frac{S}{12} g \circ g$$

   where $\circ$ is the Kulkarni-Nomizu product.
4. If (M, g) is any Riemannian manifold, the sectional curvature $K(\Pi)$ equals the Gaussian curvature of the 2-dimensional manifold M_Π obtained by following geodesics with initial direction in Π [**Le1**, Proposition 8.8].
5. One has $Rm \equiv 0$ near p (equivalently, all sectional curvatures vanish near p) if and only if some neighbourhood of p is isometric to a subset of Euclidean space [**Le1**, Theorem 7.3].
6. The sphere $S^n_R := \{x \in \mathbb{R}^{n+1}\,;\, |x| = R\}$ with its canonical metric (the metric induced by the Euclidean metric in $\mathbb{R}^{n+1}$) is a Riemannian manifold whose sectional curvatures are all equal to $1/R^2$.
7. The hyperbolic space $H^n_R = \{x \in \mathbb{R}^n\,;\, |x| < R\}$ with metric $g_{jk} = \frac{4R^4}{(R^2-|x|^2)^2}\delta_{jk}$ is a Riemannian manifold with sectional curvatures equal to $-1/R^2$.

8. The model spaces $\mathbb{R}^n, S^n_R, H^n_R$ and their quotients are the only connected complete Riemannian n-manifolds with constant sectional curvature [**Le1**, Corollary 11.13].

6.4. Curvature bounds

The purpose in this section is to indicate how curvature bounds affect various properties of manifolds. There is a large literature on this topic, see for instance [**Pe**] and the references therein. Relevant bounds include upper and lower bounds for the following quantities:

- sectional curvatures
- Ricci tensor
- scalar curvature
- diameter
- volume
- injectivity radius

Suitable bounds on these quantities put certain restrictions on e.g. the

- topological properties (compactness, fundamental group, Betti numbers, homeomorphism type)
- metric and geometric properties (diameter, volume growth, isometry group)
- analytic properties (isoperimetric/Sobolev/Poincaré inequalities, heat kernel estimates)

of the manifold in question.

A few simple ideas to keep in mind in this context:

- sectional curvature bounds are stronger than Ricci curvature bounds
- Ricci curvature bounds are stronger than scalar curvature bounds
- positive curvature causes geodesics to converge
- negative curvature causes geodesics to spread out
- curvature bounds sometimes allow to compare properties of a manifold to properties of a constant curvature manifold

Here are just a few examples of results with sectional curvature lower bounds ($K \geq a$ means that $K(\Pi) \geq a$ for all 2-planes $\Pi \subset T_pM$ for all $p \in M$):

THEOREM 6.12. *Let (M, g) be a connected complete n-dimensional Riemannian manifold.*

(1) *(Bonnet-Myers 1935) If $K \geq \delta > 0$, then M is compact and has finite fundamental group.*
(2) *(Sphere theorem, Brendle-Schoen 2007) If (M, g) is simply connected and $\frac{1}{4} < K \leq 1$, then M is diffeomorphic to S^n.*
(3) *(Finiteness of Betti numbers, Gromov 1981) If $K \geq 0$, then $\chi(M) \leq C(n)$. Moreover, if $K \geq -k^2$ and $\mathrm{diam} \leq D$, then $\chi(M) \leq C(n, D, n)$.*

Here are examples of results where the weaker Ricci curvature lower bounds are sufficient to obtain some control ($Ric \geq a$ means that $Ric(v, v) \geq a|v|^2$ for all $v \in TM$):

THEOREM 6.13. *Let (M, g) be a connected complete n-dimensional Riemannian manifold.*
(1) *(Myers 1941) If $Ric \geq \delta > 0$, then M is compact with finite fundamental group.*
(2) *(Hamilton 1982) If (M, g) is a compact simply connected 3-manifold and if $Ric > 0$, then M is diffeomorphic to S^3.*
(3) *(Bochner 1948) If (M, g) is compact oriented and $Ric \geq 0$, then $b_1(M) \leq n$.*

In the remainder of this text, we focus on lower bounds for Ricci curvature. In particular, we prove the Bochner vanishing theorem and Myers' theorem, and also discuss the important Bishop-Gromov volume comparison method. The presentation partly follows [**Pe**] and [**Zh**].

A basic tool for exploiting Ricci curvature lower bounds is the following identity due to Bochner.

LEMMA 6.14. *(Bochner identity) If $u \in C^3(M)$, then*

$$\Delta(\frac{1}{2}|\nabla u|^2) = |\nabla^2 u|^2 + \langle \nabla(\Delta u), \nabla u \rangle + Ric(\nabla u, \nabla u).$$

REMARK. The identity is often applied to harmonic functions (so $\Delta u = 0$) or to distance functions (so $|\nabla u|^2 \equiv 1$): in both cases one term drops out, the term $|\nabla^2 u|^2$ is nonnegative, and having a bound for the Ricci term will lead to very useful inequalities.

PROOF. We will use the "Ricci calculus" for tensor computations: vector fields are written as X^k and tensor fields as $T_{j_1 \cdots j_k}$, covariant derivatives are written as

$$\nabla_i T_{j_1 \cdots j_k} = (\nabla T)_{i j_1 \cdots j_k}.$$

We will also raise and lower indices freely via g, and these operations commute with each ∇_i by the compatibility of ∇ with g. Under these conventions, we have $\Delta = \nabla^i \nabla_i$ and the commutation formula for covariant derivatives acting on 1-forms is

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\theta_k = -R_{ijkl}\theta^l.$$

We now compute

$$\begin{aligned}
&\nabla^i \nabla_i (\frac{1}{2}\nabla^j u \nabla_j u) = \frac{1}{2}\nabla^i (\nabla_i \nabla^j u \nabla_j u + \nabla^j u \nabla_i \nabla_j u) \\
&= |\nabla^2 u|^2 + \nabla^i \nabla_j \nabla_i u \nabla^j u \\
&= |\nabla^2 u|^2 + \nabla_j \nabla^i \nabla_i u \nabla^j u - R^i{}_{jil} \nabla^l u \nabla^j u \\
&= |\nabla^2 u|^2 + \langle \nabla(\Delta u), \nabla u \rangle + Ric(\nabla u, \nabla u).
\end{aligned}$$

□

We now invoke the Bochner identity applied to a certain harmonic function to prove that $Ric \geq 0$ implies a bound on the first Betti number.

THEOREM 6.15. *(Bochner vanishing theorem) Suppose that (M, g) is a compact oriented n-manifold. If $Ric \geq 0$, then $b_1(M) \leq n$. Moreover, if $Ric \geq 0$ and $Ric|_p > 0$ at some point p, then $b_1(M) = 0$.*

PROOF. By Hodge theory (Theorem 5.5) we have $b_1(M) = \dim \mathcal{H}_1$, so it is enough to study harmonic 1-forms in M. Let $\omega \in \mathcal{H}_1$, so that ω is a 1-form with $d\omega = \delta\omega = 0$. We claim the Bochner-type identity

$$\Delta(\frac{1}{2}|\omega|^2) = |\nabla\omega|^2 + Ric(\omega, \omega). \tag{6.2}$$

To prove this fix a point $q \in M$, and choose a coordinate neighbourhood U of q whose image in $\mathbb{R}^n$ is a ball. Since $d\omega = 0$ in U, the Poincaré lemma (Lemma 2.9) shows that there is $u \in C^\infty(U)$ so that

$$\omega = du \text{ in } U,$$

and thus $\Delta u = -\delta du = -\delta\omega = 0$ in U. Bochner's identity applied to u in U implies (6.2) near q, but since q was arbitrary we have that (6.2) holds in M.

We now integrate (6.2) over M. Observing that $\int_M \Delta f \, dV = 0$ for any smooth function f[1], we obtain

$$\int_M (|\nabla\omega|^2 + Ric(\omega, \omega)) \, dV = 0.$$

[1]since $\int_M \Delta f \, dV = (\Delta f, 1)_{L^2} = -(df, \delta(1))_{L^2} = 0$

Since $Ric \geq 0$, both terms in the integrand are nonnegative and we get the following identities in M:

$$\nabla\omega \equiv 0, \qquad Ric(\omega,\omega) \equiv 0.$$

Writing $Y = \omega^\flat$ for the vector field corresponding to ω, the first condition means that

$$\nabla_X Y = 0$$

for all vector fields X in M. Thus Y is a *parallel vector field*, and in particular it is constant along any curve. For any $q \in M$, the vector field Y is completely determined by its value at q, so the map

$$J_q : \mathcal{H}_1 \to T_q M, \quad \omega \mapsto \omega^\flat(q)$$

is injective. This proves that $b_1(M) \leq n$. If additionally $Ric|_p > 0$ for some p, then the condition $Ric(\omega,\omega) \equiv 0$ implies that $\omega(p) = 0$, so any harmonic form is $\equiv 0$ showing that $b_1(M) = 0$. □

Next we apply the Bochner identity to an eigenfunction, in order to control the constant in an L^2 Poincaré inequality by a Ricci lower bound.

THEOREM 6.16. *(Lichnerowicz 1958) Let (M,g) be a compact oriented n-manifold. If $Ric \geq (n-1)H > 0$, then*

$$nH\|u\|^2_{L^2(M)} \leq \|\nabla u\|^2_{L^2(M)}, \quad u \in H^1(M), \quad \int_M u\,dV = 0.$$

The constant is optimal, as is shown by the sphere of radius $\frac{1}{\sqrt{H}}$.

We need a simple lemma that will also be useful later:

LEMMA 6.17. *If (M,g) is n-dimensional and if $u \in C^2(M)$, then*

$$|\nabla^2 u|^2|_p \geq \frac{(\Delta u)^2|_p}{n-m}$$

if $\nabla^2 u|_p$ has at least m zero eigenvalues where $0 \leq m \leq n-1$. Equality holds iff the remaining $n-m$ eigenvalues are all equal.

PROOF. Fix geodesic normal coordinates x near p, so the 1-forms $\{dx^1, \ldots, dx^n\}$ are orthonormal at p. Then

$$\nabla^2 u|_p = a_{jk} dx^j \otimes dx^k$$

where the matrix $A = (a_{jk})_{j,k=1}^n$ is symmetric and has eigenvalues $\lambda_1, \ldots, \lambda_n$. We may choose $\lambda_1 = \ldots = \lambda_m = 0$. Then by Cauchy-Schwarz

$$\begin{aligned}|\nabla^2 u|^2|_p &= \mathrm{tr}(A^t A) = \lambda_{m+1}^2 + \ldots + \lambda_n^2 \\ &\geq \frac{(\lambda_{m+1} + \ldots + \lambda_n)^2}{n-m} = \frac{(\mathrm{tr}(A))^2}{n-m} = \frac{(\Delta u)^2|_p}{n-m}\end{aligned}$$

with equality iff $\lambda_{m+1} = \ldots = \lambda_n$. □

Proof of Theorem 6.16. Denote by λ_1 the first positive eigenvalue of the Laplace-Beltrami operator $-\Delta$. We will prove that

$$\lambda_1 \|u\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2, \qquad u \in H^1(M), \ \int_M u \, dV = 0 \tag{6.3}$$

and

$$\lambda_1 \geq nH. \tag{6.4}$$

The result follows by combining these facts.

We may assume that M is connected (otherwise argue on each connected component). The spectral theory for the Hodge Laplacian in Chapter 5, specialized to 0-forms, shows that there is a sequence $\{\lambda_j\}_{j=0}^\infty$ with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$$

and an orthonormal basis $\{\phi_j\}_{j=0}^\infty$ of $L^2(M)$ such that

$$-\Delta \phi_j = \lambda_j \phi_j.$$

Here $\lambda_0 = 0$ is a simple eigenvalue and the corresponding eigenfunction ϕ_0 is constant, since M is connected and $\mathrm{Ker}(-\Delta))$ consists of the locally constant functions by Theorem 5.3. Now, if $u \in H^1(M)$ satisfies $(u, 1)_{L^2} = 0$, and if additionally $u = \sum_{j=0}^N c_j \phi_j$, then $c_0 = 0$ and

$$\|\nabla u\|_{L^2}^2 = (du, du)_{L^2} = -(\Delta u, u)_{L^2} = \sum_{j=1}^N \lambda_j c_j^2 \geq \lambda_1 \sum_{j=1}^N c_j^2 = \lambda_1 \|u\|_{L^2}^2.$$

Since any $u \in H^1(M)$ can be approximated in the H^1 norm by finite sums of eigenfunctions, we obtain (6.3).

Take now $u = \phi_1$ to be an eigenfunction corresponding to λ_1:

$$-\Delta u = \lambda_1 u.$$

We will prove (6.4) by applying the Bochner identity to u. Indeed, the Bochner identity gives

$$\Delta(\frac{1}{2}|\nabla u|^2) = |\nabla^2 u|^2 - \lambda_1 |\nabla u|^2 + Ric(\nabla u, \nabla u).$$

We integrate this identity over M. Since $\int_M \Delta f \, dV = 0$, we get

$$0 = \int_M (|\nabla^2 u|^2 - \lambda_1 |\nabla u|^2 + Ric(\nabla u, \nabla u)) \, dV.$$

By Lemma 6.17 (with $m = 0$) we have $|\nabla^2 u|^2 \geq \frac{(\Delta u)^2}{n} = \frac{\lambda_1^2}{n} u^2$, and by assumption $Ric(\nabla u, \nabla u) \geq (n-1)H|\nabla u|^2$. Thus we get

$$0 \geq \frac{\lambda_1^2}{n} \int_M u^2 \, dv + ((n-1)H - \lambda_1) \int_M |\nabla u|^2 \, dV.$$

Since

$$\int_M |\nabla u|^2 \, dV = (du, du)_{L^2} = (-\Delta u, u)_{L^2} = \lambda_1 \int_M u^2 \, dV,$$

we obtain (6.4). □

Our final aim is to sketch the proof the Bishop-Gromov volume comparison results. Along the way, we will also prove Myers' theorem.

THEOREM 6.18. *Let (M, g) be a complete Riemannian n-manifold, and let $Ric \geq (n-1)H$ for some $H \in \mathbb{R}$.*

1. *(Bishop volume comparison)*

$$\mathrm{Vol}_g(B(p,r)) \leq \mathrm{Vol}_H(B(r)) \text{ for } r > 0.$$

2. *(Gromov relative volume comparison)*

$$\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_H(B(r))} \text{ is nonincreasing for } r > 0.$$

Here $\mathrm{Vol}_H(B(r))$ is the volume of a ball of radius r in the model space with constant curvature H.

To prove this result, we will apply the Bochner identity to distance functions $q \mapsto d_g(q,p)$ for fixed p. Let (M, g) be a Riemannian manifold and $p \in M$. Recall from Theorems 6.10 and 6.11 that geodesic normal coordinates are defined in some neighbourhood U of p, and if (r, θ) are corresponding polar normal coordinates then r is smooth in $U \setminus \{p\}$ and Lipschitz in U. One has the following properties:

$$r(q) = d_g(q,p), \qquad \nabla r = \frac{\partial}{\partial r}, \qquad |\nabla r| = 1,$$
$$g(r,\theta) = \begin{pmatrix} 1 & 0 \\ 0 & g_0(r,\theta) \end{pmatrix}.$$

The main tool is the following result.

THEOREM 6.19. *(Laplacian comparison) If Ric* $\geq (n-1)H$, *then*

$$\Delta r \leq \Delta_H r \text{ in } U \setminus \{p\},$$

where Δ_H *is the Laplace operator of the model space with constant curvature* H.

PROOF. The Bochner identity applied to $u = r$ in $U \setminus \{p\}$ gives

$$|\nabla^2 r|^2 + \frac{\partial}{\partial r}(\Delta r) + Ric(\nabla r, \nabla r) = 0.$$

The form of the metric $g(r,\theta)$ implies $\nabla_{\partial_r} dr = 0$ upon computing Christoffel symbols. Thus for any X

$$(\nabla^2 r)(\partial_r, X) = (\nabla_{\partial_r} dr)(X) = 0,$$

which implies that one eigenvalue of $\nabla^2 r$ is zero. By Lemma 6.17,

$$|\nabla^2 r|^2 \geq \frac{(\Delta r)^2}{n-1}.$$

The condition $Ric(\nabla r, \nabla r) \geq (n-1)H|\nabla r|^2 = (n-1)H$ now implies

$$\frac{(\Delta r)^2}{n-1} + \frac{\partial}{\partial r}(\Delta r) + (n-1)H \leq 0.$$

The left hand side contains a Riccati type expression for Δr.

If we had done the computation above for a metric with constant curvature H, all the inequalities above would have been equalities. This shows that

$$\frac{(\Delta_H r)^2}{n-1} + \frac{\partial}{\partial r}(\Delta_H r) + (n-1)H = 0.$$

A simple comparison for the two Riccati ODE now implies the required inequality $\Delta r \leq \Delta_H r$. ☐

At this point we can prove Myers' theorem.

THEOREM 6.20. *(Myers 1941) Let (M, g) be a connected complete n-dimensional Riemannian manifold. If $Ric \geq \frac{n-1}{R^2}$ where $R > 0$, then (M, g) has diameter $\leq \pi R$, M is compact, and M has finite fundamental group.*

PROOF. The main point is to prove the diameter estimate. We argue by contradiction and assume that the diameter is $> \pi R$. Since (M, g) is complete, there are points $p, p_1 \in M$ and a minimizing unit speed geodesic $\gamma : [0, L] \to M$ with $\gamma(0) = p$, $\gamma(L) = p_1$, and $L > \pi R$. (Here we used the Hopf-Rinow theorem [**Pe**, Section 5.8].) Letting $r(q) = d_g(q, p)$, the fact that γ is minimizing implies that r is smooth near $\gamma((0, \pi R])$ [**Pe**, Section 5.9]. Thus

$$\Delta r \leq \Delta_H r \text{ near } \gamma(\pi R)$$

where $H = 1/R^2$. But since $R > 0$, one can compute that

$$\Delta_H r = (n-1)\sqrt{H} \cot \sqrt{H} r$$

and thus

$$\lim_{r \to \pi R-} \Delta r \leq \lim_{r \to \pi R-} \Delta_H r = -\infty.$$

This contradicts the fact that Δr was smooth near $\gamma(\pi R)$.

We have now proved that the diameter of (M, g) is $\leq \pi R$. Now $M = \exp_p(\overline{B_{\pi R}(0)})$, so M is compact as the continuous image of a compact set. To prove the statement about the fundamental group, observe that the universal cover of M is also complete and satisfies the same Ricci lower bound, hence has finite diameter and is compact. There is a bijective map between the fundamental group of M and the inverse image of any $p \in M$ in the universal cover. The last set is discrete, hence compactness of the universal cover implies that the fundamental group is finite. □

To conclude, we sketch the proof of the volume comparison results.

PROOF OF THEOREM 6.18. (Sketch) It is possible to derive other expressions for Δr. For example, writing the volume form in polar normal coordinates as

$$dV = A(r, \theta)\, dr \wedge d\theta$$

where $d\theta$ is the standard volume form on S^{n-1}, one can check (see [**Zh**] for the details) that

$$\Delta r(r,\theta) = \frac{\partial_r A(r,\theta)}{A(r,\theta)}.$$

Thus the Laplacian comparison result (Theorem 6.19) implies

$$\frac{\partial_r A(r,\theta)}{A(r,\theta)} \leq \frac{\partial_r A_H(r,\theta)}{A_H(r,\theta)}$$

where A_H is the corresponding quantity for a constant curvature H metric. The previous inequality can be written as $\partial_r(\log \frac{A}{A_H}) \leq 0$, which gives that

$$r \mapsto \frac{A(r,\theta)}{A_H(r,\theta)} \text{ is nonincreasing for each } \theta.$$

In particular, since $\frac{A}{A_H} \to 1$ as $r \to 0$ (the metric becomes Euclidean as we approach the origin in normal coordinates), we have

$$A(r,\theta) \leq A_H(r,\theta).$$

Since A and A_H are infinitesimal volume elements, integrating the last two inequalities proves the Bishop and Gromov comparison results (a) and (b) at least for small $r > 0$. An additional argument, related to looking at the set where r is not smooth (i.e. the cut locus), proves (a) and (b) for all $r > 0$. □

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