INVERSE PROBLEMS FOR ELLIPTIC EQUATIONS WITH FRACTIONAL POWER TYPE NONLINEARITIES

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Abstract. We study inverse problems for semilinear elliptic equations with fractional power type nonlinearities. Our arguments are based on the higher order linearization method, which helps us to solve inverse problems for certain nonlinear equations in cases where the solution for a corresponding linear equation is not known. By using a fractional order adaptation of this method, we show that the results of [LLLS20a, LLLS20b] remain valid for general power type nonlinearities.

Keywords. Inverse boundary value problem, Calderón problem, partial data, semilinear elliptic equations, higher order linearization, transversally anisotropic manifold.

Contents

1. Introduction

In this work we study inverse problems for semilinear elliptic equations with fractional power type nonlinearities, extending the earlier results in [LLLS20a, LLLS20b] from integer powers to fractional powers. Here, when we say r is fractional we mean $r \in \mathbb{R} \setminus \mathbb{Z}$. Let $r > 1$ be fractional and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$, for $n \geq 2$. Consider the semilinear elliptic equation

$$\begin{cases} \Delta u + q(x)|u|^{r-1}u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $q \in C^\alpha(\overline{\Omega})$ is a potential function and C^α is the space of α-Hölder continuous functions. By assuming a suitable *smallness condition* on the boundary data f, one can obtain the well-posedness of the Dirichlet problem (1.1) for small solutions (see Section 2). One can then define the corresponding *Dirichlet-to-Neumann* (DN) map Λ_q of (1.1) by

$$\Lambda_q : C^{2,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega), \qquad f \mapsto \partial_\nu u_f|_{\partial\Omega},$$

for some $0 < \alpha < 1$, where $u_f \in C^{2,\alpha}(\overline{\Omega})$ is the unique small solution of (1.1), and ν is the unit outer normal on $\partial\Omega$. We will consider the following problem:

- **Inverse Problem 1:** Determine the potential q from the knowledge of Λ_q.

A typical method in the study of inverse boundary value problems for nonlinear elliptic equations was initiated by Isakov [Isa93], where he introduced the first linearization of the given (nonlinear) DN map. More precisely, the first linearization allows one to reduce the nonlinear equations to the linear equations, and one can adapt some known results for the linear equations to solve certain inverse problems for the nonlinear equations. Meanwhile, the second order linearization has been successfully applied in solving inverse problems, see [AZ17, CNV19, KN02, Sun96, SU97].

Throughout this paper the number $r > 1$ is fractional, and the solution u is real valued but may change sign, so it is natural to consider $q(x)|u|^{r-1}u$ instead of $q(x)u^r$ to have well-defined nonlinear term. Note also that at least when $n = 1$ the case $0 < r < 1$ would roughly correspond to the second order differential equation $u'' = F(u)$, where F is not Lipschitz. In this case, it is well-known that uniqueness of solutions can fail, so the assumption $r > 1$ is reasonable. Let us write $r = k + \alpha > 1$ for some $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ in the rest of this work.

In case of $r = m \in \mathbb{N}$ and nonlinear term $q(x)u^m$, corresponding inverse problems were first investigated in [FO20, LLLS20a], and related problems have been further studied in many works. For example, the articles [LLLS20b, KU20c, KU20b] studied related inverse problems for semilinear elliptic equations with partial data. In [LL20, Lin20, LO20], the authors studied inverse problems for fractional semilinear elliptic equations. In [LZ20, KU20a, CF20, KKU20], the authors studied partial data inverse problems for the nonlinear magnetic Schrödinger and conductivity equations. The nonlinearities in these articles are typically integer power type, or holomorphic in u and ∇u (i.e. sums of integer powers).

The main tool in solving these inverse problems is based on the *higher order linearization* technique, where one introduces extra small parameters for the Dirichlet data to reduce inverse problems for nonlinear elliptic equations into statements involving solutions of simpler linear elliptic equations. In the case of nonlinearity $q(x)u^m$ where $m \in \mathbb{N}$, this just means that we are looking at the mth order Fréchet derivative of the nonlinear measurement operator. For a nonlinearity of fractional order $r = k+\alpha$, we will in some sense need to use the αth fractional derivative of the kth Fréchet derivative instead. A somewhat related method was used in [CK20] for a p-Laplace type equation. Thanks to the higher order linearization method, one may solve related inverse problems for certain semilinear elliptic equations in cases where the analogous problems for the corresponding linear equations still remain open.

Let us state our first main result to answer Inverse Problem 1:

Theorem 1.1 (The Calderón problem with full data)**.** *Let $\Omega \subset \mathbb{R}^n$ be a connected bounded domain with C^∞-smooth boundary $\partial\Omega$, for $n \geq 2$. Let $r > 1$ be a fractional number, $q_j \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, and Λ_{q_j} be the DN map of*

$$\begin{cases} \Delta u_j + q_j|u_j|^{r-1}u_j = 0 & \text{in } \Omega, \\ u_j = f & \text{on } \partial\Omega, \end{cases}$$

for $j = 1, 2$. Assume that $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$, for all $f \in C^{2,\alpha}(\partial\Omega)$ with $\|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta$, where $\delta > 0$ is a sufficiently small number. Then

$$q_1 = q_2 \text{ in } \Omega.$$

Moreover, in dimensions $n \geq 3$ the statement holds true if we only assume that $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$ whenever $\|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta$ and $f \geq 0$.

We remark that in certain applications it is natural to consider nonnegative Dirichlet data (see e.g. [RZ18]). Theorem 1.1 applies in this case when $n \geq 3$.

However, the methods for proving the other main theorems in this paper require sign-changing solutions, and we do not know if those results are valid if one only has access to measurements for nonnegative Dirichlet data.

We briefly explain the higher order linearization in the fractional power case. Let (M,g) be a compact C^∞ Riemannian manifold with a C^∞ smooth boundary ∂M. Recall that Δ_g is the Laplace-Beltrami operator, given in local coordinates by

$$\Delta_g u = \frac{1}{\det(g)^{1/2}} \sum_{a,b=1}^n \frac{\partial}{\partial x_a}\left(\det(g)^{1/2} g^{ab} \frac{\partial u}{\partial x_b}\right),$$

where $g = (g_{ab}(x))$ and $g^{-1} = (g^{ab}(x))$. Throughout this work, we assume that $g = (g_{ab})$ is uniformly elliptic. Let $q \in C^\alpha(M)$. In Proposition 2.3 we will see that by setting the Dirichlet data as

$$f = \epsilon_0 f_0 + \ldots + \epsilon_k f_k$$

and differentiating the equation (1.1) with respect to $\epsilon' = (\epsilon_1, \ldots, \epsilon_k)$ we obtain a new equation

$$\Delta_g w^{\epsilon_0}(x) = -\partial_{\epsilon_1} \cdots \partial_{\epsilon_k} \left(q(x)|u_f|^{r-1} u_f \right)\Big|_{\epsilon'=0} \text{ in } M, \tag{1.2}$$

where $w^{\epsilon_0} := \partial_{\epsilon_1} \cdots \partial_{\epsilon_k} u_f|_{\epsilon'=0}$ and $w^{\epsilon_0}|_{\partial M} = \epsilon_0 f_0|_{\partial M}$.

Furthermore, eliminating ϵ_0^α on the both sides of (1.2), by taking the limit $\epsilon_0 \to 0$, we get

$$\epsilon_0^{-\alpha} w^{\epsilon_0} \to w \text{ in } C^{2,\alpha}(M), \qquad \text{as } \epsilon_0 \to 0,$$

where w solves

$$\Delta_g w = c_r q(x) \mathrm{sgn}(v_0)^{k-1} |v_0|^\alpha v_1 \cdots v_k \text{ in } M.$$

Here c_r is the constant given by $c_r = -r(r-1)\cdots(r-(k-1))$, $\mathrm{sgn}(v_0(x))$ is the sign of $v_0(x)$, and the functions v_ℓ are harmonic in M with the corresponding boundary values f_ℓ, for $\ell = 0, 1, \ldots, k$. Moreover, we will multiply this equation by an extra auxiliary harmonic function v_{k+1} in M with its boundary data $v_{k+1}|_{\partial M} = f_{k+1}$. Now integrating over M and using integration by parts, we see that from the knowledge of the DN map for the equation $\Delta_g u + q(x)|u|^{r-1}u = 0$ in M it is possible to determine the integrals

$$c_r \int_M q(x) \mathrm{sgn}(v_0)^{k-1} |v_0|^\alpha v_1 \cdots v_{k+1} \, dV.$$

It thus suffices to choose the boundary data f_ℓ for $\ell = 0, 1, \ldots, k$, so that $v_0 \neq 0$ in M and the scalar products $v_1 \cdots v_{k+1}$ become dense in a suitable function space. This recovers the function q (see Sections 3 and 4).

Next we study the Calderón problem with partial data for elliptic equations with fractional power type nonlinearities. Let $\Omega \subset \mathbb{R}^n$ be a connected bounded domain, and $\Gamma \subset \partial\Omega$ be a nonempty relatively open subset. By using the well-posedness of (1.1) (Proposition 2.1), one can define the corresponding partial DN map Λ_q^Γ of (1.1) by

$$\Lambda_q^\Gamma : C_0^{2,\alpha}(\Gamma) \to C^{1,\alpha}(\Gamma), \qquad f \mapsto \partial_\nu u_f|_\Gamma,$$

for some $0 < \alpha < 1$, where $u_f \in C^{2,\alpha}(\overline{\Omega})$ is the unique (small) solution of (1.1) (see Section 2) with $f \in C_0^{2,\alpha}(\Gamma)$. Then our second question is:

- **Inverse Problem 2:** Determine the potential q from the knowledge of Λ_q^Γ.

Our second main result is to solve Inverse Problem 2:

Theorem 1.2 (Partial data). *Let $\Omega \subset \mathbb{R}^n$ be a connected bounded domain with C^∞-smooth boundary $\partial\Omega$, for $n \geq 2$, and $\Gamma \subset \partial\Omega$ be a nonempty relatively open subset. Let $r > 1$ be a fractional number, $q_j \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, and $\Lambda^\Gamma_{q_j}$ be the DN map of*

$$\begin{cases} \Delta u_j + q_j |u_j|^{r-1} u_j = 0 & \text{in } \Omega, \\ u_j = f & \text{on } \partial\Omega, \end{cases}$$

for $j = 1, 2$. If $\Lambda^\Gamma_{q_1}(f) = \Lambda^\Gamma_{q_2}(f)$, for all $f \in C^{2,\alpha}_0(\Gamma)$ with $\|f\|_{C^{2,\alpha}_0(\Gamma)} < \delta$, where $\delta > 0$ is a sufficiently small number, then

$$q_1 = q_2 \text{ in } \Omega.$$

Moreover, one can consider more general nonlinear terms that are (asymptotic) sums of homogeneous functions. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$.

Definition 1.1. *Let r_l, $l \geq 1$, be real numbers with $1 < r_1 < r_2 < \dots$, and let $0 < \alpha < 1$. A function $a = a(x, y) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is polyhomogeneous, written*

$$a(x, y) \sim \sum_{l=1}^{\infty} b_l(x, y),$$

if each $b_l(\,\cdot\,, y) \in C^\alpha(\overline{\Omega})$ is positively homogeneous of degree r_l with respect to the y-variable, and if for any $N \geq 1$ there is $C_N > 0$ so that the function $\beta_N := a - \sum_{l=1}^{N-1} b_l$ (with $\beta_1 = a$) is in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}, C^\alpha(\overline{\Omega}))$ and satisfies

$$\|\beta_N(\,\cdot\,, y)\|_{C^\alpha(\overline{\Omega})} + |y| \, \|\partial_y \beta_N(\,\cdot\,, y)\|_{C^\alpha(\overline{\Omega})} \leq C_N |y|^{r_N}, \qquad |y| \leq 1. \tag{1.3}$$

We will assume that $1 + \alpha \leq r_1$ (this can be arranged by decreasing α).

Note that the above definition (using $N = 1$) implies that

$$a(x, 0) = \partial_y a(x, 0) = 0. \tag{1.4}$$

A typical example of polyhomogeneous function $a(x, y)$ is a finite sum

$$a(x, y) = \sum_{l=1}^{m} q_l(x) f_l(y),$$

where $q_l(x) \in C^\alpha(\overline{\Omega})$ and $f_l(y)$ is positively homogeneous of degree r_l, i.e. $f_l(\lambda y) = \lambda^{r_l} f_l(y)$ for $y \in \mathbb{R}$ and $\lambda > 0$. One could also consider infinite sums of this type. In fact, functions $a(x, y)$ that are C^α in x, holomorphic or antiholomorphic in y, and satisfy (1.4) are polyhomogeneous with $r_l = l + 1$ just by using Taylor expansions. It is worth emphasizing that since we are always considering small solutions, only the behaviour for small $|y|$ plays a role.

We also mention that the function $f(y) = |y|^{r-1} y$, at least roughly speaking, encompasses all positively homogeneous functions. Indeed, if f is positively homogeneous of degree $r > 0$, then f is of the form

$$f(y) = \begin{cases} y^r f(1), & \text{if } y \geq 0, \\ f(-|y|) = |y|^r f(-1), & \text{if } y < 0. \end{cases}$$

The case $f(y) = |y|^{r-1} y$ is obtained by taking $f(1) = 1$ and $f(-1) = -1$. This computation also shows that if $r = k + \alpha$ where $k \geq 1$ and $\alpha \in (0, 1)$, then $f(y)$ is C^k and $f^{(k)}(y)$ is C^α.

Let us consider the following Dirichlet problem in a bounded smooth domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} \Delta u + a(x,u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $a = a(x,y)$ is a polyhomogeneous function given by Definition 1.1. By Proposition 2.1, for any sufficiently small Dirichlet data $f \in C_0^{2,\alpha}(\Gamma)$ with $\Gamma \subset \partial\Omega$, one can define the corresponding (partial) DN map via

$$\Lambda_a^\Gamma : C_0^{2,\alpha}(\Gamma) \to C^{1,\alpha}(\Gamma), \qquad f \mapsto \partial_\nu u_f|_\Gamma \,,$$

for some $0 < \alpha < 1$, where $u_f \in C^{2,\alpha}(\overline{\Omega})$ is the unique small solution of (1.5). The inverse problem is to determine the unknown function $a(x,y)$.

Theorem 1.3 (Partial data for general coefficients)**.** *Let $\Omega \subset \mathbb{R}^n$ be a connected bounded domain with C^∞-smooth boundary $\partial\Omega$, for $n \geq 2$, and $\Gamma \subset \partial\Omega$ be a nonempty relatively open subset. Let us consider the equations*

$$\Delta u + a_j(x,u) = 0 \ \textit{in } \Omega, \tag{1.6}$$

for $j = 1,2$, where $a_j(x,y) \sim \sum_{l=1}^{\infty} b_{j,l}(x,y)$ is polyhomogeneous in the sense of Definition 1.1 where the orders $1 < r_1 < r_2 < \dots$ are the same for $j = 1,2$. Let $\Lambda_{a_j}^\Gamma : C_0^{2,\alpha}(\Gamma) \to C^{1,\alpha}(\Gamma)$ be the (partial) DN maps of (1.6), for $j = 1,2$. Assume that

$$\Lambda_{a_1}^\Gamma(f) = \Lambda_{a_2}^\Gamma(f),$$

for all $f \in C_0^{2,\alpha}(\Gamma)$ with $\|f\|_{C_0^{2,\alpha}(\Gamma)} < \delta$, where $\delta > 0$ is a sufficiently small number. Then we have

$$b_{1,l}(x,y) = b_{2,l}(x,y), \quad \textit{for } x \in \Omega, \ y \in \mathbb{R} \textit{ and } l \in \mathbb{N}.$$

In particular, if $b_{j,l}$ is of the form $b_{j,l}(x,y) = q_{j,l}|y|^{r_l - 1}y$, where $q_{j,l}(x) \in C^\alpha(\overline{\Omega})$, then

$$q_{1,l}(x) = q_{2,l}(x) \ \textit{in } \Omega, \quad \textit{for } l \in \mathbb{N}.$$

Theorem 1.3 corresponds to the recovery of the coefficients of the asymptotic series expansion of $a(x,y)$ in the y-variable. Note that numbers $r_1, r_2, \dots$ could also be integers ≥ 2. Therefore, we can regard Theorem 1.3 as a generalization of the corresponding Euclidean results in [LLLS20a, LLLS20b].

Inspired by the partial data results of inverse problems for semilinear elliptic equations [LLLS20b, KU20b], one can also consider the inverse boundary value problem of recovering an obstacle and coefficients simultaneously. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a connected C^∞-smooth boundary $\partial\Omega$. Let $D \Subset \Omega$ be an open set with C^∞-smooth boundary ∂D such that $\Omega \setminus \overline{D}$ is connected. Consider the boundary value problem

$$\begin{cases} \Delta u + a(x,u) = 0 & \text{in } \Omega \setminus \overline{D}, \\ u = 0 & \text{on } \partial D, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $a = a(x,y)$ is a polyhomogeneous function defined via Definition 1.1, for $x \in \Omega \setminus \overline{D}$.

As shown in Proposition 2.1, given any Dirichlet data $f \in C^{2,\alpha}(\partial\Omega)$ with $\|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta$, for some sufficiently small number $\delta > 0$, the equation (1.7) is well-posed and admits a unique (small) solution $u \in C^{2,\alpha}(\overline{\Omega} \setminus D)$. Let $\Gamma \subset \partial\Omega$ be an arbitrarily nonempty relatively open subset, then we can define the corresponding partial DN map $\Lambda^{\Gamma}_{a,D}$ by

$$\Lambda^{\Gamma}_{a,D} : C^{2,\alpha}(\Gamma) \to C^{1,\alpha}(\Gamma), \qquad f \mapsto \partial_\nu u_f|_\Gamma ,$$

for any $f \in C^{2,\alpha}_0(\Gamma)$ with sufficiently small $\|f\|_{C^{2,\alpha}_0(\Gamma)}$, where $u_f \in C^{2,\alpha}(\overline{\Omega}\setminus D)$ is the unique solution of (1.7). The following result is analogous to [LLLS20b, Theorem 1.2] and [KU20b, Theorem 1.6].

Theorem 1.4 (Simultaneous recovery: Unknown obstacle and coefficient)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded connected domain with connected C^∞ boundary $\partial\Omega$. Let $D_1, D_2 \Subset \Omega$ be nonempty open subsets with C^∞ boundaries such that $\Omega \setminus \overline{D_j}$ are connected. For $j = 1, 2$, let $a_j = a_j(x, y)$ be polyhomogeneous functions in $y \in \mathbb{R}$, for $x \in \overline{\Omega} \setminus D_j$. Denote by $\Lambda^{\Gamma}_{a_j,D_j}$ the partial DN maps of the following Dirichlet problems*

$$\begin{cases} \Delta u_j + a_j(x, u_j) = 0 & \text{in } \Omega \setminus \overline{D_j}, \\ u_j = 0 & \text{on } \partial D_j, \\ u_j = f & \text{on } \partial\Omega \end{cases}$$

defined for any $f \in C^{2,\alpha}_0(\Gamma)$ with $\|f\|_{C^{2,\alpha}_0(\Gamma)} < \delta$, where $\delta > 0$ is a sufficiently small number. Assume that

$$\Lambda^{\Gamma}_{a_1,D_1}(f) = \Lambda^{\Gamma}_{a_2,D_2}(f), \text{ for any } \|f\|_{C^{2,\alpha}_0(\Gamma)} < \delta.$$

Then

$$D := D_1 = D_2,$$

and

$$b_{1,l}(x, y) = b_{2,l}(x, y), \qquad \text{for } x \in \Omega \setminus \overline{D},\ y \in \mathbb{R} \text{ and } l \in \mathbb{N}.$$

Remark 1.2. *It is worth emphasizing that the simultaneous recovery of an embedded obstacle and the surrounding potentials in the linear setting, for example, the linear Schrödinger equation (i.e., for the case $r = 1$ in Theorem 1.4) is an open problem. We refer readers to* [Isa90, LLLS20b] *for further discussions and* [CLL19] *for arguments in a linear nonlocal setting.*

The proof of Theorem 1.4 is similar to the proof of Theorem 1.3, and the only difference is that we need to recover the unknown obstacle first. The method to recover the unknown obstacle has been investigated in [LLLS20b, Theorem 1.2]. We will give the proof in Section 4.

We are also able to extend the geometric results in [LLLS20a] to fractional power type nonlinearities. We refer to [LLLS20a] for the introduction of these problems.

Theorem 1.5 (Simultaneous recovery of metric and potential in the plane)**.** *Let (M_1, g_1) and (M_2, g_2) be two compact connected C^∞ Riemannian manifolds with mutual C^∞ boundary ∂M and $\dim(M_1) = \dim(M_2) = 2$. For $j = 1, 2$, let Λ_{M_j,g_j,q_j} be the DN maps of*

$$\Delta_{g_j} u + q_j |u|^{r-1} u = 0 \text{ in } M_j, \tag{1.8}$$

where $r > 1$ is a fractional number. Let $0 < \alpha < 1$ and assume that

$$\Lambda_{M_1,g_1,q_1}(f) = \Lambda_{M_2,g_2,q_2}(f) \text{ on } \partial M,$$

for any $f \in C^{2,\alpha}(\partial M)$ with $\|f\|_{C^{2,\alpha}(\partial M)} \leq \delta$, where $\delta > 0$ is a sufficiently small number. Then:

(1) *There exists a conformal diffeomorphism* $J : M_1 \to M_2$ *and a positive smooth function* $\sigma \in C^\infty(M_1)$ *such that*

$$\sigma J^* g_2 = g_1 \text{ in } M_1,$$

with $J|_{\partial M} = \mathrm{Id}$ *and* $\sigma|_{\partial M} = 1$.

(2) *Moreover, one can also recover the potential up to a natural gauge invariance in the sense that*

$$\sigma q_1 = q_2 \circ J \text{ in } M_1.$$

Furthermore, as shown in [LLLS20a] for integer power type nonlinearities, one can also consider the corresponding Calderón type inverse problem on a transversally anisotropic manifold. Let us consider inverse problems for the semilinear Schrödinger equation on *transversally anisotropic* manifold with fractional power type nonlinearities. The definition of a transversally anisotropic manifold is given as follows.

Definition 1.3. *Let* (M, g) *be a compact oriented manifold with a* C^∞ *boundary and with* $\dim M \geq 3$. (M, g) *is called* transversally anisotropic *if* $(M, g) \Subset (T, g)$*, where* $T = \mathbb{R} \times M_0$ *and* $g(x) = g(x_1, x') = e(x_1) \oplus g_0(x')$ *for* $x_1 \in \mathbb{R}$ *and* $x' \in M_0$*. Here* $(\mathbb{R}, e)$ *denotes the Euclidean line and* (M_0, g_0) *stands for an* $(n-1)$*-dimensional compact manifold with a smooth boundary.*

Theorem 1.6. *Let* (M, g) *be a transversally anisotropic manifold, let* $q_j \in C^\infty(M)$*, and let* Λ_{q_j} *be the DN maps for the equations*

$$\Delta_g u + q_j |u|^{r-1} u = 0 \text{ in } M$$

for $j = 1, 2$*, where we further assume the fractional number satisfies*

$$r > 3.$$

Suppose that the DN maps satisfy

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f) \text{ on } \partial M,$$

for all f *with* $\|f\|_{C^{2,\alpha}(\partial M)} \leq \delta$*, for a sufficiently small number* $\delta > 0$ *and for some* $0 < \alpha < 1$*. Then* $q_1 = q_2$ *in* M*.*

Theorems 1.5 and 1.6 follow from the corresponding arguments in [LLLS20a] if we use the integral identity (2.11) with the choice $v_0 = 1$ in M (by taking $f_0 = 1$ on ∂M).

The structure of this article is given as follows. In Section 2, we give well-posedness results for the relevant semilinear elliptic equations and derive the integral identity which plays a crucial role in the study of our inverse problems. In Section 3, we prove global uniqueness and simultaneous recovery in the Euclidean case, i.e., Theorems 1.1-1.4. Finally, we prove Theorems 1.5-1.6 in Section 4.

2. Preliminaries

First, let us recall the definition of Hölder spaces. Let $U \subset \mathbb{R}^n$ be an open set, let $k \in \mathbb{N} \cup \{0\}$, and let $0 < \alpha < 1$. The function space $C^{k,\alpha}(\overline{U})$ consists of those real valued functions $u \in C^k(\overline{U})$ for which the norm

$$\|f\|_{C^{k,\alpha}(\overline{U})} := \sum_{|\gamma| \leq k} \|\partial^\gamma f\|_{L^\infty(U)} + \sup_{x \neq y,\ x, y \in \overline{U}} \sum_{|\gamma| = k} \frac{|\partial^\gamma f(x) - \partial^\gamma f(y)|}{|x - y|^\alpha},$$

is finite. Here $\gamma = (\gamma_1, \cdots, \gamma_n)$ is a multi-index with $\gamma_i \in \mathbb{N} \cup \{0\}$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n$. Furthermore, we also denote the space

$$C_0^{k,\alpha}(\overline{U}) := \text{closure of } C_c^\infty(U) \text{ in } C^{k,\alpha}(\overline{U}).$$

In short, we only use $C^\alpha(\overline{U})$ to denote $C^{0,\alpha}(\overline{U})$ when $k=0$. In addition, one can define Hölder spaces on any Riemannian manifold (M,g) using the Riemannian distance or via local coordinates, see e.g. [Tay11, Section 13.8 in vol. III].

2.1. **Well-posedness.** Let (M,g) be a C^∞ compact Riemannian manifold with C^∞-smooth boundary ∂M. We study the well-posedness of the following boundary value problem

$$\begin{cases}\Delta_g u + a(x,u) = 0 & \text{in } M,\\ u = f & \text{on } \partial M,\end{cases} \tag{2.1}$$

for any sufficiently small Dirichlet data $f\in C^{2,\alpha}(\partial M)$, for some $0<\alpha<1$. Let us assume that the nonlinear coefficient $a=a(x,y)\in C^{k,\alpha}_{\mathrm{loc}}(\mathbb{R},C^\alpha(M))$ for some $k\geq 1$, meaning that $y\mapsto \partial_y^j a(\,\cdot\,,y)$ is a continuous map $\mathbb{R}\to C^\alpha(M)$ for $0\leq j\leq k$ and for any $R>0$, $\left\|\partial_y^k a(\,\cdot\,,y)-\partial_y^k a(\,\cdot\,,z)\right\|_{C^\alpha}\leq C_R|y-z|^\alpha$ whenever $|y|,|z|\leq R$. Also assume that the following two conditions hold:

$$a(x,0)=0, \quad \text{for } x\in M, \tag{2.2}$$

$$\text{The map } v\mapsto \Delta_g v+\partial_y a(\,\cdot\,,0)v \text{ is injective on } H_0^1(M). \tag{2.3}$$

We prove the well-posedness of (2.1) for small Dirichlet data $f\in C^{2,\alpha}(\partial M)$.

Proposition 2.1 (Well-posedness)**.** *Let (M,g) be a compact Riemannian manifold with C^∞ boundary ∂M and let Q be the semilinear elliptic operator*

$$Q(u):=\Delta_g u + a(x,u),$$

where $a\in C^{k,\alpha}_{\mathrm{loc}}(\mathbb{R},C^\alpha(M))$ for some $k\geq 1$, $\alpha\in(0,1)$, and (2.2) and (2.3) are satisfied. There exist $\delta, C>0$ such that for any f in the set

$$U_\delta := \left\{h\in C^{2,\alpha}(\partial M)\,;\,\|h\|_{C^{2,\alpha}(\partial M)}\leq\delta\right\},$$

there is a solution $u=u_f$ of

$$\begin{cases}\Delta_g u + a(x,u) = 0 & \textit{in } M,\\ u = f & \textit{on } \partial M,\end{cases} \tag{2.4}$$

which satisfies

$$\|u\|_{C^{2,\alpha}(M)}\leq C\,\|f\|_{C^{2,\alpha}(\partial M)}. \tag{2.5}$$

The solution u_f is unique within the class $\left\{w\in C^{2,\alpha}(M)\,;\,\|w\|_{C^{2,\alpha}(M)}\leq C\delta\right\}$. In addition, there are C^k Frechét differentiable maps

$$S: U_\delta\to C^{2,\alpha}(M),\quad f\mapsto u_f,$$
$$\Lambda: U_\delta\to C^{1,\alpha}(\partial M),\quad f\mapsto \partial_\nu u_f|_{\partial M}.$$

In particular, if $a(x,u)=q(x)|u|^{r-1}u$ for a fractional number $r>1$ and $q\in C^\alpha(M)$, then the function $q(x)|u|^{r-1}u$ satisfies the condition $a(x,0)=\partial_y a(x,0)=0$, which implies that the conditions (2.2) and (2.3) hold automatically (due to the well-posedness of the Laplace equation). Hence, Proposition 2.1 implies the well-posedness of the Dirichlet problem (1.1) immediately.

For the proof of Proposition 2.1, we will need a lemma that will also be useful later.

Lemma 2.2. *Let (M,g) be a compact Riemannian manifold with C^∞ boundary ∂M, let $0<\alpha<1$, and let $b(x,y)\in C^\alpha_{\mathrm{loc}}(\mathbb{R},C^\alpha(M))$. For any $u\in C^1(M)$ one has $b(x,u(x))\in C^\alpha(M)$, and*

$$\|b(x,u+v)-b(x,u)\|_{C^\alpha(M)}=o(1),\ \textit{as } \|v\|_{C^1(M)}\to 0. \tag{2.6}$$

Proof. The assumption that $t \mapsto b(\,\cdot\,,t)$ is a C^α_{loc} function $\mathbb{R} \to C^\alpha(M)$ means that for any $R > 0$ there is $C_R > 0$ such that

$$\begin{aligned} |b(x,t)| &\leq C_R, \\ |b(x,t) - b(y,t)| &\leq C_R d_g(x,y)^\alpha, \\ |b(x,t) - b(x,s)| &\leq C_R |t-s|^\alpha, \\ |b(x,t) - b(x,s) - (b(y,t) - b(y,s))| &\leq \|b(\,\cdot\,,t) - b(\,\cdot\,,s)\|_{C^\alpha(M)}\, d_g(x,y)^\alpha \\ &\leq C_R d_g(x,y)^\alpha |t-s|^\alpha, \end{aligned}$$

whenever $x, y \in M$ and $|t|, |s| \leq R$.

Now if $u \in C^1(M)$ with $\|u\|_{L^\infty(M)} \leq R$, one has $|b(x,u(x))| \leq C_R$ and

$$\begin{aligned} |b(x,u(x)) - b(y,u(y))| &\leq |b(x,u(x)) - b(y,u(x))| + |b(y,u(x)) - b(y,u(y))| \\ &\leq C_R \left[1 + \|u\|^\alpha_{C^1(M)}\right] d_g(x,y)^\alpha. \end{aligned}$$

This shows that $b(x,u(x)) \in C^\alpha(M)$.

Let now $u, v \in C^1(M)$ with $\|u\|_{L^\infty} \leq R$ and $\|u+v\|_{L^\infty} \leq R$. Then

$$\|b(x,u+v) - b(x,u)\|_{L^\infty(M)} \leq C_R \|v\|^\alpha_{L^\infty(M)}.$$

Let us next estimate the C^α norm of $b(x,u+v) - b(x,u)$. Writing $h(x,u) := b(x,u)$ and $w_t(x) := u(x) + tv(x)$, we have

$$\begin{aligned} &|h(x,w_1(x)) - h(x,w_0(x)) - [h(y,w_1(y)) - h(y,w_0(y))]| \\ \leq &|h(x,w_1(x)) - h(x,w_0(x)) - [h(y,w_1(x)) - h(y,w_0(x))]| \\ &+ |h(y,w_1(x)) - h(y,w_0(x)) - [h(y,w_1(y)) - h(y,w_0(y))]|. \end{aligned} \tag{2.7}$$

The first absolute value on the right of (2.7) is $\leq C_R d_g(x,y)^\alpha |v(x)|^\alpha$. The second absolute value on the right of (2.7) can be estimated by grouping the terms in two different ways and using the triangle inequality: it is either $\leq C_R \|v\|^\alpha_{L^\infty(M)}$ or $\leq C_R \left(\|u\|_{C^1(M)} + \|v\|_{C^1(M)}\right)^\alpha d_g(x,y)^\alpha$.

By interpolation, this shows that for any $\beta < \alpha$ one has

$$\|b(x,u+v) - b(x,u)\|_{C^\beta(M)} = o(1), \text{ as } \|v\|_{C^1(M)} \to 0.$$

This estimate is also true for $\beta = \alpha$. This can be seen by writing

$$b = b_\epsilon + r_\epsilon,$$

where

$$b_\epsilon(x,t) = \int_{\mathbb{R}} \varphi_\epsilon(t-s) b(x,s)\, ds.$$

Here $\varphi_\epsilon(t) = \epsilon^{-n}\varphi(t/\epsilon)$ is a standard mollifier with $\varphi \in C^\infty_c((-1,1))$, $0 \leq \varphi \leq 1$, and $\int_{\mathbb{R}} \varphi(t)\, dt = 1$. Repeating the argument above for b_ϵ using a higher Hölder exponent in t, and using the estimate $\|r_\epsilon(\,\cdot\,,t)\|_{C^\alpha(M)} \leq C_R \epsilon^\alpha$ for $|t| \leq R$ which follows from the regularity of b, finally yields the estimate

$$\|b(x,u+v) - b(x,u)\|_{C^\alpha(M)} = o(1), \text{ as } \|v\|_{C^1(M)} \to 0. \qquad \square$$

Proof of Proposition 2.1. We prove the existence of solutions by using the implicit function theorem in Banach spaces [Zei86, Theorem 4.B]. Let

$$X = C^{2,\alpha}(\partial M), \quad Y = C^{2,\alpha}(M), \quad Z = C^\alpha(M) \times C^{2,\alpha}(\partial M).$$

Consider the map

$$F : X \times Y \to Z, \quad F(f,u) = (Q(u), u|_{\partial M} - f).$$

Now F indeed maps to Z, since by Lemma 2.2 the map $u \mapsto a(x,u)$ takes $C^{2,\alpha}(M)$ to $C^\alpha(M)$. Thus F is well defined.

We next show that F is a C^k map. Let $0 < m \leq k$ be an integer. If $u, v \in C^{2,\alpha}(M)$ we use the Taylor formula

(2.8)
$$
\begin{aligned}
&a(x, u+v) \\
&= \sum_{j=0}^{m-1} \frac{\partial_u^j a(x,u)}{j!} v^j + \int_0^1 \frac{\partial_u^m a(x, u+tv)}{(m-1)!} v^m (1-t)^{m-1}\, dt \\
&= \sum_{j=0}^{m} \frac{\partial_u^j a(x,u)}{j!} v^j - \frac{v^m}{m!} \partial_u^m a(x,u) + \int_0^1 \frac{\partial_u^m a(x, u+tv)}{(m-1)!} v^m (1-t)^{m-1}\, dt \\
&= \sum_{j=0}^{m} \frac{\partial_u^j a(x,u)}{j!} v^j + \frac{v^m}{(m-1)!} \int_0^1 \left[\partial_u^m a(x, u+tv) - \partial_u^m a(x,u)\right] (1-t)^{m-1}\, dt.
\end{aligned}
$$

We study the remainder term. From (2.6) with $b = \partial_u^m a$ we obtain the estimate

$$\|\partial_u^m a(x, u+tv) - \partial_u^m a(x,u)\|_{C^\alpha(M)} = o(1), \text{ if } t \in [0,1] \text{ and } \|v\|_{C^{2,\alpha}(M)} \to 0.$$

Inserting this in the Taylor formula computation (2.8) yields

$$\left\| a(x, u+v) - \sum_{j=0}^{m} \frac{\partial_u^j a(x,u)}{j!} v^j \right\|_{C^\alpha(M)} = o\left(\|v\|_{C^{2,\alpha}(M)}^m \right), \text{ as } \|v\|_{C^{2,\alpha}(M)} \to 0.$$

This shows that $u \mapsto a(x,u)$ is a C^k map $C^{2,\alpha}(M) \to C^\alpha(M)$. Since the other parts of F are linear, F is a C^k map.

Note that $F(0,0) = 0$ by (2.2). The linearization of F at $(0,0)$ in the u-variable is

$$D_u F|_{(0,0)}(v) = (\Delta_g v + \partial_u a(x,0) v, v|_{\partial M}).$$

This is a homeomorphism $Y \to Z$ by (2.3). To see this, let $(w, \phi) \in Z = C^\alpha(M) \times C^{2,\alpha}(\partial M)$, and consider the Dirichlet problem

$$
\begin{cases}
(\Delta_g + \partial_u a(x,0)) v = w & \text{in } M, \\
v = \phi & \text{on } \partial M.
\end{cases}
\tag{2.9}
$$

The solution of (2.9), if it exists, is unique by (2.3), and by using the Fredholm alternative and Schauder estimates the solution $v \in Y = C^{2,\alpha}(M)$ exists (see e.g. [Tay11, Exercise 1 in Section 13.8]) and depends continuously on the data (w, ϕ). Thus the implicit function theorem in Banach spaces [Zei86, Theorem 4.B] yields that there is $\delta > 0$, a closed ball $U_\delta = \overline{B_X(0,\delta)} \subset X$, and a C^k map $S : U \to Y$ such that whenever $\|f\|_{C^{2,\alpha}(\partial M)} \leq \delta$ we have

$$F(f, S(f)) = (0,0).$$

Since S is Lipschitz continuous and $S(0) = 0$, $u = S(f)$ satisfies

$$\|u\|_{C^{2,\alpha}(M)} \leq C \|f\|_{C^{2,\alpha}(\partial M)}.$$

Moreover, by redefining δ if necessary $u = S(f)$ is the only solution to $F(f,u) = (0,0)$ whenever $\|u\|_{C^{2,\alpha}(M)} \leq C\delta$. We have proven the existence of unique small solutions of the Dirichlet problem (2.4) and the fact that the solution operator $S : U_\delta \to C^{2,\alpha}(M)$ is a C^k map. Since the normal derivative is a linear map $C^{2,\alpha}(M) \to C^{1,\alpha}(\partial M)$, it follows that also Λ is a well defined C^k map $U_\delta \to C^{1,\alpha}(\partial M)$. $\square$

In the next proposition we present an integral identity involving the kth linearization the DN map Λ_q. Below, we write

$$(D^k f)_x(y_1, \ldots, y_k)$$

to denote the kth derivative at x of a C^k map f between Banach spaces, considered as a symmetric k-linear form acting on $(y_1, \ldots, y_k)$. We refer to [Hor85, Section 1.1], where the notation $f^{(k)}(x; y_1, \ldots, y_k)$ is used instead of $(D^k f)_x(y_1, \ldots, y_k)$.

Proposition 2.3 (Integral identity). *Let (M, g) be a compact C^∞ Riemannian manifold with a C^∞ smooth boundary ∂M. Let $q \in C^\alpha(M)$, and let Λ_q be the DN map for the semilinear elliptic equation*

$$\Delta_g u + q|u|^{r-1}u = 0 \text{ in } M, \tag{2.10}$$

where

$$r = k + \alpha, \quad k \geq 1 \text{ and } \alpha \in (0, 1).$$

Let $f_0 \in C^{2,\alpha}(\partial M)$. Then the kth linearization $(D^k \Lambda_q)_{\epsilon_0 f_0}$ of Λ_q at $\epsilon_0 f_0$ satisfies the following identity: For any $f_1, \ldots, f_{k+1} \in C^{2,\alpha}(\partial M)$ one has

$$\begin{aligned} &\lim_{\epsilon_0 \to 0} \epsilon_0^{-\alpha} \int_{\partial M} (D^k \Lambda_q)_{\epsilon_0 f_0}(f_1, \ldots, f_k) f_{k+1} \, dS \\ &= c_r \int_M q|v_0|^{r-1} v_0^{1-k} v_1 \cdots v_{k+1} \, dV, \end{aligned} \tag{2.11}$$

where c_r is the constant given by

$$c_r = -r(r-1)\cdots(r-(k-1)).$$

Here each v_ℓ, $\ell = 0, \ldots, k+1$, is a harmonic function satisfying

$$\begin{cases} \Delta_g v_\ell = 0 & \text{in } M, \\ v_\ell = f_\ell & \text{on } \partial M. \end{cases} \tag{2.12}$$

Proof. Let $f_0 \in C^{2,\alpha}(\partial M)$ and denote $h_0 = \epsilon_0 f_0$, where ϵ_0 is small. The nonlinearity $a(x, u) = q(x)|u|^{r-1}u$ satisfies the conditions in Proposition 2.1, and thus the DN map $\Lambda_q = \partial_\nu S|_{\partial M}$ is well defined for boundary data f with $\|f\|_{C^{2,\alpha}(\partial M)} \leq \delta$. Here $S : f \mapsto u_f$ is the solution operator for the Dirichlet problem of the equation (2.10).

We first compute the derivatives of Λ_q at h_0. For this it is enough to consider the derivatives of S. Let us write

$$\widetilde{f} = \widetilde{f}(x; \epsilon_1, \ldots, \epsilon_k) := \epsilon_1 f_1(x) + \ldots + \epsilon_k f_k(x).$$

Let $f = h_0 + \widetilde{f}$, then the solution

$$u_f := S(f) = S(h_0 + \epsilon_1 f_1 + \cdots + \epsilon_k f_k) \in C^{2,\alpha}(M)$$

is k times continuously differentiable with respect to the parameters $\epsilon_1, \ldots, \epsilon_k$ by Proposition 2.1. Let us denote

$$\epsilon := (\epsilon_0, \epsilon'), \quad \epsilon' := (\epsilon_1, \ldots, \epsilon_k).$$

Applying $\partial_{\epsilon_1} \cdots \partial_{\epsilon_j}\big|_{\epsilon'=0}$ to the Taylor formula for C^k maps (see e.g. [Hor85, equation (1.1.8)])

$$u_f = S(h_0 + \widetilde{f}) = \sum_{m=0}^{k} \frac{(D^m S)_{h_0}(\widetilde{f}, \ldots, \widetilde{f})}{m!} + o\left(\|\widetilde{f}\|^k_{C^{2,\alpha}(\partial M)}\right)$$

implies that $(D^m S)_{h_0}$ for $0 \leq m \leq k$ may be computed using the formula

$$(D^m S)_{h_0}(f_1, \ldots, f_m) = \partial_{\epsilon_1} \cdots \partial_{\epsilon_m} u_f\big|_{\epsilon'=0}. \tag{2.13}$$

Moreover, since S is C^k map $C^{2,\alpha}(\partial M) \to C^{2,\alpha}(M)$, since $u \mapsto q(x)|u|^{r-1}u$ is a C^k map $C^{2,\alpha}(M) \to C^\alpha(M)$ by the argument in Proposition 2.1, and since Δ_g is linear, we may differentiate the equation

$$\begin{cases} \Delta_g u_f + q(x)|u_f|^{r-1}u_f = 0 & \text{in } M, \\ u_f = f = h_0 + \widetilde{f} & \text{on } \partial M, \end{cases} \tag{2.14}$$

up to k times in the ϵ_ℓ variables at $\epsilon'=0$ (recalling that $\widetilde{f}=f(x;\epsilon')=\widetilde{f}(x;\epsilon_1,\dots,\epsilon_k)$).

Let $\ell\in\{1,\dots,k\}$. Then for any $\beta>0$ we have the identity

$$\partial_{\epsilon_\ell}\left(|u_f|^\beta u_f\right)=(\beta|u_f|^{\beta-2}u_f^2+|u_f|^\beta)\partial_{\epsilon_\ell}u_f=(\beta+1)|u_f|^\beta\partial_{\epsilon_\ell}u_f$$

so that

$$\begin{cases}\Delta_g\left(\partial_{\epsilon_\ell}u_f|_{\epsilon'=0}\right)+q(x)r|u_f|^{r-1}\partial_{\epsilon_\ell}u_f\big|_{\epsilon'=0}=0 & \text{in } M,\\ \partial_{\epsilon_\ell}u_f\big|_{\epsilon'=0}=f_\ell & \text{on } \partial M.\end{cases}\tag{2.15}$$

Thus the first linearization of the map S at h_0 is

$$v_\ell^{\epsilon_0}:=(DS)_{h_0}(f_\ell)=\partial_{\epsilon_\ell}u_f\big|_{\epsilon'=0}\tag{2.16}$$

where $v_\ell^{\epsilon_0}$ satisfies (2.15). For $\ell=1,2,\dots,k$, we also claim that

$$\lim_{\epsilon_0\to 0}v_\ell^{\epsilon_0}=v_\ell \text{ in } C^{2,\alpha}(M),\tag{2.17}$$

where v_ℓ is the harmonic function satisfying (2.12) with Dirichlet data f_ℓ. To prove (2.17), note by the Schauder estimates we have

$$\begin{aligned}\|v_\ell^{\epsilon_0}-v_\ell\|_{C^{2,\alpha}(M)}&\le C\left(\|\Delta_g(v_\ell^{\epsilon_0}-v_\ell)\|_{C^\alpha(M)}+\|\epsilon_0 f_0+f_\ell-f_\ell\|_{C^{2,\alpha}(\partial M)}\right)\\ &=C\left(\left\|q\left[r|u_f|^{r-1}\partial_{\epsilon_\ell}u_f\right]\big|_{\epsilon'=0}\right\|_{C^\alpha(M)}+\|\epsilon_0 f_0\|_{C^{2,\alpha}(\partial M)}\right)\\ &\le C\left(\left\||u_{\epsilon_0 f_0}|^{r-1}\right\|_{C^\alpha(M)}+\epsilon_0\right).\end{aligned}$$

Now $\|u_{\epsilon_0 f_0}\|_{C^{2,\alpha}(M)}\le C\epsilon_0\|f_0\|_{C^{2,\alpha}(\partial M)}$ by (2.5). Then (2.6) with $b(x,t)$ replaced by $|t|^{r-1}$ implies that $\left\||u_{\epsilon_0 f_0}|^{r-1}\right\|_{C^\alpha(M)}\to 0$ as $\epsilon_0\to 0$, proving (2.17).

Let now $2\le j\le k$. Applying $\partial_{\epsilon_1}\cdots\partial_{\epsilon_j}\big|_{\epsilon'=0}$ to (2.14) gives that

$$\begin{cases}\Delta_g\left(\partial_{\epsilon_1}\cdots\partial_{\epsilon_j}u_f\big|_{\epsilon'=0}\right)=-\,\partial_{\epsilon_1}\cdots\partial_{\epsilon_j}\left(q(x)|u|^{r-1}u\right)\big|_{\epsilon'=0} & \text{in } M,\\ \partial_{\epsilon_1}\cdots\partial_{\epsilon_j}u_f\big|_{\epsilon'=0}=0 & \text{on } \partial M,\end{cases}$$

Since $r>k$, the fact that u_f is k times continuously Frechét differentiable in ϵ' gives that

$$\lim_{\epsilon_0\to 0}\partial_{\epsilon_1}\cdots\partial_{\epsilon_j}\left(q(x)|u|^{r-1}u\right)\big|_{\epsilon'=0}=0.$$

By an argument similar to the one above using Schauder estimates we obtain

$$\lim_{\epsilon_0\to 0}\partial_{\epsilon_1}\cdots\partial_{\epsilon_j}u_f\big|_{\epsilon'=0}=0.$$

Let us consider the kth mixed derivative $w^{\epsilon_0}:=\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}u_f|_{\epsilon'=0}$ further. It satisfies the equation

$$\begin{cases}\Delta_g w^{\epsilon_0}=-\,\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}\left(q(x)|u|^{r-1}u\right)\big|_{\epsilon'=0} & \text{in } M,\\ w^{\epsilon_0}=0 & \text{on } \partial M,\end{cases}\tag{2.18}$$

We wish to multiply (2.18) by $\epsilon_0^{-\alpha}$ and take the limit as $\epsilon_0\to 0$. Since $f(t)=|t|^{r-1}t$ for $r=k+\alpha$ satisfies the homogeneity relation $f(\lambda t)=\lambda^r f(t)$ for $\lambda>0$, we have that

$$\frac{d^k}{dy^k}\left(|y|^{r-1}y\right)=r(r-1)\cdots(r-(k-1))|y|^{r-1}y^{1-k}=-c_r|y|^{r-1}y^{1-k}.$$

Using Faà di Bruno's formula, see [Har06], we find that

$$\begin{aligned}\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}\left(|u_f|^{r-1}u_f\right)\big|_{\epsilon'=0} &= \sum_{\sigma\in P} c_\sigma |u_f|^{r-1}u_f^{1-|\sigma|}\prod_{\delta\in\sigma}\partial^\delta_{\epsilon'}u_f\Big|_{\epsilon'=0}\\ &= c_r|u_f|^{r-1}u_f^{1-k}\left(\partial_{\epsilon_1}u_f\right)\cdots\left(\partial_{\epsilon_k}u_f\right)|_{\epsilon'=0}\\ &\quad+\sum_{\substack{\sigma\in P,\\ |\sigma|<k}} c_\sigma |u_f|^{r-1}u_f^{1-|\sigma|}\prod_{\delta\in\sigma}\partial^\delta_{\epsilon'}u_f\Big|_{\epsilon'=0},\end{aligned}\tag{2.19}$$

where P contains all partitions of $\{1,\dots,k\}$ and the product over $\delta\in\sigma$ runs over all sets in the partition σ. The number $|\sigma|$ denotes the cardinality of the set σ and $\partial^\delta_{\epsilon'}$ is the usual multi-index notation for partial derivatives in ϵ'.

Observe that $u_f|_{\epsilon'=0}$ solves the nonlinear equation (2.10) with boundary value $h_0=\epsilon_0 f_0$. By continuity and uniqueness of solutions, we have that

$$\epsilon_0^{-1}u_f\big|_{\epsilon'=0}\to v_0 \text{ in } C^{2,\alpha}(M), \qquad \text{as } \epsilon_0\to 0.\tag{2.20}$$

Then note that $|\sigma|<k$ implies that the products

$$\prod_{\delta\in\sigma}\partial^\delta_{\epsilon'}u_f\Big|_{\epsilon'=0}$$

are bounded in $C^\alpha(M)$ as $\epsilon_0\to 0$, because the solution operator S is continuously k-Fréchet differentiable and the Hölder space $C^\alpha(M)$ is an algebra. Next, since the function $g(y)=|y|^{r-1}y^{1-|\sigma|}$ is homogeneous of degree $k-|\sigma|+\alpha\geq 1+\alpha$, Euler's homogeneous function theorem shows that it belongs to $C^1(\mathbb{R})$. Since the composition of $C^1(\mathbb{R})$ function with a $C^{2,\alpha}(M)$ function is at least $C^\alpha(M)$, we have that

$$\epsilon^{-\alpha}|u_f|^{r-1}u_f^{1-|\sigma|}\Big|_{\epsilon'=0}=\epsilon_0^{k-|\sigma|}\left|\frac{u_f}{\epsilon_0}\right|^{r-1}\left(\frac{u_f}{\epsilon_0}\right)^{1-|\sigma|}\Bigg|_{\epsilon'=0}\to 0 \quad \text{in } C^\alpha(M)\tag{2.21}$$

as $\epsilon_0\to 0$. By using (2.17), (2.20) and (2.21), we see that after multiplying (2.19) by $\epsilon_0^{-\alpha}$ and taking the limit $\epsilon_0\to 0$, only the first term on the right hand side of (2.19) survives. To analyze this first term in the right-hand side of (2.19), observe that $g(y)=|y|^{r-1}y^{1-k}$ belongs to $C^\alpha(\mathbb{R})$ and u_f is in $C^{2,\alpha}(M)$, so the composition $|u_f|^{r-1}u_f^{1-k}$ is in $C^\alpha(M)$. Recall again from (2.16) that $\partial_{\epsilon_\ell}u_f|_{\epsilon'=0}\to v_\ell$ in $C^{2,\alpha}(M)$ as $\epsilon_0\to 0$ for all $\ell=1,2,\dots,k$. Due to the continuity of the solution map S, we finally have in C^α the limit

$$\lim_{\epsilon_0\to 0}\epsilon_0^{-\alpha}\,\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}\left(q|u_f|^{r-1}u_f\right)\big|_{\epsilon'=0}=-c_r q|v_0|^{r-1}v_0^{1-k}v_1\cdots v_k.\tag{2.22}$$

Integrating the equation (2.18) against the harmonic function v_{k+1}, we have

$$\int_{\partial M}(\partial_\nu w^{\epsilon_0})f_{k+1}\,dS=-\int_M \partial_{\epsilon_1}\cdots\partial_{\epsilon_k}\left(q(x)|u_f|^{r-1}u_f\right)\big|_{\epsilon'=0}v_{k+1}\,dV.$$

Since $\Lambda_q=\partial_\nu S$ where ∂_ν is linear, the formula (2.13) gives that $\partial_\nu w^{\epsilon_0}|_{\partial M}=(D^k\Lambda_q)_{\epsilon_0 f_0}(f_1,\dots,f_k)$. Now (2.22) yields

$$\lim_{\epsilon_0\to 0}\epsilon_0^{-\alpha}\int_{\partial M}(D^k\Lambda_q)_{\epsilon_0 f_0}(f_1,\dots,f_k)f_{k+1}\,dS=c_r\int_M q|v_0|^{r-1}v_0^{1-k}v_1\cdots v_k\,dV$$

as required. □

It is easy to see that the integral identity also holds for any $f\in C^{2,\alpha}_0(\Gamma)$, for any open subset $\Gamma\subset\partial M$. The following result is an easy consequence of the preceding proposition. For simplicity we only state the result in Euclidean domains.

Corollary 2.4 (Integral identity with partial data). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$, for $n \geq 2$, and let $\Gamma \subset \partial\Omega$ be a nonempty relatively open subset. Let $q \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, and let Λ_q^Γ be the partial data DN map for the semilinear elliptic equation*

$$\begin{cases} \Delta u + q|u|^{r-1}u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $r = k + \alpha$ with $k \geq 1$ and $\alpha \in (0,1)$. The kth linearization $D^k\Lambda_q^\Gamma$ of Λ_q^Γ satisfies the following identity: For any $f_0, f_1, \ldots, f_{k+1} \in C_0^{2,\alpha}(\Gamma)$, one has

$$\begin{aligned} \lim_{\epsilon_0 \to 0} \int_{\partial\Omega} \epsilon_0^{-\alpha} \left(D^k\Lambda_q^\Gamma\right)_{\epsilon_0 f_0} (f_1, \ldots, f_k) f_{k+1} \, dS \\ = c_r \int_\Omega q|v_0|^{r-1} v_0^{1-k} v_1 \cdots v_{k+1} \, dx, \end{aligned} \tag{2.23}$$

where $c_r = -r(r-1)\cdots(r-(k-1))$. Here each v_ℓ, $\ell = 0, \ldots, k+1$, is a harmonic function satisfying

$$\Delta v_\ell = 0 \text{ in } \Omega \quad \text{and} \quad v_\ell = f_\ell \text{ on } \partial\Omega.$$

The result follows immediately from Proposition 2.3, even if the Dirichlet data is supported in a relatively open subset $\Gamma \subset \partial\Omega$.

It is worth mentioning that even in the case $1 < r < 2$ we can use *two* boundary functions f_0 and f_1. A suitable choice of the Dirichlet data f_0 allows us to get rid of the nonlinear term $|v_0|^\alpha$, if necessary, while still retaining the ability to choose f_1 and the auxiliary function f_2 in an appropriate way.

Remark 2.5. *We mention that for nonlinearities $a(x,u) = q(x)|u|^\alpha u$ where $q \in C^\alpha(M)$ and $\alpha \in (0,1)$, one can prove that the solution of*

$$\begin{cases} \Delta u_\epsilon + q|u_\epsilon|^\alpha u_\epsilon = 0 & \text{in } M, \\ u_\epsilon = \epsilon f & \text{on } \partial M, \end{cases}$$

where $f \in C^{2,\alpha}(\partial M)$ and $\epsilon > 0$ is small, has the asymptotic expansion

$$u_\epsilon = \epsilon v + \epsilon^{1+\alpha} w + O(\epsilon^{1+2\alpha}),$$

where v is the harmonic function satisfying

$$\begin{cases} \Delta v = 0 & \text{in } M, \\ v = f & \text{on } \partial M, \end{cases}$$

and w is the solution of

$$\begin{cases} \Delta w = -q|v|^\alpha v & \text{in } M, \\ w = 0 & \text{on } \partial M. \end{cases}$$

One could use such one-parameter asymptotic expansions to give alternative proofs of some of our full data inverse problems. However, we will instead use Proposition 2.3 and Corollary 2.4, which are based on multiparameter expansions and will lead to more general results. For our proof of Theorem 1.6 it is crucial to use Proposition 2.3 with $k \geq 3$.

3. Global uniqueness in Euclidean space

In this section, let us prove our main Euclidean results. Recall that we are considering real-valued solutions. In order to apply the density results [FKSU09, LLLS20a] involving products of complex-valued harmonic functions, let us start with the following simple lemma also used in [LLLS20b]:

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$, for $n \geq 2$. Let $f \in L^\infty(\Omega)$, $v_1, v_2 \in L^2(\Omega)$, and $v_3, \ldots, v_k \in L^\infty(\Omega)$ be complex valued functions where $k \geq 2$. Then*

$$\int_\Omega f v_1 \cdots v_k \, dx = \sum_{j=1}^{2^k} \int_\Omega c_j f w_1^{(j)} \cdots w_k^{(j)} \, dx$$

where $c_j \in \{\pm 1, \pm i\}$ and $w_1^{(j)} \in \{\mathrm{Re}(v_1), \mathrm{Im}(v_1)\}, \cdots, w_k^{(j)} \in \{\mathrm{Re}(v_k), \mathrm{Im}(v_k)\}$ for $1 \leq j \leq 2^k$.

Proof. The result follows by writing

$$\int_M f v_1 \cdots v_k \, dx = \int_M f(\mathrm{Re}(v_1) + i\mathrm{Im}(v_1)) \cdots (\mathrm{Re}(v_k) + i\mathrm{Im}(v_k)) \, dx$$

and by multiplying out the right hand side. □

Lemma 3.1 also holds on Riemannian manifolds (M, g), which will be applied in Section 4.

Proof of Theorem 1.1. Since $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$ for all small f and since Λ_{q_j} is a C^k map by Proposition 2.1, one has

$$\left(D^k \Lambda_{q_1}\right)_{\epsilon_0 f_0} (f_1, \ldots, f_k) = \left(D^k \Lambda_{q_2}\right)_{\epsilon_0 f_0} (f_1, \ldots, f_k)$$

for all $f_0, \ldots, f_{k+1} \in C^{2,\alpha}(\partial\Omega)$ and for ϵ_0 small. The integral identity (2.23) applied with q_1 and q_2 implies that

$$\int_\Omega (q_1 - q_2) |v_0|^{r-1} v_0^{1-k} v_1 \cdots v_{k+1} \, dx = 0$$

for any real-valued harmonic functions $v_0, \ldots, v_{k+1} \in C^{2,\alpha}(\overline{\Omega})$. Let $v_0 = v_3 = \ldots = v_{k+1} = 1$ be constant functions in Ω. Then

$$\int_\Omega (q_1 - q_2) v_1 v_2 \, dx = 0 \tag{3.1}$$

whenever $v_j \in C^{2,\alpha}(\overline{\Omega})$ are real-valued and harmonic. Since the real and imaginary parts of a complex valued harmonic function are harmonic, it follows from Lemma 3.1 that (3.1) remains true for complex valued harmonic functions.

Now let $v_1(x) = e^{(-\zeta + i\xi)\cdot x}$ and $v_2(x) = e^{(\zeta + i\xi)\cdot x}$ be Calderón's exponential solutions (see [Cal80]), which are harmonic, and where $\zeta, \xi \in \mathbb{R}^n$ with $|\zeta| = |\xi|$ and $\zeta \cdot \xi = 0$. Then we have

$$\begin{aligned} 0 &= \int_\Omega (q_1 - q_2) v_1 v_2 \, dx \\ &= \int_\Omega (q_1 - q_2) e^{(-\zeta + i\xi)\cdot x} e^{(\zeta + i\xi)\cdot x} \, dx \\ &= \int_\Omega (q_1 - q_2) e^{2i\xi\cdot x} \, dx. \end{aligned} \tag{3.2}$$

Thus, via (3.2), we obtain that the Fourier transform of the difference $q_1 - q_2$ at -2ξ is zero. Since $\xi \in \mathbb{R}^n$ can be chosen arbitrarily, we must have $q_1 = q_2$ as desired.

Let us give another proof of this result when $n \geq 3$ and when we only assume that $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$ for all small f with $f \geq 0$. As before, let $f_0 = f_3 = \ldots = f_{k+1} = 1$ so that $v_0 = v_3 = \ldots = v_{k+1} = 1$ in Ω. Then (3.1) holds whenever $f_1, f_2 \geq 0$. Let $x \notin \overline{\Omega}$ and choose the boundary values f_1, f_2 so that $v_1(y) = v_2(y) = |x - y|^{2-n}$. Then $v_1, v_2 > 0$ are harmonic in Ω. Inserting these solutions to (3.1) and writing $q = q_1 - q_2$, we see that

$$\int_\Omega |x - y|^{4-2n} q(y) \, dy = 0$$

for $x \notin \overline{\Omega}$. By [Isa90, page 79], the knowledge of the Riesz potential

$$I_\beta \mu(x) = \int_\Omega |x-y|^\beta \, d\mu(y),$$

for $x \notin \overline{\Omega}$ uniquely determines the measure $\mu(y)$ in Ω, when $\beta \neq 2k$ and $\beta + n \neq 2k+2$ for all $k = 0, 1, \ldots$. Since these conditions are satisfied for $\beta = 4-2n$, we see that $q = 0$ by setting $d\mu(y) = q(y)\, dy$ above. Isakov [Isa90] credits M. Riesz [Rie38] and M. M. Lavrentiev [Lav67] for the first results about determination of a measure from the Riesz potential. $\square$

Proof of Theorem 1.2. Since the DN maps satisfy $\Lambda^\Gamma_{q_1}(f) = \Lambda^\Gamma_{q_2}(f)$ for any sufficiently small Dirichlet data $f \in C^{2,\alpha}_0(\Gamma)$, we have for any $f_0, \ldots, f_{k+1} \in C^{2,\alpha}_0(\Gamma)$

$$\lim_{\epsilon_0 \to 0} \epsilon_0^{-\alpha} \int_{\partial\Omega} \left(D^k \Lambda^\Gamma_{q_1} - D^k \Lambda^\Gamma_{q_2} \right)_{\epsilon_0 f_0} (f_1, \ldots, f_k) f_{k+1} \, dS = 0. \tag{3.3}$$

Therefore, by subtracting the integral identity (2.23) for $q = q_1, q_2$ and inserting (3.3), one has

$$\int_\Omega (q_1 - q_2) |v_0|^{r-1} v_0^{1-k} v_1 \ldots v_{k+1} \, dx = 0,$$

where v_ℓ are the solutions of (2.12) in Ω for $\ell = 0, 1, \ldots, k+1$ with $v_\ell|_{\partial\Omega} = f_\ell$. Write $F := (q_1 - q_2)|v_0|^{r-1} v_0^{1-k} v_3 \ldots v_{k+1}$, so that we have

$$\int_\Omega F v_1 v_2 \, dx = 0.$$

By applying Lemma 3.1, we see that the last identity is valid for complex-valued harmonic functions $v_1, v_2 \in C^{2,\alpha}(\overline{\Omega})$ with $\mathrm{supp}(v_\ell|_{\partial\Omega}) \subset \Gamma$. On the other hand, via the density result of [FKSU09], one can choose $\{v_1 v_2\}$ to form a dense subset in $L^1(\Omega)$ with $\mathrm{supp}(v_1|_{\partial\Omega}), \mathrm{supp}(v_2|_{\partial\Omega}) \subset \Gamma$. This implies that $F = 0$ in Ω. Finally, by choosing $f_0, f_3, \ldots, f_{k+1} \not\equiv 0$ to be nonnegative Dirichlet data supported in Γ, we see that $v_0, v_3, \ldots, v_{k+1}$ are positive in Ω by the maximum principle. Thus one can conclude that $q_1 = q_2$ in Ω. $\square$

Next we prove Theorem 1.3.

Proof of Theorem 1.3. Via Proposition 2.1, let $u_j \in C^{2,\alpha}(\overline{\Omega})$, for $j = 1, 2$, be the unique (small) solutions to

$$\begin{cases} \Delta u_j + a_j(x, u_j) = 0 & \text{in } \Omega, \\ u_j = \epsilon_0 f_0 + \epsilon_1 f_1 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $\epsilon_\ell \geq 0$ are small parameters and $f_\ell \in C^{2,\alpha}_0(\Gamma)$, for $\ell = 0, 1$. Then, as in equation (2.16) in the proof of Proposition 2.3, we have that the first linearization of the solution map S_j to (3.4), $j = 1, 2$, at $h_0 := \epsilon_0 f_0$ satisfies

$$v^{\epsilon_0}_{j,1} := (DS_j)_{h_0}(f_1) = \partial_{\epsilon_1} u_j|_{\epsilon_1 = 0}$$

where $v^{\epsilon_0}_{j,1}$ satisfies

$$\begin{cases} \Delta v^{\epsilon_0}_{j,1} = -\partial_y a_j(x, u_j|_{\epsilon_1=0}) v^{\epsilon_0}_{j,1} & \text{in } \Omega, \\ v^{\epsilon_0}_{j,1} = f_1 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

for $j = 1, 2$. Analogously to (2.17) in the proof of Proposition 2.3, one has

$$v^{\epsilon_0}_{j,1} \to v_1 \text{ in } C^{2,\alpha}(\overline{\Omega}), \qquad \text{as } \epsilon_0 \to 0,$$

where v_1 solves $\Delta v_1 = 0$ in Ω and $v_1|_{\partial\Omega} = f_1$.

Fix $f_2 \in C_0^{2,\alpha}(\Gamma)$ and let v_2 solve $\Delta v_2 = 0$ in Ω with $v_2|_{\partial\Omega} = f_2$. Since $\Lambda^\Gamma_{a_1}(f) = \Lambda^\Gamma_{a_2}(f)$ for any sufficiently small $f \in C_0^{2,\alpha}(\Gamma)$, integration by parts and (3.5) yield that

$$\begin{aligned}
0 &= \partial_{\epsilon_1}|_{\epsilon_1=0}\left(\int_{\partial\Omega} f_2\left(\Lambda^\Gamma_{a_1} - \Lambda^\Gamma_{a_2}\right)(\epsilon_0 f_0 + \epsilon_1 f_1)\, dS\right)\\
&= \partial_{\epsilon_1}|_{\epsilon_1=0}\left(\int_\Omega v_2\left(\Delta u_1 - \Delta u_2\right) dx\right) + \partial_{\epsilon_1}|_{\epsilon_1=0}\left(\int_\Omega \nabla v_2 \cdot \nabla\left(u_1 - u_2\right) dx\right)\\
&= -\int_\Omega v_2\, \partial_{\epsilon_1}|_{\epsilon_1=0}\left(a_1(x,u_1) - a_2(x,u_2)\right) dx\\
&\quad + \partial_{\epsilon_1}|_{\epsilon_1=0}\left(\int_{\partial\Omega} \partial_\nu v_2\left(u_1|_{\partial\Omega} - u_2|_{\partial\Omega}\right) dS\right)\\
&= -\int_\Omega v_2\left(\partial_y a_1(x,u_1|_{\epsilon_1=0})v^{\epsilon_0}_{1,1} - \partial_y a_2(x,u_2|_{\epsilon_1=0})v^{\epsilon_0}_{2,1}\right) dx\\
&\quad + \int_{\partial\Omega} \partial_\nu v_2\left(f_1 - f_1\right) dS\\
&= -\int_\Omega v_2\left(\partial_y a_1(x,u_1|_{\epsilon_1=0})v^{\epsilon_0}_{1,1} - \partial_y a_2(x,u_2|_{\epsilon_1=0})v^{\epsilon_0}_{2,1}\right) dx.
\end{aligned}\tag{3.6}$$

For $j = 1, 2$, the function

$$w_j := u_j|_{\epsilon_1=0}$$

now solves

$$\begin{cases}\Delta w_j + a_j(x, w_j) = 0 & \text{in } \Omega,\\ w_j = \epsilon_0 f_0 & \text{on } \partial\Omega.\end{cases}$$

By (2.5) we have

$$\|w_j\|_{C^{2,\alpha}(\overline{\Omega})} \le C\epsilon_0\, \|f_0\|_{C^{2,\alpha}(\partial\Omega)}\,.$$

Since $\Delta(w_j - \epsilon_0 v_0) = -a_j(x, w_j)$ in Ω with $w_j - \epsilon_0 v_0|_{\partial\Omega} = 0$, Schauder estimates imply that

$$\|w_j - \epsilon_0 v_0\|_{C^{2,\alpha}(\overline{\Omega})} \le C\, \|a_j(x, w_j)\|_{C^{\alpha}(\overline{\Omega})}\,.$$

Using the Taylor formula as in (2.8) together with the conditions

$$a_j(x, 0) = \partial_y a_j(x, 0) = 0$$

gives that

$$a_j(x, w_j(x)) = w_j(x)\int_0^1 \left(\partial_y a_j(x, tw_j(x)) - \partial_y a_j(x, 0)\right) dt.$$

We may now apply (2.6) with b replaced by a_j to obtain that

$$\begin{aligned}
\|w_j - \epsilon_0 v_0\|_{C^{2,\alpha}(\overline{\Omega})} &\le C\, \|w_j\|_{C^\alpha(\overline{\Omega})}\int_0^1 \|\partial_y a_j(x, tw_j) - \partial_y a_j(x, 0)\|_{C^\alpha(\overline{\Omega})}\, dt\\
&= o(\epsilon_0)
\end{aligned}\tag{3.7}$$

as $\epsilon_0 \to 0$.

We have by assumption $a_j(x, y) \sim \sum_{l=1}^{\infty} b_{j,l}(x, y)$, where each $b_{j,l}(\,\cdot\,, y) \in C^\alpha(\overline{\Omega})$ is homogeneous of order $r_l > 1$ with respect to the variable $y \in \mathbb{R}$, for $l \ge 1$. Let us also write $\beta_{j,N} := a_j - \sum_{l=1}^{N-1} b_{j,l}$ for $j = 1, 2$ and $N \ge 1$, with $\beta_{j,1} = a_j$. Then $\beta_{j,N}$

is in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}, C^\alpha(\overline{\Omega}))$ as in Definition 1.1. It follows from (1.3) that, in particular,

$$\left\|\partial_y a_j(\,\cdot\,,y) - \sum_{l=1}^{N-1} \partial_y b_{j,l}(\,\cdot\,,y)\right\|_{L^\infty(\Omega)} \leq C_N |y|^{r_N-1}, \qquad |y| \leq 1,$$

for $j = 1, 2$.

We apply the above with $N = 2$ and $y = w_j(x) = u_j(x)|_{\epsilon_1=0}$ to have for $x \in \overline{\Omega}$, for $j = 1, 2$ that

$$|\partial_y a_j(x, w_j) - \partial_y b_{j,1}(x, w_j)| \leq C_2 \, |w_j|^{r_2-1} \leq C\epsilon_0^{r_2-1}.$$

Multiplying this by $\epsilon_0^{-r_1+1}$ and using the facts that $r_2 > r_1$ and $\partial_y b_{j,1}(x, y)$ is homogeneous of order $r_1 - 1$ in y, we obtain in $L^\infty(\Omega)$ that

$$\lim_{\epsilon_0\to 0} \epsilon_0^{-r_1+1} \partial_y a_j(x, w_j) = \lim_{\epsilon_0\to 0} \partial_y b_{j,1}(x, \epsilon_0^{-1} w_j) = \partial_y b_{j,1}(x, v_0).$$

Here in the last equality we additionally used (3.7). Recall that we also have that the limit $\lim_{\epsilon_0\to 0} v_{j,1}^{\epsilon_0} = v_1$ in $C^{2,\alpha}(\overline{\Omega})$, for both $j = 1, 2$. Hence, we obtain

$$\begin{aligned} 0 &= \lim_{\epsilon_0\to 0} \epsilon_0^{-r_1+1} \int_\Omega v_2 \left[\partial_y a_1(x, u_1|_{\epsilon_1=0}) v_{1,1}^{\epsilon_0} - \partial_y a_2(x, u_2|_{\epsilon_1=0}) v_{2,1}^{\epsilon_0}\right] dx \\ &= \int_\Omega \left[\partial_y b_{1,1}(x, v_0) - \partial_y b_{2,1}(x, v_0)\right] v_1 v_2 \, dx. \end{aligned}$$

Via the density result of [FKSU09], products $v_1 v_2$ of pairs of harmonic functions with boundary values supported in $\Gamma \subset \partial\Omega$ are dense in $L^1(\Omega)$. Therefore, we must have

$$\partial_y b_{1,1}(x, v_0) = \partial_y b_{2,1}(x, v_0), \text{ for } x \in \Omega.$$

In addition, notice that the boundary value $f_0 \in C_0^{2,\alpha}(\Gamma)$ has been arbitrary so far. Let $x_0 \in \Omega$, let $y_0 \in \mathbb{R}$ and let us choose by Runge approximation (see e.g. [LLS19, Proposition A.2]) a boundary value $f_0 = f_{0,x_0} \in C_0^\infty(\Gamma)$ so that

$$v_0(x_0) = y_0. \tag{3.8}$$

We deduce that

$$\partial_y b_{1,1}(x_0, y_0) = \partial_y b_{2,1}(x_0, y_0)$$

for any $x_0 \in \Omega$ and any y_0. Thus we have $\partial_y b_{1,1} = \partial_y b_{2,1}$. By Euler's homogeneous function theorem, we have

$$b_{1,1}(x, y) = \frac{y}{r_1} \partial_y b_{1,1}(x, y) = \frac{y}{r_1} \partial_y b_{2,1}(x, y) = b_{2,1}(x, y),$$

where $r_1 > 1$ is the degree of homogeneity for $b_{j,1}(x, y)$ with respect to the y-variable, for $j = 1, 2$. Thus $b_{1,1} = b_{2,1}$.

We proceed by induction on the index $l \in \mathbb{N}$ of $b_{j,l}$, $j = 1, 2$, to show that $b_{1,l} = b_{2,l}$ for any $l \in \mathbb{N}$. We have already shown the case $l = 1$. Let us then make the induction assumption that $b_{1,l} = b_{2,l}$ for $l = 1, \ldots, L$, for some $L \in \mathbb{N}$. Then, we have that

$$
\begin{aligned}
&|(\partial_y a_1(x,y)-\partial_y a_2(x,y))-(\partial_y b_{1,L+1}(x,y)-\partial_y b_{2,L+1}(x,y))|\\
=&\Bigg|(\partial_y a_1(x,y)-\partial_y a_2(x,y))-\sum_{l=1}^{L}\partial_y b_{1,l}(x,y)+\sum_{l=1}^{L}\partial_y b_{2,l}(x,y)\\
&\quad-\left(\partial_y b_{1,L+1}(x,y)-\partial_y b_{2,L+1}(x,y)\right)\Bigg|\\
=&\left|\left(\partial_y a_1(x,y)-\sum_{l=1}^{L+1}\partial_y b_{1,l}(x,y)\right)-\left(\partial_y a_2(x,y)-\sum_{l=1}^{L+1}\partial_y b_{2,l}(x,y)\right)\right|\\
=&\,|\partial_y\beta_{1,L+2}(x,y)-\partial_y\beta_{2,L+2}(x,y)|\leq 2C_{L+2}|y|^{r_{L+2}-1}.
\end{aligned}
$$

Here we used the induction assumption in the first equality. Applying this for $y=w_j(x)=u_j(x)|_{\epsilon_1=0}$ we have for $x\in\overline{\Omega}$, and for $j=1,2$, that

$$
|(\partial_y a_1(x,w_j)-\partial_y a_2(x,w_j))-(\partial_y b_{1,L+1}(x,w_j)-\partial_y b_{2,L+1}(x,w_j))|\leq C\epsilon_0^{r_{L+2}-1},
$$

for some constant $C>0$. Here we used again $\|w_j\|_{C^{2,\alpha}(\overline{\Omega})}\leq C\epsilon_0\,\|f_0\|_{C^{2,\alpha}(\partial\Omega)}$

Therefore, by using (3.7), homogeneity and $r_{L+2}>r_{L+1}$, we obtain in $L^\infty(\Omega)$ that

$$
\begin{aligned}
&\lim_{\epsilon_0\to 0}\epsilon_0^{-r_{L+1}+1}\left(\partial_y a_1(x,u_1|_{\epsilon_1=0})-\partial_y a_2(x,u_2|_{\epsilon_1=0})\right)\\
=&\lim_{\epsilon_0\to 0}\left(\partial_y b_{1,L+1}(x,\epsilon_0^{-1}w_1)-\partial_y b_{2,L+1}(x,\epsilon_0^{-1}w_2)\right)\\
=&\partial_y b_{1,L+1}(x,v_0)-\partial_y b_{2,L+1}(x,v_0).
\end{aligned}
$$

By repeating the arguments we used to prove the special case $N=2$, which especially use the integral identity (3.6) and [FKSU09], we obtain

$$
\partial_y b_{1,L+1}=\partial_y b_{2,L+1}.
$$

By Euler's homogeneous function theorem again, we then have $b_{1,L+1}=b_{2,L+1}$ in Ω as desired, which concludes the induction step and the proof of the theorem. □

Remark 3.2. *In the previous proof we recovered the expansion coefficients $b_l(x,y)$ of the potential $a\sim\sum_{l=1}^{\infty}b_l$ at arbitrary point $(x_0,y_0)\in\Omega\times\mathbb{R}$. This was done by using Runge approximation (see (3.8)) to select a boundary value f_0 so that the corresponding solution v_0 satisfies $v_0(x_0)=y_0$. This is slightly different from earlier results in [LLLS20a, LLLS20b, KU20b], where one recovers the Taylor coefficients $\tilde{b}_l(x,y):=\partial_y^l\tilde{a}(x,y)$ of an unknown smooth potential $\tilde{a}(x,y)$ only at $y=0$, $x\in\Omega$.*

In the end of this section, let us prove the simultaneous recovery of an obstacle and a potential.

Proof of Theorem 1.4. For $\ell=0,1$, let $\epsilon_\ell\geq 0$ be sufficiently small parameters, and $f_\ell\in C_0^{2,\alpha}(\Gamma)$. Consider the Dirichlet data $f=\epsilon_0 f_0+\epsilon_1 f_1$ and let $u_j=u_j(x)$ be the solution of

$$
\begin{cases}
\Delta u_j+a_j(x,u_j)=0 & \text{in }\Omega,\\
u_j=0 & \text{on }\partial D_j,\\
u_j=f & \text{on }\partial\Omega,
\end{cases}
\tag{3.9}
$$

for $j=1,2$, where $a_j=a_j(x,z)$ are polyhomogeneous in the sense of Definition 1.1 with $x\in\Omega\setminus\overline{D_j}$. We first show that $D_1=D_2$ and then recover the coefficients similarly as in the proof of Theorem 1.3.

Step 1. Recovering the obstacle.

As in the proof of Proposition 2.3, see (2.16), we have that the first linearization of the solution map S_j to (3.9), $j = 1, 2$, at $h_0 := \epsilon_0 f_0$ satisfies

$$v_{j,\ell}^{\epsilon_0} := (DS_j)_{h_0}(f_\ell) = \partial_{\epsilon_\ell} u_j|_{\epsilon'=0} \,,$$

where $v_{j,\ell}^{\epsilon_0}$ is the solution of

$$\begin{cases} \Delta_g v_{j,\ell}^{\epsilon_0} = -\partial_y a_j(x, u_j|_{\epsilon'=0}) v_{j,\ell}^{\epsilon_0} & \text{in } \Omega, \\ v_{j,\ell}^{\epsilon_0} = 0 & \text{on } \partial D_j, \\ v_{j,\ell}^{\epsilon_0} = f_\ell & \text{on } \partial\Omega. \end{cases}$$

Analogously to (2.17) in the proof of Proposition 2.3, one has

$$v_{j,\ell}^{\epsilon_0} \to v_j^{(\ell)} \text{ in } C^{2,\alpha}(\overline{\Omega} \setminus D_j), \quad \text{ as } \epsilon_0 \to 0,$$

where

$$\begin{cases} \Delta v_j^{(\ell)} = 0 & \text{in } \Omega \setminus \overline{D_j}, \\ v_j^{(\ell)} = 0 & \text{on } \partial D_j, \\ v_j^{(\ell)} = f_\ell & \text{on } \partial\Omega \end{cases}$$

for $j = 1, 2$ and $\ell = 0, 1$. The rest of the proof is the analogous to the proof of [LLLS20b, Theorem 1.2]. (See also [KU20b, Theorem 1.6].) For the sake of completeness, we offer details of the proof below.

Let G be the connected connected component of $\Omega \setminus (\overline{D_1 \cup D_2})$, whose boundary contains $\partial\Omega$. Consider the function $\widetilde{v}^{(\ell)} := v_1^{(\ell)} - v_2^{(\ell)}$, which solves

$$\begin{cases} \Delta \widetilde{v}^{(\ell)} = 0 & \text{in } G, \\ \widetilde{v}^{(\ell)} = \partial_\nu \widetilde{v}^{(\ell)} = 0 & \text{on } \Gamma, \end{cases}$$

where we have used that $\Lambda^{\Gamma}_{a_1,D_1}(f) = \Lambda^{\Gamma}_{a_2,D_2}(f)$, which holds for all sufficiently small Dirichlet data $f \in C_0^{2,\alpha}(\Gamma)$. By the unique continuation of harmonic functions this yields that $\widetilde{v}^{(\ell)} = 0$ in G. That is, for $\ell = 0, 1$, we have

$$v_1^{(\ell)} = v_2^{(\ell)} \text{ in } G. \tag{3.10}$$

We use a contradiction argument to prove $D_1 = D_2$. For this, let us assume that $D_1 \neq D_2$. Note that the connected component $G \neq \emptyset$. By using [LLLS20b, Lemma A.3], there exists

$$x_1 \in \partial G \cap (\Omega \setminus \overline{D_1}) \cap \partial D_2.$$

Since $x_1 \in \partial D_2$, we have $v_2^{(\ell)}(x_1) = 0$. By (3.10) and continuity, we also have that $v_1^{(\ell)}(x_1) = 0$. Note that x_1 is an interior point of the open set $\Omega \setminus \overline{D_1}$.

We next fix one of the boundary values f_ℓ to be non-negative and not identically 0. Since $v_1^{(\ell)}(x_1) = 0$, the maximum principle implies that $v_1^{(\ell)} \equiv 0$ in $\Omega \setminus \overline{D_1}$, which contradicts to the assumption that $v_1^{(\ell)} = f_\ell$ on $\partial\Omega$ is not identically zero (because the harmonic function $v_1^{(\ell)}$ is continuous up to boundary). This shows that

$$D := D_1 = D_2.$$

Step 2. Recovering the coefficient.

Since we have proved that $D_1 = D_2 = D$, it follows that the partial data Dirichlet-to-Neumann maps for the equations $\Delta u + a_j(x, u) = 0$ in $\Omega \setminus \overline{D}$ agree on Γ. Applying Theorem 1.3 in the connected set $\Omega \setminus \overline{D}$ then implies that $b_{1,l} = b_{2,l}$ for all $l \in \mathbb{N}$. This concludes the proof. $\square$

4. Global uniqueness in Riemannian manifolds

In this last section of this paper, we prove Theorem 1.5 and Theorem 1.6. In our earlier work [LLLS20a], we proved similar theorems for power type nonlinearities, with integer exponents. We begin with the proof of Theorem 1.5.

Proof of Theorem 1.5. The proof is similar to the proof of [LLLS20a, Theorem 1.2]. We first recover the manifold and the its conformal class by the first linearization. After that we use the integral identity (2.11) to recover the potential.

Step 1. Recovering the conformal manifold.

By using Proposition 2.1, the equality $\Lambda_{M_1,g_1,q_1}(f) = \Lambda_{M_2,g_2,q_2}(f)$, for all $f \in C^{2,\alpha}(\partial M)$ with $\|f\|_{C^{2,\alpha}(\partial M)} \leq \delta$, where $\delta > 0$ is a sufficiently small number, implies

$$(D\Lambda_{M_1,g_1,q_1})_0 = (D\Lambda_{M_2,g_2,q_2})_0 .$$

Here, for $j = 1, 2$, the maps $\left(D\Lambda_{M_j,g_j,q_j}\right)_0$ are the DN maps of the linearizations of the equations $\Delta_{g_j} u_j + q_j|u_j|^{r-1}u_j = 0$ in M_j at a boundary value $f = 0$. This implies that the DN maps on ∂M of the first linearized equation

$$\begin{cases} \Delta_{g_j} v_j = 0 & \text{in } M_j, \\ v_j = f & \text{on } \partial M \end{cases}$$

agree on ∂M. That is, we know the DN maps on ∂M of the anisotropic Calderón problem on two-dimensional Riemannian manifolds. Thus, as noted in the proof of [LLLS20a, Theorem 1.2], we may use [LLS19, Theorem 5.1] to determine the manifold and the Riemannian metric up to a conformal transformation: There exists a C^∞ smooth diffeomorphism $J : M_1 \to M_2$ such that

$$\sigma J^* g_2 = g_1 \text{ in } M_1$$

with $J|_{\partial M} = \mathrm{Id}$. Here the function $\sigma \in C^\infty(M_1)$ is positive with $\sigma|_{\partial M} = 1$.

Step 2. Recovering the potential.

Let us transform the equation $\Delta_{g_2} u_2 + q_2|u_2|^{r-1}u_2 = 0$ from the manifold (M_2, g_2) into the manifold (M_1, g_1) as follows. We denote in M_1

$$\widetilde{q}_2 = \sigma^{-1}(q_2 \circ J) \equiv \sigma^{-1} J^* q_2.$$

Let u_2 be the solution to

$$\begin{cases} \Delta_{g_2} u_2 + q_2|u_2|^{r-1}u_2 = 0 & \text{in } M_2, \\ u_2 = f & \text{on } \partial M, \end{cases} \tag{4.1}$$

where $f \in C^{2,\alpha}(\partial M)$ with $\|f\|_{C^{2,\alpha}(\partial M)} \leq \delta$, $\delta > 0$ sufficiently small. Let us define

$$\widetilde{u}_2 := J^* u_2 \equiv u_2 \circ J,$$

in M_1. Then $\widetilde{u}_2$ satisfies in M_1

$$\begin{aligned} &\Delta_{g_1} \widetilde{u}_2 + \widetilde{q}_2|\widetilde{u}_2|^{r-1}\widetilde{u}_2 \\ =&\Delta_{\sigma J^* g_2} \widetilde{u}_2 + \widetilde{q}_2|\widetilde{u}_2|^{r-1}\widetilde{u}_2 \\ =&\sigma^{-1}\Delta_{J^* g_2} \widetilde{u}_2 + \sigma^{-1}(J^* q_2)|\widetilde{u}_2|^{r-1}\widetilde{u}_2 \\ =&\sigma^{-1} J^*(\Delta_{g_2} u_2) + \sigma^{-1}(J^* q_2)|J^* u_2|^{r-1} J^* u_2 \\ =&\sigma^{-1} J^* \left(\Delta_{g_2} u_2 + q_2|u_2|^{r-1}u_2\right). \end{aligned}$$

Here we used the conformal invariance of the Laplace-Beltrami operator in two dimensions and the coordinate invariance of Laplace-Beltrami operator in the second and third equality respectively. Therefore, one has

$$\begin{cases} \Delta_{g_1}\widetilde{u}_2 + \widetilde{q}_2|\widetilde{u}_2|^{r-1}\widetilde{u}_2 = 0 & \text{in } M_1, \\ \widetilde{u}_2 = f & \text{on } \partial M, \end{cases} \tag{4.2}$$

where we have used that u_2 is the solution of (4.1), $f \in C^{2,\alpha}(\partial M)$ and $J|_{\partial M} = \mathrm{Id}$.

Let u_1 be the solution to the nonlinear equation $\Delta_{g_1}u_1 + q_1|u_1|^{r-1}u_1 = 0$ in M_1 with potential q_1 and boundary data f. We show next that

$$\partial_{\nu_1}u_1 = \partial_{\nu_1}\widetilde{u}_2 \text{ on } \partial M. \tag{4.3}$$

Via the assumption that $\Lambda_{M_1,g_1,q_1}(f) = \Lambda_{M_2,g_2,q_2}(f)$, it follows that if $u_1 = u_2 = f \in C^{2,\alpha}(\partial M)$ on ∂M, then

$$\partial_{\nu_1}u_1 = \partial_{\nu_2}u_2 \text{ on } \partial M. \tag{4.4}$$

We compute that

$$\partial_{\nu_2}u_2 = \nu_2 \cdot du_2 = \nu_2 \cdot d(u_2 \circ J \circ J^{-1}) = (J_*^{-1}\nu_2)\cdot d\widetilde{u}_2 = \nu_1 \cdot d\widetilde{u}_2 = \partial_{\nu_1}\widetilde{u}_2, \tag{4.5}$$

where $\cdot$ denotes the canonical pairing between vectors and covectors, and d is the exterior derivative of a function. For example $\nu_2 \cdot du_2 = g(\nu_2, \nabla u_2) = \sum_{k=1}^2 \nu_2^k\, \partial_k u_2$. We used that $J : M_1 \to M_2$ is conformal diffeomorphism, $\sigma J^* g_2 = g_1$, with $J|_{\partial M} = \mathrm{Id}$ and $\sigma|_{\partial M} = 1$ in (4.5). Combining (4.4) and (4.5), we have (4.3) as claimed.

We have by (4.3) that

$$\Lambda_{M_1,g_1,q_1}(f) = \partial_{\nu_1}u_1 = \partial_{\nu_1}\widetilde{u}_2 = \widetilde{\Lambda}_{M_1,g_1,\widetilde{q}_2}(f), \tag{4.6}$$

for all $f \in C^{2,\alpha}(\partial M)$ with $\|f\|_{C^{2,\alpha}(\partial M)} \le \delta$, where $\widetilde{\Lambda}_{M_1,g_1,\widetilde{q}_2}$ denotes the DN map of the Dirichlet problem (4.2) on ∂M.

We apply Proposition 2.3 on (M_1, g_1), the DN maps Λ_{M_1,g_1,q_1} and $\widetilde{\Lambda}_{M_1,g_1,\widetilde{q}_2}$, which agree by (4.6). By Proposition 2.1 we have

$$\lim_{\epsilon_0\to 0} \epsilon_0^{-\alpha}\left(D^k\Lambda_{M_1,g_1,q_1}\right)\Big|_{\epsilon_0 f_0} = \lim_{\epsilon_0\to 0}\epsilon_0^{-\alpha}\left(D^k\widetilde{\Lambda}_{M_1,g_1,\widetilde{q}_2}\right)\Big|_{\epsilon_0 f_0} \text{ on } \partial M,$$

and by Proposition 2.3

$$\int_{M_1}(q_1 - \widetilde{q}_2)|v_0|^{r-1}v_0^{1-k}v_1\cdots v_{k+1}\,dV = 0,$$

where $v_0, v_1, \cdots, v_k \in C^{2,\alpha}(M_1)$ are harmonic functions in (M_1, g_1) with $r = k + \alpha > 1$. We can choose $v_0 = v_1 = \cdots = v_{k-2} = 1$ in M_1, hence

$$\int_{M_1}(q_1 - \widetilde{q}_2)v_{k-1}v_k\,dV = 0$$

for any harmonic functions v_{k-1} and v_k in M_1.

By choosing v_{k-1} and v_k to be complex geometrical optics solutions constructed in [GT11] (see the proof of Proposition 5.1 in [GT11]), we conclude that

$$q_1 = \widetilde{q}_2 \text{ in } M_1.$$

We point out that the construction in [GT11] can be simplified in our case where v_{k-1} and v_k are harmonic. In such case, Carleman estimates are not needed and the construction in [GST19] would suffice. We have proven the claim. $\square$

Proof of Theorem 1.6. Let us write $r = k+\alpha$, $k \in \mathbb{N}$, $k \ge 3$ and $\alpha \in (0,1)$. For $j = 1, 2$, consider Λ_{q_j} to be the DN map for the equation $\Delta_g u_j + q_j|u_j|^{r-1}u_j = 0$ in M. If $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$ for any sufficiently small $f \in C^{2,\alpha}(\partial M)$, then by Proposition 2.1

$$\lim_{\epsilon_0\to 0}\epsilon_0^{-\alpha}\left(D^k\Lambda_{q_1}\right)_{\epsilon_0 f_0} = \lim_{\epsilon_0\to 0}\epsilon_0^{-\alpha}\left(D^k\Lambda_{q_2}\right)_{\epsilon_0 f_0}.$$

Hence, by Proposition 2.3, we have

$$\int_M (q_1 - q_2)|v_0|^{r-1} v_0^{1-k} v_1 \cdots v_{k+1}\, dV = 0,$$

where $v_j \in C^{2,\alpha}(M)$ are harmonic functions in M. Therefore, by choosing $v_0 \equiv 1$ and by using [LLLS20a, Proposition 5.1], one obtains that $q_1 = q_2$ in M, as desired. □

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