

INVERSE SCATTERING AT FIXED ENERGY ON SURFACES WITH EUCLIDEAN ENDS

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ABSTRACT. On a fixed Riemann surface (M_0, g_0) with N Euclidean ends and genus g , we show that, under a topological condition, the scattering matrix $S_V(\lambda)$ at frequency $\lambda > 0$ for the operator $\Delta + V$ determines the potential V if $V \in C^{1,\alpha}(M_0) \cap e^{-\gamma d(\cdot, z_0)^j} L^\infty(M_0)$ for all $\gamma > 0$ and for some $j \in \{1, 2\}$, where $d(z, z_0)$ denotes the distance from z to a fixed point $z_0 \in M_0$. The topological condition is given by $N \geq \max(2g + 1, 2)$ for $j = 1$ and by $N \geq g + 1$ if $j = 2$. In \mathbb{R}^2 this implies that the operator $S_V(\lambda)$ determines any $C^{1,\alpha}$ potential V such that $V(z) = O(e^{-\gamma|z|^2})$ for all $\gamma > 0$.

1. INTRODUCTION

The purpose of this paper is to prove an inverse scattering result at fixed frequency $\lambda > 0$ in dimension 2. The typical question one can ask is to show that the scattering matrix $S_V(\lambda)$ for the Schrödinger operator $\Delta + V$ determines the potential. This is known to be false if V is only assumed to be Schwartz, by the example of Grinevich-Novikov [6], but it is also known to be true for exponentially decaying potentials (i.e. $V \in e^{-\gamma|z|} L^\infty(\mathbb{R}^2)$ for some $\gamma > 0$) with norm smaller than a constant depending on the frequency λ , see Novikov [15]. For other partial results we refer to [2], [10], [19], [20], [21]. The determinacy of V from $S_V(\lambda)$ when V is compactly supported, without any smallness assumption on the norm, follows from the recent work of Bukhgeim [1] on the inverse boundary problem after a standard reduction to the Dirichlet-to-Neumann operator on a large sphere (see [25] for this reduction).

In dimensions $n \geq 3$, it is proved in Novikov [16] (see also [3] for the case of magnetic Schrödinger operators) that the scattering matrix at a fixed frequency λ determines an exponentially decaying potential. When V is compactly supported this also follows directly from the result by Sylvester-Uhlmann [22] on the inverse boundary problem, by reducing to the Dirichlet-to-Neumann operator on a large sphere. Melrose [14] gave a direct proof of the last result based on the methods of [22], and this proof was extended to exponentially decaying potentials in [26] and to the magnetic case in [17]. In the geometric scattering setting, [11, 12] reconstruct the asymptotic expansion of a potential or metrics from the scattering operator at fixed frequency on asymptotically Euclidean/hyperbolic manifolds. Further results of this type are given in [27, 28].

The method for proving the determinacy of V from $S_V(\lambda)$ in [14, 26] is based on the construction of complex geometric optics solutions $u(z) = e^{\rho \cdot z}(1 + r(\rho, z))$ of $(\Delta + V - \lambda^2)u = 0$ with $\rho \in \mathbb{C}^n, z \in \mathbb{R}^n$, and the density of the oscillating scattering solutions $u_{sc}(z) = \int_{S^{n-1}} \Phi_V(\lambda, z, \omega) f(\omega) d\omega$ within those complex geometric optics solutions, where $\Phi_V(\lambda, z, \omega) = e^{i\lambda\omega \cdot z} + e^{-i\lambda\omega \cdot z} |z|^{-\frac{1}{2}(n-1)} a(\lambda, z, \omega)$ are the perturbed plane wave solutions (here $\omega \in S^{n-1}$ and $a \in L^\infty$). Unlike when $n \geq 3$, the problem in dimension 2 is that the set of complex geometrical optics solutions of this type is not large enough to show that the Fourier transform of $V_1 - V_2$ is 0.

The real novelty in the recent work of Bukhgeim [1] in dimension 2 is the construction of new complex geometric optics solutions (at least on a bounded domain $\Omega \subset \mathbb{C}$) of $(\Delta + V_i)u_i = 0$ of the form $u_1 = e^{\Phi/h}(1 + r_1(h))$ and $u_2 = e^{-\Phi/h}(1 + r_2(h))$ with $0 < h \ll 1$ where Φ is a holomorphic function in \mathbb{C} with a unique non-degenerate critical point at a fixed $z_0 \in \mathbb{C}$ (for instance $\Phi(z) = (z - z_0)^2$), and $\|r_j(h)\|_{L^p}$ is small as $h \rightarrow 0$ for $p > 1$. These solutions allow to use stationary phase at z_0 to get

$$\int_{\Omega} (V_1 - V_2)u_1\bar{u}_2 = C(V_1(z_0) - V_2(z_0))h + o(h), \quad C \neq 0$$

as $h \rightarrow 0$ and thus, if the Dirichlet-to-Neumann operators on $\partial\Omega$ are the same, then $V_1(z_0) = V_2(z_0)$.

One of the problems to extend this to inverse scattering is that a holomorphic function in \mathbb{C} with a non-degenerate critical point needs to grow at least quadratically at infinity, which would somehow force to consider potentials V having Gaussian decay. On the other hand, if we allow the function to be meromorphic with simple poles, then we can construct such functions, having a single critical point at any given point p , for instance by considering $\Phi(z) = (z - p)^2/z$. Of course, with such Φ we then need to work on $\mathbb{C} \setminus \{0\}$, which is conformal to a surface with no hole but with 2 Euclidean ends, and Φ has linear growth in the ends. In general, on a surface with genus g and N Euclidean ends, we can use the Riemann-Roch theorem to construct holomorphic functions with linear or quadratic growth in the ends, the dimension of the space of such functions depending on g, N .

In the present work, we apply this idea to obtain an inverse scattering result for $\Delta_{g_0} + V$ on a fixed Riemann surface (M_0, g_0) with Euclidean ends, under some topological condition on M_0 and some decay condition on V .

Theorem 1.1. *Let (M_0, g_0) be a non-compact Riemann surface with genus g and N ends isometric to $\mathbb{R}^2 \setminus \{|z| \leq 1\}$ with metric $|dz|^2$. Let V_1 and V_2 be two potentials in $C^{1,\alpha}(M_0)$ with $\alpha > 0$, and such that $S_{V_1}(\lambda) = S_{V_2}(\lambda)$ for some $\lambda > 0$. Let $d(z, z_0)$ denote the distance between z and a fixed point $z_0 \in M_0$.*

- (i) *If $N \geq \max(2g + 1, 2)$ and $V_i \in e^{-\gamma d(\cdot, z_0)} L^\infty(M_0)$ for all $\gamma > 0$, then $V_1 = V_2$.*
- (ii) *If $N \geq g + 1$ and $V_i \in e^{-\gamma d(\cdot, z_0)^2} L^\infty(M_0)$ for all $\gamma > 0$, then $V_1 = V_2$.*

In \mathbb{R}^2 , where $g = 0$ and $N = 1$, we have an immediate corollary:

Corollary 1.2. *Let $\lambda > 0$ and let $V_1, V_2 \in C^{1,\alpha}(\mathbb{R}^2) \cap e^{-\gamma|z|^2} L^\infty(\mathbb{R}^2)$ for all $\gamma > 0$. If the scattering matrices satisfy $S_{V_1}(\lambda) = S_{V_2}(\lambda)$, then $V_1 = V_2$.*

This is an improvement on the result of Bukhgeim [1] which shows identifiability for compactly supported functions, and in a certain sense on the result of Novikov [15] since it is assumed there that the potential has to be of small L^∞ norm.

The structure of the paper is as follows. In Section 2 we employ the Riemann-Roch theorem and a transversality argument to construct Morse holomorphic functions on (M_0, g_0) with linear or quadratic growth in the ends. Section 3 considers Carleman estimates with harmonic weights on (M_0, g_0) , where suitable convexification and weights at the ends are required since the surface is non compact. Complex geometrical optics solutions are constructed in Section 4. Section 5 discusses direct scattering theory on surfaces with Euclidean ends and contains the proof that scattering solutions are dense in the set of suitable solutions, and Section 6 gives the proof of Theorem 1.1. Finally, there is an appendix discussing a Paley-Wiener type result for functions with Gaussian decay which is needed to prove density of scattering solutions.

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2. HOLOMORPHIC MORSE FUNCTIONS ON A SURFACE WITH EUCLIDEAN ENDS

2.1. Riemann surfaces with Euclidean ends. Let (M_0, g_0) be a non-compact connected smooth Riemannian surface with N ends E_1, \dots, E_N which are Euclidean, i.e. isometric to $\mathbb{C} \setminus \{|z| \leq 1\}$ with metric $|dz|^2$. By using a complex inversion $z \rightarrow 1/z$, each end is also isometric to a pointed disk

$$E_i \simeq \{|z| \leq 1, z \neq 0\} \text{ with metric } \frac{|dz|^2}{|z|^4}$$

thus conformal to the Euclidean metric on the pointed disk. The surface M_0 can then be compactified by adding the points corresponding to $z = 0$ in each pointed disk corresponding to an end E_i , we obtain a closed Riemann surface M with a natural complex structure induced by that of M_0 , or equivalently a smooth conformal class on M induced by that of M_0 . Another way of thinking is to say that M_0 is the closed Riemann surface M with N points e_1, \dots, e_N removed. The Riemann surface M has holomorphic charts $z_\alpha : U_\alpha \rightarrow \mathbb{C}$ and we will denote by z_1, \dots, z_N the complex coordinates corresponding to the ends of M_0 , or equivalently to the neighbourhoods of the points e_i . The Hodge star operator \star acts on the cotangent bundle T^*M , its eigenvalues are $\pm i$ and the respective eigenspaces $T_{1,0}^*M := \ker(\star + i\text{Id})$ and $T_{0,1}^*M := \ker(\star - i\text{Id})$ are sub-bundles of the complexified cotangent bundle $\mathbb{C}T^*M$ and the splitting $\mathbb{C}T^*M = T_{1,0}^*M \oplus T_{0,1}^*M$ holds as complex vector spaces. Since \star is conformally invariant on 1-forms on M , the complex structure depends only on the conformal class of g . In holomorphic coordinates $z = x + iy$ in a chart U_α , one has $\star(udx + vdy) = -vdx + udy$ and

$$T_{1,0}^*M|_{U_\alpha} \simeq \mathbb{C}dz, \quad T_{0,1}^*M|_{U_\alpha} \simeq \mathbb{C}d\bar{z}$$

where $dz = dx + idy$ and $d\bar{z} = dx - idy$. We define the natural projections induced by the splitting of $\mathbb{C}T^*M$

$$\pi_{1,0} : \mathbb{C}T^*M \rightarrow T_{1,0}^*M, \quad \pi_{0,1} : \mathbb{C}T^*M \rightarrow T_{0,1}^*M.$$

The exterior derivative d defines the de Rham complex $0 \rightarrow \Lambda^0 \rightarrow \Lambda^1 \rightarrow \Lambda^2 \rightarrow 0$ where $\Lambda^k := \Lambda^k T^*M$ denotes the real bundle of k -forms on M . Let us denote $\mathbb{C}\Lambda^k$ the complexification of Λ^k , then the ∂ and $\bar{\partial}$ operators can be defined as differential operators $\partial : \mathbb{C}\Lambda^0 \rightarrow T_{1,0}^*M$ and $\bar{\partial} : \mathbb{C}\Lambda^0 \rightarrow T_{0,1}^*M$ by

$$\partial f := \pi_{1,0} df, \quad \bar{\partial} f := \pi_{0,1} df,$$

they satisfy $d = \partial + \bar{\partial}$ and are expressed in holomorphic coordinates by

$$\partial f = \partial_z f dz, \quad \bar{\partial} f = \partial_{\bar{z}} f d\bar{z},$$

with $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$. Similarly, one can define the ∂ and $\bar{\partial}$ operators from $\mathbb{C}\Lambda^1$ to $\mathbb{C}\Lambda^2$ by setting

$$\partial(\omega_{1,0} + \omega_{0,1}) := d\omega_{0,1}, \quad \bar{\partial}(\omega_{1,0} + \omega_{0,1}) := d\omega_{1,0}$$

if $\omega_{0,1} \in T_{0,1}^*M$ and $\omega_{1,0} \in T_{1,0}^*M$. In coordinates this is simply

$$\partial(udz + vd\bar{z}) = \partial v \wedge d\bar{z}, \quad \bar{\partial}(udz + vd\bar{z}) = \bar{\partial}u \wedge dz.$$

If g is a metric on M whose conformal class induces the complex structure $T_{1,0}^*M$, there is a natural operator, the Laplacian acting on functions and defined by

$$\Delta f := -2i \star \bar{\partial} \partial f = d^* d$$

where d^* is the adjoint of d through the metric g and \star is the Hodge star operator mapping Λ^2 to Λ^0 and induced by g as well.

2.2. Holomorphic functions. We are going to construct Carleman weights given by holomorphic functions on M_0 which grow at most linearly or quadratically in the ends. We will use the Riemann-Roch theorem, following ideas of [7], however, the difference in the present case is that we have very little freedom to construct these holomorphic functions, simply because there is just a finite dimensional space of such functions by Riemann-Roch. For the convenience of the reader, and to fix notations, we recall the usual Riemann-Roch index theorem (see Farkas-Kra [5] for more details). A divisor D on M is an element

$$D = ((p_1, n_1), \dots, (p_k, n_k)) \in (M \times \mathbb{Z})^k, \text{ where } k \in \mathbb{N}$$

which will also be denoted $D = \prod_{i=1}^k p_i^{n_i}$ or $D = \prod_{p \in M} p^{\alpha(p)}$ where $\alpha(p) = 0$ for all p except $\alpha(p_i) = n_i$. The inverse divisor of D is defined to be $D^{-1} := \prod_{p \in M} p^{-\alpha(p)}$ and the degree of the divisor D is defined by $\deg(D) := \sum_{i=1}^k n_i = \sum_{p \in M} \alpha(p)$. A non-zero meromorphic function on M is said to have divisor D if $(f) := \prod_{p \in M} p^{\text{ord}(p)}$ is equal to D , where $\text{ord}(p)$ denotes the order of p as a pole or zero of f (with positive sign convention for zeros). Notice that in this case we have $\deg(f) = 0$. For divisors $D' = \prod_{p \in M} p^{\alpha'(p)}$ and $D = \prod_{p \in M} p^{\alpha(p)}$, we say that $D' \geq D$ if $\alpha'(p) \geq \alpha(p)$ for all $p \in M$. The same exact notions apply for meromorphic 1-forms on M . Then we define for a divisor D

$$\begin{aligned} r(D) &:= \dim(\{f \text{ meromorphic function on } M; (f) \geq D\} \cup \{0\}), \\ i(D) &:= \dim(\{u \text{ meromorphic 1 form on } M; (u) \geq D\} \cup \{0\}). \end{aligned}$$

The Riemann-Roch theorem states the following identity: for any divisor D on the closed Riemann surface M of genus g ,

$$(1) \quad r(D^{-1}) = i(D) + \deg(D) - g + 1.$$

Notice also that for any divisor D with $\deg(D) > 0$, one has $r(D) = 0$ since $\deg(f) = 0$ for all f meromorphic. By [5, Th. p70], let D be a divisor, then for any non-zero meromorphic 1-form ω on M , one has

$$(2) \quad i(D) = r(D(\omega)^{-1})$$

which is thus independent of ω . For instance, if $D = 1$, we know that the only holomorphic function on M is 1 and one has $1 = r(1) = r((\omega)^{-1}) - g + 1$ and thus $r((\omega)^{-1}) = g$ if ω is a non-zero meromorphic 1 form. Now if $D = (\omega)$, we obtain again from (1)

$$g = r((\omega)^{-1}) = 2 - g + \deg((\omega))$$

which gives $\deg((\omega)) = 2(g - 1)$ for any non-zero meromorphic 1-form ω . In particular, if D is a divisor such that $\deg(D) > 2(g - 1)$, then we get $\deg(D(\omega)^{-1}) = \deg(D) - 2(g - 1) > 0$ and thus $i(D) = r(D(\omega)^{-1}) = 0$, which implies by (1)

$$(3) \quad \deg(D) > 2(g - 1) \implies r(D^{-1}) = \deg(D) - g + 1 \geq g.$$

Now we deduce the

Lemma 2.1. *Let e_1, \dots, e_N be distinct points on a closed Riemann surface M with genus g , and let z_0 be another point of $M \setminus \{e_1, \dots, e_N\}$. If $N \geq \max(2g + 1, 2)$, the following hold true:*

- (i) *there exists a meromorphic function f on M with at most simple poles, all contained in $\{e_1, \dots, e_N\}$, such that $\partial f(z_0) \neq 0$,*
- (ii) *there exists a meromorphic function h on M with at most simple poles, all contained in $\{e_1, \dots, e_N\}$, such that z_0 is a zero of order at least 2 of h .*

Proof. Let first $g \geq 1$, so that $N \geq 2g + 1$. By the discussion before the Lemma, we know that there are at least $g + 2$ linearly independent (over \mathbb{C}) meromorphic functions f_0, \dots, f_{g+1} on M with at most simple poles, all contained in $\{e_1, \dots, e_{2g+1}\}$. Without loss of generality, one can set $f_0 = 1$ and by linear combinations we can assume that $f_1(z_0) = \dots = f_{g+1}(z_0) = 0$. Now consider the divisor $D_j = e_1 \dots e_{2g+1} z_0^{-j}$ for $j = 1, 2$, with degree $\deg(D_j) = 2g + 1 - j$, then by the Riemann-Roch formula (more precisely (3))

$$r(D_j^{-1}) = g + 2 - j.$$

Thus, since $r(D_1^{-1}) > r(D_2^{-1}) = g$ and using the assumption that $g \geq 1$, we deduce that there is a function in $\text{span}(f_1, \dots, f_{g+1})$ which has a zero of order 2 at z_0 and a function which has a zero of order exactly 1 at z_0 . The same method clearly works if $g = 0$ by taking two points e_1, e_2 instead of just e_1 . \square

If we allow double poles instead of simple poles, the proof of Lemma 2.1 shows the

Lemma 2.2. *Let e_1, \dots, e_N be distinct points on a closed Riemann surface M with genus g , and let z_0 be another point of $M \setminus \{e_1, \dots, e_N\}$. If $N \geq g + 1$, then the following hold true:*

- (i) *there exists a meromorphic function f on M with at most double poles, all contained in $\{e_1, \dots, e_N\}$, such that $\partial f(z_0) \neq 0$,*
- (ii) *there exists a meromorphic function h on M with at most double poles, all contained in $\{e_1, \dots, e_N\}$, such that z_0 is a zero of order at least 2 of h .*

2.3. Morse holomorphic functions with prescribed critical points. We follow in this section the arguments used in [7] to construct holomorphic functions with non-degenerate critical points (i.e. Morse holomorphic functions) on the surface M_0 with genus g and N ends, such that these functions have at most linear growth (resp. quadratic growth) in the ends if $N \geq \max(2g + 1, 2)$ (resp. if $N \geq g + 1$). We let \mathcal{H} be the complex vector space spanned by the meromorphic functions on M with divisors larger or equal to $e_1^{-1} \dots e_N^{-1}$ (resp. by $e_1^{-2} \dots e_N^{-2}$) if we work with functions having linear growth (resp. quadratic growth), where $e_1, \dots, e_N \in M$ are points corresponding to the ends of M_0 as explained in Section 2. Note that \mathcal{H} is a complex vector space of complex dimension greater or equal to $N - g + 1$ (resp. $2N - g + 1$) for the $e_1^{-1} \dots e_N^{-1}$ divisor (resp. the $e_1^{-2} \dots e_N^{-2}$ divisor). We will also consider the real vector space H spanned by the real parts and imaginary parts of functions in \mathcal{H} , this is a real vector space which admits a Lebesgue measure. We now prove the following

Lemma 2.3. *The set of functions $u \in H$ which are not Morse in M_0 has measure 0 in H , in particular its complement is dense in H .*

Proof. We use an argument very similar to that used by Uhlenbeck [24]. We start by defining $m : M_0 \times H \rightarrow T^*M_0$ by $(p, u) \mapsto (p, du(p)) \in T_p^*M_0$. This is clearly a smooth map, linear in the second variable, moreover $m_u := m(\cdot, u) = (\cdot, du(\cdot))$ is smooth on M_0 . The map u is a

Morse function if and only if m_u is transverse to the zero section, denoted $T_0^*M_0$, of T^*M_0 , i.e. if

$$\text{Image}(D_p m_u) + T_{m_u(p)}(T_0^*M_0) = T_{m_u(p)}(T^*M_0), \quad \forall p \in M_0 \text{ such that } m_u(p) = (p, 0).$$

This is equivalent to the fact that the Hessian of u at critical points is non-degenerate (see for instance Lemma 2.8 of [24]). We recall the following transversality result, the proof of which is contained in [24, Th.2] by replacing Sard-Smale theorem by the usual finite dimensional Sard theorem:

Theorem 2.4. *Let $m : X \times H \rightarrow W$ be a C^k map and X, W be smooth manifolds and H a finite dimensional vector space, if $W' \subset W$ is a submanifold such that $k > \max(1, \dim X - \dim W + \dim W')$, then the transversality of the map m to W' implies that the complement of the set $\{u \in H; m_u \text{ is transverse to } W'\}$ in H has Lebesgue measure 0.*

We want to apply this result with $X := M_0$, $W := T^*M_0$ and $W' := T_0^*M_0$, and with the map m as defined above. We have thus proved our Lemma if one can show that m is transverse to W' . Let (p, u) such that $m(p, u) = (p, 0) \in W'$. Then identifying $T_{(p,0)}(T^*M_0)$ with $T_p M_0 \oplus T_p^* M_0$, one has

$$Dm_{(p,u)}(z, v) = (z, dv(p) + \text{Hess}_p(u)z)$$

where $\text{Hess}_p(u)$ is the Hessian of u at the point p , viewed as a linear map from $T_p M_0$ to $T_p^* M_0$ (note that this is different from the covariant Hessian defined by the Levi-Civita connection). To prove that m is transverse to W' we need to show that $(z, v) \rightarrow (z, dv(p) + \text{Hess}_p(u)z)$ is onto from $T_p M_0 \oplus H$ to $T_p M_0 \oplus T_p^* M_0$, which is realized if the map $v \rightarrow dv(p)$ from H to $T_p^* M_0$ is onto. But from Lemma 2.1, we know that there exists a meromorphic function f with real part $v = \text{Re}(f) \in H$ such that $v(p) = 0$ and $dv(p) \neq 0$ as an element of $T_p^* M_0$. We can then take $v_1 := v$ and $v_2 := \text{Im}(f)$, which are functions of H such that $dv_1(p)$ and $dv_2(p)$ are linearly independent in $T_p^* M_0$ by the Cauchy-Riemann equation $\bar{\partial}f = 0$. This shows our claim and ends the proof by using Theorem 2.4. \square

In particular, by the Cauchy-Riemann equation, this Lemma implies that the set of Morse functions in \mathcal{H} is dense in \mathcal{H} . We deduce

Proposition 2.1. *There exists a dense set of points p in M_0 such that there exists a Morse holomorphic function $f \in \mathcal{H}$ on M_0 which has a critical point at p .*

Proof. Let p be a point of M_0 and let u be a holomorphic function with a zero of order at least 2 at p , the existence is ensured by Lemma 2.1. Let $B(p, \eta)$ be a any small ball of radius $\eta > 0$ near p , then by Lemma 2.3, for any $\epsilon > 0$, we can approach u by a holomorphic Morse function $u_\epsilon \in \mathcal{H}_\epsilon$ which is at distance less than ϵ of u in a fixed norm on the finite dimensional space \mathcal{H} . Rouché's theorem for $\partial_z u_\epsilon$ and $\partial_z u$ (which are viewed as functions locally near p) implies that $\partial_z u_\epsilon$ has at least one zero of order exactly 1 in $B(p, \eta)$ if ϵ is chosen small enough. Thus there is a Morse function in \mathcal{H} with a critical point arbitrarily close to p . \square

Remark 2.5. *In the case where the surface M has genus 0 and N ends, we have an explicit formula for the function in Proposition 2.1: indeed M_0 is conformal to $\mathbb{C} \setminus \{e_1, \dots, e_{N-1}\}$ for some $e_i \in \mathbb{C}$ - i.e. the Riemann sphere minus N points - then the function $f(z) = (z - z_0)^2 / (z - e_1)$ with $z_0 \notin \{e_1, \dots, e_{N-1}\}$ has z_0 for unique critical point in $\mathbb{C} \setminus \{e_1, \dots, e_{N-1}\}$ and it is non-degenerate.*

We end this section by the following Lemmas which will be used for the amplitude of the complex geometric optics solutions but not for the phase.

Lemma 2.6. *For any $p_0, p_1, \dots, p_n \in M_0$ some points of M_0 and $L \in \mathbb{N}$, then there exists a function $a(z)$ holomorphic on M_0 which vanishes to order L at all p_j for $j = 1, \dots, n$ and such that $a(p_0) \neq 0$. Moreover $a(z)$ can be chosen to have at most polynomial growth in the ends, i.e. $|a(z)| \leq C|z|^J$ for some $J \in \mathbb{N}$.*

Proof. It suffices to find on M some meromorphic function with divisor greater or equal to $D := e_1^{-J} \dots e_N^{-J} p_1^L \dots p_n^L$ but not greater or equal to Dp_0 and this is insured by Riemann-Roch theorem as long as $JN - nL \geq 2g$ since then $r(D) = -g + 1 + JN - nL$ and $r(Dp_0) = -g + JN - nL$. \square

Lemma 2.7. *Let $\{p_0, p_1, \dots, p_n\} \subset M_0$ be a set of $n+1$ disjoint points. Let $c_0, c_1, \dots, c_K \in \mathbb{C}$, $L \in \mathbb{N}$, and let z be a complex coordinate near p_0 such that $p_0 = \{z = 0\}$. Then there exists a holomorphic function f on M_0 with zeros of order at least L at each p_j , such that $f(z) = c_0 + c_1 z + \dots + c_K z^K + O(|z|^{K+1})$ in the coordinate z . Moreover f can be chosen so that there is $J \in \mathbb{N}$ such that, in the ends, $|\partial_z^\ell f(z)| = O(|z|^J)$ for all $\ell \in \mathbb{N}_0$.*

Proof. The proof goes along the same lines as in Lemma 2.6. By induction on K and linear combinations, it suffices to prove it for $c_0 = \dots = c_{K-1} = 0$. As in the proof of Lemma 2.6, if J is taken large enough, there exists a function with divisor greater or equal to $D := e_1^{-J} \dots e_N^{-J} p_0^{K-1} p_1^L \dots p_n^L$ but not greater or equal to Dp_0 . Then it suffices to multiply this function by c_K times the inverse of the coefficient of z^K in its Taylor expansion at $z = 0$. \square

2.4. Laplacian on weighted spaces. Let x be a smooth positive function on M_0 , which is equal to $|z|^{-1}$ for $|z| > r_0$ in the ends $E_i \simeq \{z \in \mathbb{C}; |z| > 1\}$, where r_0 is a large fixed number. We now show that the Laplacian Δ_{g_0} on a surface with Euclidean ends has a right inverse on the weighted spaces $x^{-J}L^2(M_0)$ for $J \notin \mathbb{N}$ positive.

Lemma 2.8. *For any $J > -1$ which is not an integer, there exists a continuous operator G mapping $x^{-J}L^2(M_0)$ to $x^{-J-2}L^2(M_0)$ such that $\Delta_{g_0}G = \text{Id}$.*

Proof. Let $g_b := x^2 g_0$ be a metric conformal to g_0 . The metric g_b in the ends can be written $g_b = dx^2/x^2 + d\theta_{S^1}^2$ by using radial coordinates $x = |z|^{-1}, \theta = z/|z| \in S^1$, this is thus a b-metric in the sense of Melrose [13], giving the surface a geometry of surface with cylindrical ends. Let us define for $m \in \mathbb{N}_0$

$$H_b^m(M_0) := \{u \in L^2(M_0; d\text{vol}_{g_b}); (x\partial_x)^j \partial_\theta^k u \in L^2(M_0; d\text{vol}_{g_b}) \text{ for all } j+k \leq m\}.$$

The Laplacian has the form $\Delta_{g_b} = -(x\partial_x)^2 + \Delta_{S^1}$ in the ends, and the indicial roots of Δ_{g_b} in the sense of Section 5.2 of [13] are given by the complex numbers λ such that $x^{-i\lambda}\Delta_{g_b}x^{i\lambda}$ is not invertible as an operator acting on the circle S_θ^1 . Thus the indicial roots are the solutions of $\lambda^2 + k^2 = 0$ where k^2 runs over the eigenvalues of Δ_{S^1} , that is, $k \in \mathbb{Z}$. The roots are simple at $\pm ik \in i\mathbb{Z} \setminus \{0\}$ and 0 is a double root. In Theorem 5.60 of [13], Melrose proves that Δ_{g_b} is Fredholm on $x^a H_b^2(M_0)$ if and only if $-a$ is not the imaginary part of some indicial root, that is here $a \notin \mathbb{Z}$. For $J > 0$, the kernel of Δ_{g_b} on the space $x^J H_b^2(M_0)$ is clearly trivial by an energy estimate. Thus $\Delta_{g_b} : x^{-J} H_b^0(M_0) \rightarrow x^{-J} H_b^{-2}(M_0)$ is surjective for $J > 0$ and $J \notin \mathbb{Z}$, and the same then holds for $\Delta_{g_b} : x^{-J} H_b^2(M_0) \rightarrow x^{-J} H_b^0(M_0)$ by elliptic regularity.

Now we can use Proposition 5.64 of [13], which asserts, for all positive $J \notin \mathbb{Z}$, the existence of a pseudodifferential operator G_b mapping continuously $x^{-J}H_b^0(M_0)$ to $x^{-J}H_b^2(M_0)$ such that $\Delta_{g_b}G_b = \text{Id}$. Thus if we set $G = G_b x^{-2}$, we have $\Delta_{g_0}G = \text{Id}$ and G maps continuously $x^{-J+1}L^2(M_0)$ to $x^{-J-1}L^2(M_0)$ (note that $L^2(M_0) = xH_b^0(M_0)$). \square

3. CARLEMAN ESTIMATE FOR HARMONIC WEIGHTS WITH CRITICAL POINTS

3.1. The linear weight case. In this section, we prove a Carleman estimate using harmonic weights with non-degenerate critical points, in a way similar to [7]. Here however we need to work on a non compact surface and with weighted spaces. We first consider a Morse holomorphic function $\Phi \in \mathcal{H}$ obtained from Proposition 2.1 with the condition that Φ has linear growth in the ends, which corresponds to the case where $V \in e^{-\gamma/x}L^\infty(M_0)$ for all $\gamma > 0$. The Carleman weight will be the harmonic function $\varphi := \text{Re}(\Phi)$. We let x be a positive smooth function on M_0 such that $x = |z|^{-1}$ in the complex charts $\{z \in \mathbb{C}; |z| > 1\} \simeq E_i$ covering the end E_i .

Let $\delta \in (0, 1)$ be small and let us take $\varphi_0 \in x^{-\alpha}L^2(M_0)$ a solution of $\Delta_{g_0}\varphi_0 = x^{2-\delta}$, a solution exists by Proposition 2.8 if $\alpha > 1 + \delta$. Actually, by using Proposition 5.61 of [13], if we choose $\alpha < 2$, then it is easy to see that φ_0 is smooth on M_0 and has polyhomogeneous expansion as $|z| \rightarrow \infty$, with leading asymptotic in the end E_i given by $\varphi_0 = -x^{-\delta}/\delta^2 + c_i \log(x) + d_i + O(x)$ for some c_i, d_i which are smooth functions in S^1 . For $\epsilon > 0$ small, we define the convexified weight $\varphi_\epsilon := \varphi - \frac{h}{\epsilon}\varphi_0$.

We recall from the proof of Proposition 3.1 in [7] the following estimate which is valid in any compact set $K \subset M_0$: for all $w \in C_0^\infty(K)$, we have

$$(4) \quad \frac{C}{\epsilon} \left(\frac{1}{h} \|w\|_{L^2}^2 + \frac{1}{h^2} \|w|d\varphi|\|_{L^2}^2 + \frac{1}{h^2} \|w|d\varphi_\epsilon|\|_{L^2}^2 + \|dw\|_{L^2(K)}^2 \right) \leq \|e^{\varphi_\epsilon/h} \Delta_g e^{-\varphi_\epsilon/h} w\|_{L^2}^2$$

where C depends on K but not on h and ϵ .

So for functions supported in the end E_i , it clearly suffices to obtain a Carleman estimate in $E_i \simeq \mathbb{R}^2 \setminus \{|z| \leq 1\}$ by using the Euclidean coordinate z of the end.

Proposition 3.1. *Let $\delta \in (0, 1)$, and φ_ϵ as above, then there exists $C > 0$ such that for all $\epsilon \gg h > 0$ small enough, and all $u \in C_0^\infty(E_i)$*

$$h^2 \|e^{\varphi_\epsilon/h} (\Delta - \lambda^2) e^{-\varphi_\epsilon/h} u\|_{L^2}^2 \geq \frac{C}{\epsilon} (\|x^{1-\frac{\delta}{2}} u\|_{L^2}^2 + h^2 \|x^{1-\frac{\delta}{2}} du\|_{L^2}^2).$$

Proof. The metric g_0 can be extended to \mathbb{R}^2 to be the Euclidean metric and we shall denote by Δ the flat positive Laplacian on \mathbb{R}^2 . Let us write $P := \Delta_{g_0} - \lambda^2$, then the operator $P_h := h^2 e^{\varphi_\epsilon/h} P e^{-\varphi_\epsilon/h}$ is given by

$$P_h = h^2 \Delta - |d\varphi_\epsilon|^2 + 2h \nabla \varphi_\epsilon \cdot \nabla - h \Delta \varphi_\epsilon - h^2 \lambda^2,$$

following the notation of [4, Chap. 4.3], it is a semiclassical operator in $S^0(\langle \xi \rangle^2)$ with semiclassical full Weyl symbol

$$\sigma(P_h) := |\xi|^2 - |d\varphi_\epsilon|^2 - h^2 \lambda^2 + 2i \langle d\varphi_\epsilon, \xi \rangle = a + ib.$$

We can define $A := (P_h + P_h^*)/2 = h^2 \Delta - |d\varphi_\epsilon|^2 - h^2 \lambda^2$ and $B := (P_h - P_h^*)/2i = -2ih \nabla \varphi_\epsilon \cdot \nabla + ih \Delta \varphi_\epsilon$ which have respective semiclassical full symbols a and b , i.e. $A = \text{Op}_h(a)$ and $B =$

$\text{Op}_h(b)$ for the Weyl quantization. Notice that A, B are symmetric operators, thus for all $u \in C_0^\infty(E_i)$

$$(5) \quad \|(A + iB)u\|^2 = \langle (A^2 + B^2 + i[A, B])u, u \rangle.$$

It is easy to check that the operator $ih^{-1}[A, B]$ is a semiclassical differential operator in $S^0(\langle \xi \rangle^2)$ with full semiclassical symbol

$$(6) \quad \{a, b\}(\xi) = 4(D^2\varphi_\epsilon(d\varphi_\epsilon, d\varphi_\epsilon) + D^2\varphi_\epsilon(\xi, \xi))$$

Let us now decompose the Hessian of φ_ϵ in the basis $(d\varphi_\epsilon, \theta)$ where θ is a covector orthogonal to $d\varphi_\epsilon$ and of norm $|d\varphi_\epsilon|$. This yields coordinates $\xi = \xi_0 d\varphi_\epsilon + \xi_1 \theta$ and there exist smooth functions M, N, K so that

$$D^2\varphi_\epsilon(\xi, \xi) = |d\varphi_\epsilon|^2(M\xi_0^2 + N\xi_1^2 + 2K\xi_0\xi_1).$$

Notice that φ_ϵ has a polyhomogeneous expansion at infinity of the form

$$\varphi_\epsilon(z) = \gamma \cdot z + \frac{h}{\epsilon} \frac{r^\delta}{\delta^2} + c_1 \log(r) + c_2 + c_3 r^{-1} + O(r^{-2})$$

where $r = |z|, \omega = z/r, \gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ and c_i are some smooth functions on S^1 depending on h ; in particular we have

$$d\varphi_\epsilon = \gamma_1 dz_1 + \gamma_2 dz_2 + O(r^{-1+\delta}), \quad \partial_z^\alpha \partial_{\bar{z}}^\beta \varphi_\epsilon(z) = O(r^{-2+\delta}) \quad \text{for all } \alpha + \beta \geq 2$$

which implies that $M, N, K \in r^{-2+\delta} L^\infty(E_i)$. Then one can write

$$\begin{aligned} \{a, b\} &= 4|d\varphi_\epsilon|^2(M + M\xi_0^2 + N\xi_1^2 + 2K\xi_0\xi_1) \\ &= 4(N(a + h^2\lambda^2) + ((M - N)\xi_0 + 2K\xi_1)b/2 + (N + M)|d\varphi_\epsilon|^2) \end{aligned}$$

and since $M + N = \text{Tr}(D^2\varphi_\epsilon) = -\Delta\varphi_\epsilon = h\Delta\varphi_0/\epsilon$ we obtain

$$(7) \quad \begin{aligned} \{a, b\} &= 4|d\varphi_\epsilon|^2(c(z)(a + h^2\lambda^2) + \ell(z, \xi)b + \frac{h}{\epsilon}r^{-2+\delta}), \\ c(z) &= \frac{N}{|d\varphi_\epsilon|^2}, \quad \ell(z, \xi) = \frac{(M - N)\xi_0 + 2K\xi_1}{2|d\varphi_\epsilon|^2}. \end{aligned}$$

Now, we take a smooth extension of $|d\varphi_\epsilon|^2, a(z, \xi), \ell(z, \xi)$ and r to $z \in \mathbb{R}^2$, this can be done for instance by extending r as a smooth positive function on \mathbb{R}^2 and then extending $d\varphi$ and $d\varphi_0$ to smooth non vanishing 1-forms on \mathbb{R}^2 (not necessarily exact) so that $|d\varphi_\epsilon|^2$ is smooth positive (for small h) and polynomial in h and a, ℓ are of the same form as in $\{|z| > 1\}$. Let us define the symbol and quantized differential operator on \mathbb{R}^2

$$e := 4|d\varphi_\epsilon|^2(c(z)(a + h^2\lambda^2) + \ell(z, \xi)b), \quad E := \text{Op}_h(e)$$

and write

$$(8) \quad \begin{aligned} ih^{-1}r^{1-\frac{\delta}{2}}[A, B]r^{1-\frac{\delta}{2}} &= hF + r^{1-\frac{\delta}{2}}Er^{1-\frac{\delta}{2}} - \frac{h}{\epsilon}(A^2 + B^2), \\ \text{with } F &:= h^{-1}r^{1-\frac{\delta}{2}}(ih^{-1}[A, B] - E)r^{1-\frac{\delta}{2}} + \frac{1}{\epsilon}(A^2 + B^2). \end{aligned}$$

We deduce from (6) and (7) the following

Lemma 3.2. *The operator F is a semiclassical differential operator in the class $S^0(\langle \xi \rangle^4)$ with semiclassical principal symbol*

$$\sigma(F)(\xi) = \frac{4|d\varphi|^2}{\epsilon} + \frac{1}{\epsilon}(|\xi|^2 - |d\varphi|^2)^2 + \frac{4}{\epsilon}(\langle \xi, d\varphi \rangle)^2.$$

By the semiclassical Gårding estimate, we obtain the

Corollary 3.3. *The operator F of Lemma 3.2 is such that there is a constant C so that*

$$\langle Fu, u \rangle \geq \frac{C}{\epsilon} (\|u\|_{L^2}^2 + h^2 \|du\|_{L^2}^2).$$

Proof. It suffices to use that $\sigma(F)(\xi) \geq \frac{C'}{\epsilon}(1+|\xi|^4)$ for some $C' > 0$ and use the semiclassical Gårding estimate. \square

So by writing $\langle i[A, B]u, u \rangle = \langle ir^{1-\frac{\delta}{2}}[A, B]r^{1-\frac{\delta}{2}}r^{-1+\frac{\delta}{2}}u, r^{-1+\frac{\delta}{2}}u \rangle$ in (5) and using (8) and Corollary 3.3, we obtain that there exists $C > 0$ such that for all $u \in C_0^\infty(E_i)$

$$(9) \quad \begin{aligned} \|P_h u\|_{L^2}^2 &\geq \langle (A^2 + B^2)u, u \rangle + \frac{Ch^2}{\epsilon} (\|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^2 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2) + h \langle Eu, u \rangle \\ &\quad - \frac{h^2}{\epsilon} (\|A(r^{-1+\frac{\delta}{2}}u)\|_{L^2}^2 + \|B(r^{-1+\frac{\delta}{2}}u)\|_{L^2}^2). \end{aligned}$$

We observe that $h^{-1}[A, r^{-1+\frac{\delta}{2}}]r^{1+\frac{\delta}{2}} \in S^0(\langle \xi \rangle)$ and $h^{-1}[B, r^{-1+\frac{\delta}{2}}]r^{1+\frac{\delta}{2}} \in hS^0(1)$, and thus

$$\|A(r^{-1+\frac{\delta}{2}}u)\|_{L^2}^2 + \|B(r^{-1+\frac{\delta}{2}}u)\|_{L^2}^2 \leq C' (\|Au\|_{L^2}^2 + \|Bu\|_{L^2}^2 + h^2 \|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^4 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2)$$

for some $C' > 0$. Taking h small, this implies with (9) that there exists a new constant $C > 0$ such that

$$(10) \quad \|P_h u\|_{L^2}^2 \geq \frac{1}{2} \langle (A^2 + B^2)u, u \rangle + \frac{Ch^2}{\epsilon} (\|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^2 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2) + h \langle Eu, u \rangle.$$

It remains to deal with $h \langle Eu, u \rangle$: we first write $E = 4|d\varphi_\epsilon|^2(c(z)(A + h^2\lambda^2) + \text{Op}_h(\ell)B) + hr^{-1+\frac{\delta}{2}}Sr^{-1+\frac{\delta}{2}}$ where S is a semiclassical differential operator in the class $S^0(\langle \xi \rangle)$ by the decay estimates on $c(z), \ell(z, \xi)$ as $z \rightarrow \infty$, then by Cauchy-Schwartz (and with $L := \text{Op}_h(\ell)$)

$$\begin{aligned} |\langle hEu, u \rangle| &\leq Ch (\|Au\|_{L^2} + h^2 \|r^{-1+\frac{\delta}{2}}u\|_{L^2} + h \|Sr^{-1+\frac{\delta}{2}}u\|_{L^2}) \|r^{-1+\frac{\delta}{2}}u\|_{L^2} + Ch \|Bu\|_{L^2} \|Lu\|_{L^2} \\ &\leq \frac{1}{4} \|Au\|_{L^2}^2 + h^2 \|Sr^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + Ch^2 \|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + \frac{1}{4} \|Bu\|_{L^2}^2 + Ch^2 \|Lu\|_{L^2}^2 \end{aligned}$$

where C is a constant independent of h, ϵ but may change from line to line. Now we observe that $Lr^{1-\frac{\delta}{2}}$ and S are in $S^0(\langle \xi \rangle)$ and thus

$$\|Sr^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + \|Lu\|_{L^2}^2 \leq C (\|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^2 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2),$$

which by (10) implies that there exists $C > 0$ such that for all $\epsilon \gg h > 0$ with ϵ small enough

$$\|P_h u\|_{L^2}^2 \geq \frac{Ch^2}{\epsilon} (\|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^2 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2)$$

for all $u \in C_0^\infty(E_i)$. The proof is complete. \square

Combining now Proposition 3.1 and (4), we obtain

Proposition 3.4. *Let (M_0, g_0) be a Riemann surface with Euclidean ends with x a boundary defining function of the radial compactification \bar{M}_0 and let $\varphi_\epsilon = \varphi - \frac{h}{\epsilon}\varphi_0$ where φ is a harmonic function with non-degenerate critical points and linear growth on M_0 and φ_0 satisfies $\Delta_{g_0}\varphi_0 = x^{2-\delta}$ as above. Then for all $V \in x^{1-\frac{\delta}{2}}L^\infty(M_0)$ there exists an $h_0 > 0$, ϵ_0 and $C > 0$ such that for all $0 < h < h_0$, $h \ll \epsilon < \epsilon_0$ and $u \in C_0^\infty(M_0)$, we have*

$$(11) \quad \frac{1}{h}\|x^{1-\frac{\delta}{2}}u\|_{L^2}^2 + \frac{1}{h^2}\|x^{1-\frac{\delta}{2}}u|d\varphi|\|_{L^2}^2 + \|x^{1-\frac{\delta}{2}}du\|_{L^2}^2 \leq C\epsilon\|e^{\varphi_\epsilon/h}(\Delta_g + V - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{L^2}^2$$

Proof. As in the proof of Proposition 3.1 in [7], by taking ϵ small enough, we see that the combination of (4) and Proposition 3.1 shows that for any $w \in C_0^\infty(M_0)$,

$$\begin{aligned} \frac{C}{\epsilon} \left(\frac{1}{h}\|x^{1-\frac{\delta}{2}}w\|_{L^2}^2 + \frac{1}{h^2}\|x^{1-\frac{\delta}{2}}w|d\varphi|\|_{L^2}^2 + \frac{1}{h^2}\|x^{1-\frac{\delta}{2}}w|d\varphi_\epsilon|\|_{L^2}^2 + \|x^{1-\frac{\delta}{2}}dw\|_{L^2}^2 \right) \\ \leq \|e^{\frac{\varphi_\epsilon}{h}}(\Delta - \lambda^2)e^{-\frac{\varphi_\epsilon}{h}}w\|_{L^2}^2 \end{aligned}$$

which ends the proof. \square

3.2. The quadratic weight case for surfaces. In this section, φ has quadratic growth at infinity, which corresponds to the case where $V \in e^{-\gamma/x^2}L^\infty$ for all $\gamma > 0$. The proof when φ has quadratic growth at infinity is even simpler than the linear growth case. We define $\varphi_0 \in x^{-2}L^\infty$ to be a solution of $\Delta_{g_0}\varphi_0 = 1$, this is possible by Lemma 2.8 and one easily obtains from Proposition 5.61 of [13] that $\varphi_0 = -x^{-2}/4 + O(x^{-1})$ as $x \rightarrow 0$. We let $\varphi_\epsilon := \varphi - \frac{h}{\epsilon}\varphi_0$ which satisfies $\Delta_{g_0}\varphi_\epsilon/h = -1/\epsilon$.

If $K \subset M_0$ is a compact set, the Carleman estimate (4) in K is satisfied by Proposition 3.1 of [7], it then remains to get the estimate in the ends E_1, \dots, E_N . But the exact same proof as in Lemma 3.1 and Lemma 3.2 of [7] gives directly that for any $w \in C_0^\infty(E_i)$

$$(12) \quad \frac{C}{\epsilon} \left(\frac{1}{h}\|w\|_{L^2}^2 + \frac{1}{h^2}\|w|d\varphi|\|_{L^2}^2 + \frac{1}{h^2}\|w|d\varphi_\epsilon|\|_{L^2}^2 + \|dw\|_{L^2}^2 \right) \leq \|e^{\varphi_\epsilon/h}\Delta_{g_0}e^{-\varphi_\epsilon/h}w\|_{L^2}^2$$

for some $C > 0$ independent of ϵ, h and it suffices to glue the estimates in K and in the ends E_i as in Proposition 3.1 of [7], to obtain (12) for any $w \in C_0^\infty(M_0)$. Then by using triangle inequality

$$\|e^{\varphi_\epsilon/h}(\Delta_{g_0} + V - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{L^2} \leq \|e^{\varphi_\epsilon/h}\Delta_{g_0}e^{-\varphi_\epsilon/h}u\|_{L^2} + C\|u\|_{L^2}$$

for some C depending on $\lambda, \|V\|_{L^\infty}$, we see that the $V - \lambda^2$ term can be absorbed by the left hand side of (12) and we finally deduce

Proposition 3.5. *Let (M_0, g_0) be a Riemann surface with Euclidean ends and let $\varphi_\epsilon = \varphi - \frac{h}{\epsilon}\varphi_0$ where φ is a harmonic function with non-degenerate critical points and quadratic growth on M_0 and φ_0 satisfies $\Delta_{g_0}\varphi_0 = 1$ with $\varphi_0 \in x^{-2}L^\infty(M_0)$. Then for all $V \in L^\infty$ there exists an $h_0 > 0$, ϵ_0 and $C > 0$ such that for all $0 < h < h_0$, $h \ll \epsilon < \epsilon_0$ and $u \in C_0^\infty(M_0)$*

$$\frac{C}{\epsilon} \left(\frac{1}{h}\|u\|_{L^2}^2 + \frac{1}{h^2}\|u|d\varphi|\|_{L^2}^2 + \|du\|_{L^2}^2 \right) \leq \|e^{\varphi_\epsilon/h}(\Delta_{g_0} + V - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{L^2}^2.$$

The main difference with the linear weight case is that one can use a convexification which has quadratic growth at infinity which allows to absorb the λ^2 term, while it was not the case for the linearly growing weights.

4. COMPLEX GEOMETRIC OPTICS ON A RIEMANN SURFACE WITH EUCLIDEAN ENDS

As in [1, 9, 7], the method for identifying the potential at a point p is to construct complex geometric optic solutions depending on a small parameter $h > 0$, with phase a Morse holomorphic function with a non-degenerate critical point at p , and then to apply the stationary phase method. Here, in addition, we need the phase to be of linear growth at infinity if $V \in e^{-\gamma/x}L^\infty$ for all $\gamma > 0$ while the phase has to be of quadratic growth at infinity if $V \in e^{-\gamma/x^2}L^\infty$ for all $\gamma > 0$.

4.1. The surface case. We shall now assume that M_0 is a non-compact surface with genus g with N ends equipped with a metric g_0 which is Euclidean in the ends, and V is a $C^{1,\alpha}$ function in M_0 . Moreover, if $V \in e^{-\gamma/x}L^\infty$ for all $\gamma > 0$, we ask that $N \geq \max(2g + 1, 2)$ while if $V \in e^{-\gamma/x^2}L^\infty$ for all $\gamma > 0$, we assume that $N \geq g + 1$. As above, let us use a smooth positive function x which is equal to 1 in a large compact set of M_0 and is equal to $x = |z|^{-1}$ in the regions $|z| > r_0$ of the ends $E_i \simeq \{z \in \mathbb{C}; |z| > 1\}$, where r_0 is a fixed large number. This function is a boundary defining function of the radial compactification of M_0 in the sense of Melrose [13]. To construct the complex geometric optics solutions, we will need to work with the weighted spaces $x^{-\alpha}L^2(M_0)$ where $\alpha \in \mathbb{R}_+$.

Let \mathcal{H} be the finite dimensional complex vector space defined in the beginning of Section 2.3. Choose $p \in M_0$ such that there exists a Morse holomorphic function $\Phi = \varphi + i\psi \in \mathcal{H}$ on M_0 , with a critical point at p ; there is a dense set of such points by Proposition 2.1. The purpose of this section is to construct solutions u on M_0 of $(\Delta - \lambda^2 + V)u = 0$ of the form

$$(13) \quad u = e^{\Phi/h}(a + r_1 + r_2)$$

for $h > 0$ small, where $a \in x^{-J+1}L^2$ with $J \in \mathbb{R}_+ \setminus \mathbb{N}$ is a holomorphic function on M_0 , obtained by Lemma 2.6, such that $a(p) \neq 0$ and a vanishing to order L (for some fixed large L) at all other critical points of Φ , and finally r_1, r_2 will be remainder terms which are small as $h \rightarrow 0$ and have particular properties near the critical points of Φ . More precisely, $e^{\varphi_0/\epsilon}r_2$ will be a $o_{L^2}(h)$ and r_1 will be a $O_{x^{-J}L^2}(h)$ but with an explicit expression, which can be used to obtain sufficient information in order to apply the stationary phase method.

4.1.1. Construction of r_1 . We want to construct $r_1 = O_{x^{-J}L^2}(h)$ which satisfies

$$e^{-\Phi/h}(\Delta_{g_0} - \lambda^2 + V)e^{\Phi/h}(a + r_1) = O_{x^{-J}L^2}(h)$$

for some large $J \in \mathbb{R}_+ \setminus \mathbb{N}$ so that $a \in x^{-J+1}L^2$.

Let G be the operator of Lemma 2.8, mapping continuously $x^{-J+1}L^2(M_0)$ to $x^{-J-1}L^2(M_0)$. Then clearly $\bar{\partial}\partial G = \frac{i}{2}\star^{-1}$ when acting on $x^{-J+1}L^2$, here \star^{-1} is the inverse of \star mapping functions to 2-forms. First, we will search for r_1 satisfying

$$(14) \quad e^{-2i\psi/h}\partial e^{2i\psi/h}r_1 = -\partial G(a(V - \lambda^2)) + \omega + O_{x^{-J}H^1}(h)$$

with $\omega \in x^{-J}L^2(M_0)$ a holomorphic 1-form on M_0 and $\|r_1\|_{x^{-J}L^2} = O(h)$. Indeed, using the fact that Φ is holomorphic we have

$$e^{-\Phi/h}\Delta_{g_0}e^{\Phi/h} = -2i\star\bar{\partial}e^{-\Phi/h}\partial e^{\Phi/h} = -2i\star\bar{\partial}e^{-\frac{1}{h}(\Phi-\bar{\Phi})}\partial e^{\frac{1}{h}(\Phi-\bar{\Phi})} = -2i\star\bar{\partial}e^{-2i\psi/h}\partial e^{2i\psi/h}$$

and applying $-2i\star\bar{\partial}$ to (14), this gives

$$e^{-\Phi/h}(\Delta_{g_0} + V)e^{\Phi/h}r_1 = -a(V - \lambda^2) + O_{x^{-J}L^2}(h).$$

Writing $-\partial G(a(V - \lambda^2)) =: c(z)dz$ in local complex coordinates, $c(z)$ is $C^{2,\alpha}$ by elliptic regularity and we have $2i\partial_{\bar{z}}c(z) = a(V - \lambda^2)$, therefore $\partial_z\partial_{\bar{z}}c(p') = \partial_{\bar{z}}^2c(p') = 0$ at each critical point $p' \neq p$ by construction of the function a . Therefore, we deduce that at each critical point $p' \neq p$, $c(z)$ has Taylor series expansion $\sum_{j=0}^2 c_j z^j + O(|z|^{2+\alpha})$. That is, all the lower order terms of the Taylor expansion of $c(z)$ around p' are polynomials of z only. By Lemma 2.7, and possibly by taking J larger, there exists a holomorphic function $f \in x^{-J}L^2$ such that $\omega := \partial f$ has Taylor expansion equal to that of $\partial G(a(V - \lambda^2))$ at all critical points $p' \neq p$ of Φ . We deduce that, if $b := -\partial G(a(V - \lambda^2)) + \omega = b(z)dz$, we have

$$(15) \quad \begin{aligned} |\partial_{\bar{z}}^m \partial_z^\ell b(z)| &= O(|z|^{2+\alpha-\ell-m}), & \text{for } \ell + m \leq 2, & \text{ at critical points } p' \neq p \\ |b(z)| &= O(|z|), & & \text{if } p' = p. \end{aligned}$$

Now, we let $\chi_1 \in C_0^\infty(M_0)$ be a cutoff function supported in a small neighbourhood U_p of the critical point p and identically 1 near p , and $\chi \in C_0^\infty(M_0)$ is defined similarly with $\chi = 1$ on the support of χ_1 . We will construct r_1 to be a sum $r_1 = r_{11} + hr_{12}$ where r_{11} is a compactly supported approximate solution of (14) near the critical point p of Φ and r_{12} is correction term supported away from p . We define locally in complex coordinates centered at p and containing the support of χ

$$(16) \quad r_{11} := \chi e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 b)$$

where $Rf(z) := -(2\pi i)^{-1} \int_{\mathbb{R}^2} \frac{1}{\bar{z}-\bar{\xi}} f d\bar{\xi} \wedge d\xi$ for $f \in L^\infty$ compactly supported is the classical Cauchy operator inverting locally ∂_z (r_{11} is extended by 0 outside the neighbourhood of p). The function r_{11} is in $C^{3,\alpha}(M_0)$ and we have

$$(17) \quad \begin{aligned} e^{-2i\psi/h} \partial(e^{2i\psi/h} r_{11}) &= \chi_1(-\partial G(a(V - \lambda^2)) + \omega) + \eta \\ \text{with } \eta &:= e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 b) \partial \chi. \end{aligned}$$

We then construct r_{12} by observing that b vanishes to order $2 + \alpha$ at critical points of Φ other than p (from (15)), and $\partial \chi = 0$ in a neighbourhood of any critical point of ψ , so we can find r_{12} satisfying

$$(18) \quad 2ir_{12}\partial\psi = (1 - \chi_1)b.$$

This is possible since both $\partial\psi$ and the right hand side are valued in $T_{1,0}^*M_0$ and $\partial\psi$ has finitely many isolated zeroes on M_0 : r_{12} is then a function which is in $C^{2,\alpha}(M_0 \setminus P)$ where $P := \{p_1, \dots, p_n\}$ is the set of critical points other than p , it extends to a function in $C^{1,\alpha}(M_0)$ and it satisfies in local complex coordinates z at each p_j

$$|\partial_{\bar{z}}^\beta \partial_z^\gamma r_{12}(z)| \leq C|z|^{1+\alpha-\beta-\gamma}, \quad \beta + \gamma \leq 2$$

by using also the fact that $\partial\psi$ can be locally be considered as a smooth function with a zero of order 1 at each p_j . Moreover $b \in x^{-J}H^2(M_0)$ thus $r_1 \in x^{-J}H^2(M_0)$ and we have

$$e^{-2i\psi/h} \partial(e^{2i\psi/h} r_1) = b + h\partial r_{12} + \eta = -\partial G(a(V - \lambda^2)) + \omega + h\partial r_{12} + \eta.$$

Lemma 4.1. *The following estimates hold true*

$$\begin{aligned} \|\eta\|_{H^2(M_0)} &= O(|\log h|), & \|\eta\|_{H^1(M_0)} &\leq O(h|\log h|), & \|x^J \partial r_{12}\|_{H^1(M_0)} &= O(1), \\ \|x^J r_1\|_{L^2} &= O(h), & \|x^J (r_1 - h\tilde{r}_{12})\|_{L^2} &= o(h) \end{aligned}$$

where \tilde{r}_{12} solves $2i\tilde{r}_{12}\partial\psi = b$.

Proof. The proof is exactly the same as the proof of Lemma 4.2 in [8], except that one needs to add the weight x^J to have bounded integrals. \square

As a direct consequence, we have

Corollary 4.2. *With $r_1 = r_{11} + hr_{12}$, there exists $J > 0$ such that*

$$\|e^{-\Phi/h}(\Delta_{g_0} - \lambda^2 + V)e^{\Phi/h}(a + r_1)\|_{x^{-J}L^2(M_0)} = O(h|\log h|).$$

4.1.2. *Construction of r_2 .* In this section, we complete the construction of the complex geometric optic solutions. We deal with the general case of surfaces and we shall show the following

Proposition 4.1. *If φ_0 is the subharmonic function constructed in Section 3, then for ϵ small enough there exist solutions to $(\Delta_{g_0} - \lambda^2 + V)u = 0$ of the form $u = e^{\Phi/h}(a + r_1 + r_2)$ with $r_1 = r_{11} + hr_{12}$ constructed in the previous section and $r_2 \in e^{-\varphi_0/\epsilon}L^2$ satisfying $\|e^{\varphi_0/\epsilon}r_2\|_{L^2} \leq Ch^{3/2}|\log h|$.*

This is a consequence of the following Lemma (which follows from the Carleman estimate obtained in Section 3 above)

Lemma 4.3. *Let $\delta \in (0, 1)$, $V \in x^{1-\frac{\delta}{2}}L^\infty(M_0)$, and $\varphi_\epsilon = \varphi - \frac{h}{\epsilon}\varphi_0$ a weight with linear growth at infinity as in Proposition 3.4. For all $f \in L^2(M_0)$ and all $h > 0$ small enough, there exists a solution $v \in L^2(M_0)$ to the equation*

$$(19) \quad e^{-\varphi_\epsilon/h}(\Delta_g - \lambda^2 + V)e^{\varphi_\epsilon/h}v = x^{1-\frac{\delta}{2}}f$$

satisfying

$$\|v\|_{L^2(M_0)} \leq Ch^{\frac{1}{2}}\|f\|_{L^2(M_0)}.$$

If φ_ϵ has quadratic growth at infinity, the same result is true when $V \in L^\infty(M_0)$ but $x^{1-\frac{\delta}{2}}f$ can be replaced by $f \in L^2$ in (19).

Proof. The proof is based on a duality argument. Let $P_h := e^{\varphi_\epsilon/h}(\Delta_g - \lambda^2 + V)e^{-\varphi_\epsilon/h}$ and for all $h > 0$ the real vector space $\mathcal{A} := \{u \in x^{-1+\frac{\delta}{2}}H^1(M_0); P_h u \in L^2(M_0)\}$ equipped with the real scalar product

$$(u, w)_\mathcal{A} := \langle P_h u, P_h w \rangle_{L^2}.$$

By the Carleman estimate of Proposition 3.4, the space \mathcal{A} is a Hilbert space equipped with the scalar product above if $h < h_0$, and thus the linear functional $L : w \rightarrow \int_{M_0} x^{1-\frac{\delta}{2}}fw \, d\text{vol}_{g_0}$ on \mathcal{A} is continuous with norm bounded by $Ch^{\frac{1}{2}}\|f\|_{L^2}$ by Proposition 3.4, and by Riesz theorem there is an element $u \in \mathcal{A}$ such that $(\cdot, u)_\mathcal{A} = L$ and with norm bounded by the norm of L . It remains to take $v := P_h u$ which solves $P_h^* v = x^{1-\frac{\delta}{2}}f$ where $P_h^* = e^{-\varphi_\epsilon/h}(\Delta_g - \lambda^2 + V)e^{\varphi_\epsilon/h}$ is the adjoint of P_h and v satisfies the desired norm estimate. The proof when the weight φ_ϵ has quadratic growth at infinity is the same, but improves slightly due to the Carleman estimate of Proposition 3.5. \square

Proof of Proposition 4.1. We first solve the equation

$$(\Delta + V - \lambda^2)e^{\varphi_\epsilon/h}\tilde{r}_2 = x^{1-\frac{\delta}{2}}e^{\varphi_\epsilon/h}\left(x^{-1+\frac{\delta}{2}}e^{-\varphi_\epsilon/h}(\Delta + V - \lambda^2)e^{\Phi/h}(a + r_1)\right)$$

by using Lemma 4.3 and the fact that for J large, there is $C > 0$ such that for all $h < h_0$

$$\|x^{-1+\frac{\delta}{2}}e^{-\varphi_\epsilon/h}(\Delta + V - \lambda^2)e^{\Phi/h}(a + r_1)\|_{L^2} \leq C\|x^J e^{-\Phi/h}(\Delta - \lambda^2 + V)e^{\Phi/h}(a + r_1)\|_{L^2}$$

since $x^{-J-1}e^{\varphi_0/\epsilon} \in L^\infty(M_0)$ for all J (recall that $\varphi_0 \sim -x^{-\delta}/\delta^2$ as $x \rightarrow 0$). But now the right hand side is bounded by $O(h|\log h|)$ according to Corollary 4.2, therefore we set $r_2 := -e^{-i\psi/h-\varphi_0/\epsilon}\tilde{r}_2$ which satisfies $(\Delta_{g_0} - \lambda^2 + V)e^{\Phi/h}(a + r_1 + r_2) = 0$ and, by Lemma 4.3, the norm estimate $\|e^{\varphi_0/\epsilon}r_2\|_{L^2} \leq O(h^{3/2}|\log h|)$. \square

5. SCATTERING ON SURFACE WITH EUCLIDEAN ENDS

Let (M_0, g_0) be a surface with Euclidean ends and $V \in e^{-\gamma/x}L^\infty(M_0)$ for some γ . The scattering theory in this setting is described for instance in Melrose [14], here we will follow this presentation (see also Section 3 in Uhlmann-Vasy [26] for the \mathbb{R}^n case). First, using standard methods in scattering theory, we define the resolvent on the continuous spectrum as follows

Lemma 5.1. *The resolvent $R_V(\lambda) := (\Delta_{g_0} + V - \lambda^2)^{-1}$ admits a meromorphic extension from $\{\text{Im}(\lambda) < 0\}$ to $\{\text{Im}(\lambda) \leq A, \text{Re}(\lambda) \neq 0\}$, as a family of operators mapping $e^{-\gamma/x}L^2(M_0)$ to $e^{\gamma/x}L^2(M_0)$ for any $\gamma > A$. Moreover, for $\lambda \in \mathbb{R} \setminus \{0\}$ not a pole, $R_V(\lambda)$ maps continuously $x^\alpha L^2$ to $x^{-\alpha}L^2$ for any $\alpha > 1/2$.*

Proof. The statement is known for $V = 0$ and $M_0 = \mathbb{R}^2$ by using the explicit formula of the resolvent convolution kernel on \mathbb{R}^2 in terms of Hankel functions (see for instance [14]), we shall denote $R_0(\lambda)$ this continued resolvent. More precisely, for all $A > 0$, the operator $R_0(\lambda)$ continues analytically from $\{\text{Im}(\lambda) < 0\}$ to $\{\text{Im}(\lambda) \leq A, \text{Re}(\lambda) \neq 0\}$ as a family of bounded operators mapping $e^{-\gamma/x}L^2$ to $e^{\gamma/x}L^2$ for any $\gamma > A$. Now we can set $\chi \in C_0^\infty(M_0)$ such that $1 - \chi$ is supported in the ends E_i , and let $\chi_0, \chi_1 \in C_0^\infty(M_0)$ such that $(1 - \chi_0) = 1$ on the support of $(1 - \chi)$ and $\chi_1 = 1$ on the support of χ . Let $\lambda_0 \in -i\mathbb{R}_+$ with $i\lambda_0 \gg 0$, then the resolvent $R_0(\lambda_0)$ is well defined from $L^2(M_0)$ to $H^2(M_0)$ since the Laplacian is essentially self-adjoint [23, Proposition 8.2.4], and we have a parametrix

$$E(\lambda) := (1 - \chi_0)R_0(\lambda)(1 - \chi) + \chi_1 R_0(\lambda_0)\chi$$

which satisfies

$$(\Delta_{g_0} - \lambda^2 + V)E(\lambda) = 1 + K(\lambda),$$

$$K(\lambda) := ([\Delta_{g_0}, \chi_1] - (\lambda^2 - \lambda_0^2)\chi_1)R_0(\lambda_0)\chi - [\Delta_{g_0}, \chi_0]R_0(\lambda)(1 - \chi) + VE(\lambda),$$

where here we use the notation $R_0(\lambda)$ for an integral kernel on M_0 , which in the charts $\{z \in \mathbb{R}^2; |z| > 1\}$ corresponding the ends E_1, \dots, E_N , is given by the integral kernel of $(\Delta_{\mathbb{R}^2} - \lambda^2)^{-1}$. Using the explicit expression of the convolution kernel of $R_0(\lambda)$ in the ends (see for instance Section 1.5 of [14]) and the decay assumption on V , it is direct to see that for $\text{Im}(\lambda) < A, \text{Re}(\lambda) \neq 0$, the map $\lambda \mapsto K(\lambda)$ is a compact analytic family of bounded operators from $e^{-\gamma/x}L^2$ to $e^{-\gamma/x}L^2$ for any $\gamma > A$. Moreover $1 + K(\lambda_0)$ is invertible since $\|K(\lambda_0)\|_{L^2 \rightarrow L^2} \leq 1/2$ if $i\lambda_0$ is large enough. Then by analytic Fredholm theory, the resolvent $R_V(\lambda)$ has an meromorphic extension to $\text{Im}(\lambda) < A, \text{Re}(\lambda) \neq 0$ as a bounded operator from $e^{-\gamma/x}L^2$ to $e^{\gamma/x}L^2$ if $\gamma > A$, given by

$$R_V(\lambda) = E(\lambda)(1 + K(\lambda))^{-1}.$$

Now $(1 + K(\lambda))^{-1} = 1 + Q(\lambda)$ for some $Q(\lambda) = -K(\lambda)(1 + K(\lambda))^{-1}$ mapping $e^{-\gamma/x}L^2$ to itself for any $\gamma > A$, which proves the mapping properties of $R_V(\lambda)$ on exponential weighted spaces. For the mapping properties on $\{\operatorname{Re}(\lambda) = 0\}$, a similar argument works. \square

A corollary of this Lemma is the mapping property

Corollary 5.2. *For $\lambda \in \mathbb{R} \setminus \{0\}$ not a pole of $R_V(\lambda)$, and $f \in e^{-\gamma/x}L^\infty$ for some $\gamma > 0$, then there exists $v \in C^\infty(\partial\overline{M}_0)$ such that*

$$R_V(\lambda)f - x^{\frac{1}{2}}e^{-i\lambda/x}v \in L^2.$$

Proof. Using the expression $R_V(\lambda) = E(\lambda)(1 + Q(\lambda))$ of the proof of Lemma 5.1, it suffices to know the mapping property of $E(\lambda)$ on $e^{-\gamma/x}L^2$, but since outside a compact set (i.e. in the ends) $E(\lambda)$ is given by the free resolvent on \mathbb{R}^2 , this amounts to proving the statement in \mathbb{R}^2 , which is well-known: for instance, this is proved for $f \in C_0^\infty(\mathbb{R}^2)$ in Section 1.7 [14] but the proof extends easily to $f \in e^{-\gamma/x}L^\infty(\mathbb{R}^2)$ since the only used assumption on f for applying a stationary phase argument is actually that the Fourier transform $\hat{f}(z)$ has a holomorphic extension in a complex neighbourhood of \mathbb{R}^2 . \square

We also have a boundary pairing, the proof of which is exactly the same as [14, Lemma 2.2] (see also Proposition 3.1 of [26]).

Lemma 5.3. *For $\lambda > 0$ and $V \in e^{-\gamma/x}L^\infty(M_0)$, if $u_\pm \in x^{-\alpha}L^2(M_0)$ for some $\alpha > 1/2$ and $(\Delta_{g_0} - \lambda^2 + V)u_\pm \in x^\alpha L^2(M_0)$ with*

$$u_+ - x^{\frac{1}{2}}e^{i\lambda/x}f_{++} - x^{\frac{1}{2}}e^{-i\lambda/x}f_{+-} \in L^2, \quad u_- - x^{\frac{1}{2}}e^{i\lambda/x}f_{-+} - x^{\frac{1}{2}}e^{-i\lambda/x}f_{--} \in L^2$$

for some $f_{\pm\pm} \in C^\infty(\partial\overline{M}_0)$, then

$$\langle u_+, (\Delta_{g_0} + V - \lambda^2)u_- \rangle - \langle (\Delta_{g_0} + V - \lambda^2)u_+, u_- \rangle = 2i\lambda \int_{\partial\overline{M}_0} (f_{++}\overline{f_{-+}} - f_{+-}\overline{f_{--}})$$

where the volume form on $\partial\overline{M}_0 \simeq \sqcup_{i=1}^N S^1$ is induced by the metric $x^2g|_{T\partial\overline{M}_0}$.

As a corollary, the same exact arguments as in Sections 2.2 to 2.5 in [14] show ¹

Corollary 5.4. *The operator $R_V(\lambda)$ is analytic on $\lambda \in \mathbb{R} \setminus \{0\}$ as a bounded operator from $x^\alpha L^2$ to $x^{-\alpha}L^2$ if $\alpha > 1/2$.*

In \mathbb{R}^2 there is a Poisson operator $P_0(\lambda)$ mapping $C^\infty(S^1)$ to $x^{-\alpha}L^2(\mathbb{R}^2)$ for $\alpha > 1/2$, which satisfies that for any $f_+ \in C^\infty(S^1)$ there exists $f_- \in C^\infty(S^1)$ such that

$$P_0(\lambda)f_+ - x^{\frac{1}{2}}e^{i\lambda/x}f_+ - x^{\frac{1}{2}}e^{-i\lambda/x}f_- \in L^2, \quad (\Delta - \lambda^2)P_0(\lambda)f_+ = 0.$$

We can therefore define in our case a similar Poisson operator $P_V(\lambda)$ mapping $C^\infty(\partial\overline{M}_0)$ to $x^{-\alpha}L^2$ for $\alpha > 1/2$, by

$$(20) \quad P_V(\lambda)f_+ := (1 - \chi)P_0(\lambda)f_+ - R_V(\lambda)(\Delta_{g_0} + V - \lambda^2)(1 - \chi)P_0(\lambda)f_+$$

where $1 - \chi \in C^\infty(M_0)$ equals 1 in the ends E_i and $P_0(\lambda)$ denotes here the Schwartz kernel of the Poisson operator on \mathbb{R}^2 pulled back to each of the Euclidean ends E_i of M_0 in the

¹In [14], a unique continuation is used for Schwartz solutions of $(\Delta + V - \lambda^2)u = 0$ when V is a compactly supported potential on \mathbb{R}^n but the same result is also true in our setting, this is a consequence of a standard Carleman estimate.

obvious way. Then, since $(\Delta_{g_0} + V - \lambda^2)(1 - \chi)P_0(\lambda)f_+ \in e^{-\gamma/x}L^2$ for all $\gamma > 0$, it suffices to use Corollaries 5.2 and 5.4 to see that it defines an analytic Poisson operator $P_V(\lambda)$ on $\lambda \in \mathbb{R} \setminus \{0\}$ satisfying that for all $f_+ \in C^\infty(\partial\overline{M}_0)$, there exists $f_- \in C^\infty(\partial\overline{M}_0)$ such that

$$(21) \quad P_V(\lambda)f_+ - x^{\frac{1}{2}}e^{i\lambda/x}f_+ - x^{\frac{1}{2}}e^{-i\lambda/x}f_- \in L^2, \quad (\Delta + V - \lambda^2)P_V(\lambda)f_+ = 0.$$

Moreover, it is easily seen to be the unique solution of (21): indeed, if two such solutions exist then the difference is a solution u with asymptotic $x^{\frac{1}{2}}e^{-i\lambda/x}f_- + L^2$ for some $f_- \in C^\infty(\partial\overline{M}_0)$, but applying Lemma 5.3 with $u_- = u_+ = u$ shows that $f_- = 0$, thus $u \in L^2$, which implies $u = 0$ by Corollary 5.4.

Definition 5.5. *The scattering matrix $S_V(\lambda) : C^\infty(\partial\overline{M}_0) \rightarrow C^\infty(\partial\overline{M}_0)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ is defined to be the map $S_V(\lambda)f_+ := f_-$ where f_- is given by the asymptotic*

$$P_V(\lambda)f_+ = x^{\frac{1}{2}}e^{i\lambda/x}f_+ + x^{\frac{1}{2}}e^{-i\lambda/x}f_- + g, \quad \text{with } g \in L^2.$$

We remark that, using Lemma 5.3 and the uniqueness of the Poisson operator, one easily deduces for $\lambda \in \mathbb{R} \setminus \{0\}$

$$(22) \quad S_V(\lambda)^* = S_V(-\lambda) = S_V(\lambda)^{-1}$$

where the scalar product on $L^2(\partial\overline{M}_0)$ is induced by the metric $x^2g_0|_{T\partial\overline{M}_0}$.

We can now state a density result similar to Proposition 3.3 of [26]:

Proposition 5.6. *If $V \in e^{-\gamma_0/x}L^\infty(M_0)$ (resp. $V \in e^{-\gamma_0/x^2}L^\infty(M_0)$) for some $\gamma_0 > 0$, and $\lambda \in \mathbb{R} \setminus \{0\}$, then for any $0 < \gamma < \gamma' < \gamma_0$ the set*

$$\mathcal{F} := \{P_V(\lambda)f_+; f_+ \in C^\infty(\partial\overline{M}_0)\}$$

is dense in the null space of $\Delta_{g_0} + V - \lambda^2$ in $e^{\gamma/x}L^2(M_0)$ for the topology of $e^{\gamma'/x}L^2(M_0)$ (resp. in $e^{\gamma/x^2}L^2(M_0)$ for the topology of $e^{\gamma'/x^2}L^2(M_0)$).

Proof. First assume $V \in e^{-\gamma_0/x}L^\infty(M_0)$. Let $w \in e^{-\gamma'/x}L^2$ be orthogonal to \mathcal{F} , and set $u_- := R_V(\lambda)w$ and $u_+ = P_V(\lambda)f_{++}$ for some $f_{++} \in C^\infty(\partial\overline{M}_0)$. Then, define $f_{--} \in C^\infty(\partial\overline{M}_0)$ by $R_V(\lambda)w - x^{\frac{1}{2}}e^{-i\lambda/x}f_{--} \in L^2$, and from Lemma 5.3 we obtain $\langle f_{++}, f_{--} \rangle = 0$ since $\langle w, P_V(\lambda)f_{++} \rangle = 0$ by assumption. Since $f_{+-} = S_V(\lambda)f_{++}$ is arbitrary, then $f_{--} = 0$ and $u_- \in L^2$. In particular, from the parametrix constructed in the proof of Lemma 5.1

$$R_V(\lambda)w - (1 - \chi_0)R_0(\lambda)(1 - \chi)(1 + Q(\lambda))w \in L^2$$

with $(1 + Q(\lambda))w \in e^{-\gamma'/x}L^2$. Since in each end, $R_0(\lambda)$ is the integral kernel of the free resolvent of the Euclidean Laplacian on \mathbb{R}^2 and $(1 - \chi_0)$ and $(1 - \chi)$ are supported in the ends, we can view the term $(1 - \chi_0)R_0(\lambda)(1 - \chi)(1 + Q(\lambda))w$ as a disjoint sum (over the ends) of functions on \mathbb{R}^2 of the form

$$(23) \quad (1 - \chi_0(z)) \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{iz\xi} (\xi^2 - \lambda^2 - i0)^{-1} \hat{f}(\xi) d\xi$$

where in each end E_i , $f = (1 - \chi)(1 + Q(\lambda))w \in e^{-\gamma'/x}L^2(E_i)$ can be considered as a function in $e^{-\gamma'|z|}L^2(\mathbb{R}^2)$. By the Paley-Wiener theorem, \hat{f} is holomorphic in a strip $U = \{|\text{Im}(\xi)| < \gamma'\}$ with bound $\sup_{\eta \leq \gamma} \|\hat{f}(\cdot + i\eta)\|_{L^2(\mathbb{R}^2)} < \infty$ for all $\gamma < \gamma'$, so the fact that (23) is in L^2 implies that \hat{f} vanishes at the real sphere $\{\xi \in \mathbb{R}^2; \xi^2 = \lambda^2\}$, and thus there exists h holomorphic in U such that $\hat{f}(\xi) = (\xi^2 - \lambda^2)h(\xi)$ (see e.g. the proof of Lemma 2.5 in [17]), and satisfying the same types of L^2 estimates as \hat{f} in U on lines $\text{Im}(\xi) = \text{cst}$. By the Paley-Wiener theorem

again, we deduce that (23) is in $e^{-\gamma|z|}L^2$ and thus $R_V(\lambda)w \in e^{-\gamma/x}L^2(M_0)$ for any $\gamma < \gamma'$. Then if $v \in e^{\gamma/x}L^2(M_0)$ and $(\Delta_{g_0} + V - \lambda^2)v = 0$, one has by integration by parts

$$0 = \langle R_V(\lambda)w, (\Delta_{g_0} + V - \lambda^2)v \rangle = \langle w, v \rangle$$

which ends the proof in the case $V \in e^{-\gamma_0/x}L^\infty(M_0)$. The quadratic decay case $V \in e^{-\gamma_0/x^2}L^\infty(M_0)$ is exactly similar but instead of Paley-Wiener theorem, we use Corollary 7.3 and the inclusions $e^{-\gamma'/x^2}L^2 \subset e^{-\gamma''/x^2}L^1 \cap e^{-\gamma''/x^2}L^2$ and $e^{-\gamma'/x^2}L^\infty \subset e^{-\gamma/x^2}L^2$ for all $\gamma < \gamma'' < \gamma'$. \square

6. IDENTIFYING THE POTENTIAL

6.1. The case of a surface. On a Riemann surface (M_0, g_0) with N Euclidean ends and genus g , we assume that $V_1, V_2 \in C^{1,\alpha}(M_0)$ are two real valued potentials such that the respective scattering operators $S_{V_1}(\lambda)$ and $S_{V_2}(\lambda)$ agree for a fixed $\lambda > 0$. We also assume that for all $\gamma > 0$

$$V_1, V_2 \in \begin{cases} e^{-\gamma/x}L^\infty(M_0) & \text{if } N \geq \max(2g + 1, 2) \\ e^{-\gamma/x^2}L^\infty(M_0) & \text{if } N \geq g + 1. \end{cases}$$

By considering the asymptotics of $u_1 := P_{V_1}(\lambda)f_1$ and $P_{V_2}(-\lambda)f_2$ for $f_i \in C^\infty(\partial\overline{M}_0)$ we easily have by integration by parts that

$$(24) \quad \begin{aligned} \int_{M_0} (V_1 - V_2)u_1\overline{u_2} \, d\text{vol}_{g_0} &= -2i\lambda \int_{\partial\overline{M}_0} S_{V_1}(\lambda)f_1 \cdot \overline{f_2} - f_1 \cdot \overline{S_{V_2}(-\lambda)f_2} \\ &= -2i\lambda \int_{\partial\overline{M}_0} (S_{V_1}(\lambda) - S_{V_2}(\lambda))f_1 \cdot \overline{f_2} = 0 \end{aligned}$$

by using (22). From Proposition 5.6, this implies by density that, if $V \in e^{-\gamma/x}L^\infty$ (resp. $V \in e^{-\gamma/x^2}L^\infty$ for all $\gamma > 0$), then for all solutions u_i of $(\Delta_{g_0} + V_i - \lambda^2)u_i = 0$ in $e^{\gamma/x}L^2(M_0)$ (resp. $u_i \in e^{\gamma'/x^2}L^2(M_0)$) for some $\gamma' > 0$, we have

$$(25) \quad \int_{M_0} (V_1 - V_2)u_1\overline{u_2} \, d\text{vol}_{g_0} = 0.$$

We shall now use our complex geometric optics solutions as special solutions in the weighted space $e^{-\gamma'/hx}L^2(M_0)$ (resp. $e^{-\gamma'/hx^2}L^2(M_0)$) for some $\gamma' > 0$ if $V \in e^{-\gamma/x}L^\infty$ (resp. $V \in e^{-\gamma/x^2}L^\infty$) for all $\gamma > 0$.

Let $p \in M_0$ be such that, using Proposition 2.1, we can choose a holomorphic Morse function $\Phi = \varphi + i\psi$ with linear or quadratic growth on M_0 (depending on the topological assumption), with a critical point at p . Then for the complex geometric optics solutions u_1, u_2 with phase Φ constructed in Section 4, the identity (25) holds true. We will then deduce the

Proposition 6.1. *Let $\lambda \in (0, \infty)$ and assume that $S_{V_1}(\lambda) = S_{V_2}(\lambda)$, then $V_1(p) = V_2(p)$.*

Proof. Let u_1 and u_2 be solutions on M_0 to

$$(\Delta_g + V_j - \lambda^2)u_j = 0$$

constructed in Section 4 with phase Φ for u_1 and $-\Phi$ for u_2 , thus of the form

$$u_1 = e^{\Phi/h}(a + r_1^1 + r_2^1), \quad u_2 = e^{-\Phi/h}(a + r_1^2 + r_2^2).$$

We have the identity

$$\int_{M_0} u_1(V_1 - V_2)\overline{u_2} \, d\text{vol}_{g_0} = 0$$

Then by using the estimates in Lemma 4.1 and Proposition 4.1 we have, as $h \rightarrow 0$,

$$\int_{M_0} e^{2i\psi/h}|a|^2(V_1 - V_2) \, d\text{vol}_{g_0} + h \int_{M_0} e^{2i\psi/h}(\overline{a\tilde{r}_{12}^1} + \overline{a\tilde{r}_{12}^2})(V_1 - V_2) \, d\text{vol}_{g_0} + o(h) = 0$$

where $\tilde{r}_{12}^j \in L^\infty(M_0)$ are defined in Lemma 4.1, with the superscript j referring to the solution for the potential V_j ; in particular these functions \tilde{r}_{12}^j are independent of h .

By splitting $V_i(\cdot) = (V_i(\cdot) - V_i(p)) + V_i(p)$ and using the $C^{1,\alpha}$ regularity assumption on V_i , one can use stationary phase for the $V_i(p)$ term and integration by parts to gain a power of h for the $V_i(\cdot) - V_i(p)$ term (see the proof of Lemma 5.4 in [8] for details) to deduce

$$\int_{M_0} e^{2i\psi/h}|a|^2(V_1 - V_2) \, d\text{vol}_{g_0} = Ch(V_1(p) - V_2(p)) + o(h)$$

for some $C \neq 0$. Therefore,

$$Ch(V_1(p) - V_2(p)) + h \int_{M_0} e^{2i\psi/h}(\overline{a\tilde{r}_{12}^1} + \overline{a\tilde{r}_{12}^2})(V_1 - V_2) \, d\text{vol}_{g_0} = o(h).$$

Now to deal with the middle terms, it suffices to apply a Riemann-Lebesgue type argument like Lemma 5.3 of [8] to deduce that it is a $o(h)$. The argument is simply to approximate the amplitude in the $L^1(M_0)$ norm by a smooth compactly supported function and then use stationary phase to deal with the smooth function. We have thus proved that $V_1(p) = V_2(p)$ by taking $h \rightarrow 0$. \square

7. APPENDIX

To obtain mapping properties of the resolvent of $\Delta_{\mathbb{R}^2}$ acting on functions with Gaussian decay, we shall give two Lemmas on Fourier transforms of functions with Gaussian decay.

Lemma 7.1. *Let $f(z) \in e^{-\gamma|z|^2}L^2(\mathbb{R}^2)$ for some $\gamma > 0$. Then the Fourier transform $\hat{f}(\xi)$ extends analytically to \mathbb{C}^2 and for all $\xi, \eta \in \mathbb{R}^2$,*

$$\|\hat{f}(\xi + i\eta)\|_{L^2(\mathbb{R}^2, d\xi)} \leq 2\pi e^{\frac{|\eta|^2}{4\gamma}} \|e^{\gamma|z|^2} f\|_{L^2(\mathbb{R}^2)}.$$

If $f(z) \in e^{-\gamma|z|^2}L^1(\mathbb{R}^2)$ for some $\gamma > 0$ then

$$\sup_{\xi \in \mathbb{R}^2} |\hat{f}(\xi + i\eta)| \leq e^{\frac{|\eta|^2}{4\gamma}} \|e^{\gamma|z|^2} f\|_{L^1(\mathbb{R}^2)}.$$

Proof. The first statement is clear. For the bound, we write

$$\hat{f}(\xi + i\eta) = e^{\frac{|\eta|^2}{4\gamma}} \int_{\mathbb{R}^2} e^{-i\xi \cdot z} e^{-\gamma|z - \frac{\eta}{2\gamma}|^2} e^{\gamma|z|^2} f(z) dz = e^{\frac{|\eta|^2}{4\gamma}} \mathcal{F}_{z \rightarrow \xi}(e^{-\gamma|z - \frac{\eta}{2\gamma}|^2} e^{\gamma|z|^2} f(z)).$$

But the function $e^{-\gamma|z - \frac{\eta}{2\gamma}|^2} e^{\gamma|z|^2} f(z)$ is in $L^2(\mathbb{R}^2, dz)$ and its norm is bounded uniformly by $\|e^{\gamma|z|^2} f\|_{L^2}$, thus it suffices to use the Plancherel theorem to obtain the desired bound. The L^∞ bound is similar. \square

Lemma 7.2. *Let $F(\xi + i\eta)$ be a complex analytic function on $\mathbb{R}^2 + i\mathbb{R}^2 = \mathbb{C}^2$ such that there is $C > 0$ and $\gamma > 0$ with*

$$\|F(\xi + i\eta)\|_{L^2(\mathbb{R}^2, d\xi)} \leq C e^{\frac{|\eta|^2}{4\gamma}} \text{ and } \sup_{\xi \in \mathbb{R}^2} |F(\xi + i\eta)| \leq C e^{\frac{|\eta|^2}{4\gamma}}.$$

If F vanishes on the real submanifold $\{|\xi|^2 = \lambda^2\}$, then $\mathcal{F}_{\xi \rightarrow z}^{-1}\left(\frac{F(\xi)}{|\xi|^2 - \lambda^2}\right) \in e^{-\gamma|z|^2} L^\infty(\mathbb{R}^2)$.

Proof. First by analyticity of F , one has that F vanishes on the complex hypersurface $M_\lambda := \{\zeta \in \mathbb{C}^2; \zeta \cdot \zeta = \lambda^2\}$ (see for instance the proof of Lemma 2.5 of [17]), and in particular $G(\zeta) = F(\zeta)/(\zeta \cdot \zeta - \lambda^2)$ is an analytic function on \mathbb{C}^2 . We will first prove that for each $\eta \in \mathbb{R}^2$, $G(\xi + i\eta) \in L^1(\mathbb{R}^2, d\xi) \cap L^\infty(\mathbb{R}^2, d\xi)$ and

$$(26) \quad \|G(\xi + i\eta)\|_{L^1(\mathbb{R}^2, d\xi)} \leq C e^{\frac{|\eta|^2}{4\gamma}}.$$

If $|\eta| \leq 2$ we choose the disc $B := \{\xi \in \mathbb{R}^2; |\xi|^2 < 2(4 + \lambda^2)\}$ and let $\zeta := \xi + i\eta$. Then $\|G(\xi + i\eta)\|_{L^1(B, d\xi)}$ and $\|(\zeta \cdot \zeta - \lambda^2)^{-1}\|_{L^2(\mathbb{R}^2 \setminus B, d\xi)}$ are uniformly bounded for $|\eta| \leq 2$, and we obtain by Cauchy-Schwarz that (26) holds for $|\eta| \leq 2$. For the case $|\eta| > 2$ we define $U_\eta := \{\xi \in \mathbb{R}^2; |\zeta \cdot \zeta - \lambda^2| > |\eta|\}$ and note that

$$\begin{aligned} \sup_{|\eta| > 2} \|(\zeta \cdot \zeta - \lambda^2)^{-1}\|_{L^1(\mathbb{R}^2 \setminus U_\eta, d\xi)} &< \infty, \\ \sup_{|\eta| > 2} \|(\zeta \cdot \zeta - \lambda^2)^{-1}\|_{L^2(U_\eta, d\xi)} &< \infty. \end{aligned}$$

These results follow by decomposing the integration sets to parts where one can change coordinates $\xi_1 + i\xi_2$ to $\tilde{\xi}_1 + i\tilde{\xi}_2 := \zeta \cdot \zeta - \lambda^2$, and by evaluating simple integrals. Then (26) follows from Cauchy-Schwarz and the estimates for F .

Let $\eta = 2\gamma z$, we use a contour deformation from \mathbb{R}^2 to $2i\gamma z + \mathbb{R}^2$ in \mathbb{C}^2 ,

$$\int_{\mathbb{R}^2} e^{iz \cdot \xi} G(\xi) d\xi = \int_{\mathbb{R}^2} e^{iz \cdot (\xi + 2i\gamma z)} G(\xi + 2i\gamma z) d\xi,$$

which is justified by the fact that $G(\xi + i\eta) \in L^1(\mathbb{R}^2 \times K, d\xi d\eta)$ for any compact set K in \mathbb{R}^2 by the uniform bound (26). Now using (26) again shows that

$$\left| \int_{\mathbb{R}^2} e^{iz \cdot \xi} G(\xi) d\xi \right| \leq C e^{-\gamma|z|^2}$$

which ends the proof. □

Corollary 7.3. *Let $f(z) \in e^{-\gamma|z|^2} L^2(\mathbb{R}^2) \cap e^{-\gamma|z|^2} L^1(\mathbb{R}^2)$ for some $\gamma > 0$. Assume that its Fourier transform $\hat{f}(\xi)$ vanishes on the sphere $\{|\xi| = |\lambda|\}$, then one has*

$$\mathcal{F}_{\xi \rightarrow z}^{-1}\left(\frac{\hat{f}(\xi)}{|\xi|^2 - \lambda^2}\right) \in e^{-\gamma|z|^2} L^\infty(\mathbb{R}^2).$$

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