On the work of Joonas Ilmavirta

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This article describes the thesis work of PhD Joonas Ilmavirta, who was awarded the Stefan Banach doctoral dissertation prize in 2015. Ilmavirta received his PhD in Mathematics in 2014 at the University of Jyväskylä, Finland. He began his PhD studies under my supervision in 2012 in the inverse problems group at the Department of Mathematics and Statistics in Jyväskylä.

The PhD thesis of Ilmavirta is concerned with inverse problems and in particular broken ray tomography. Inverse problems often arise in imaging methods, such as X-ray computed tomography (whose underlying mathematical model is the X-ray or Radon transform [Na01]) and electrical impedance tomography (whose mathematical model is the inverse conductivity problem posed by A.P. Calderón [Ca80]). The field of inverse problems has been actively studied in the last 30 years, and it is now an important and topical part of both pure and applied mathematics. We refer to the textbooks [Is06], [Ki11] and the survey [Uh14] for more information on the mathematical theory of inverse problems.

Broken ray tomography arose in my recent work with C.E. Kenig [KS13], where we showed that Calderón's inverse conductivity problem with partial data reduces to the invertibility of a broken ray transform in certain geometries. The broken ray transform encodes the integrals of a function over all broken lines or geodesics that reflect on a specified part of the boundary, and the task is to recover the function from these integrals whenever possible. This is a generalization of standard X-ray tomography, and besides the early work of Mukhometov [Mu88] not so much was known about the topic before.

Ilmavirta obtained his MSc in physics but had studied an equivalent amount of mathematics when he began as a PhD student in 2012. From the very beginning, he demonstrated a great amount of originality and efficiency. The first problem I suggested to him was the inversion of the broken ray transform in the unit disc, and he obtained his first results almost immediately. During this work, he independently derived Cormack's formulas for the Radon transform (from a paper [Co63] for which Cormack was awarded the Nobel prize in medicine) and an inversion procedure for a generalized Abel transform, without having seen these facts before.

The question studied in the first paper is easy to state in full detail:

Question. Let $\mathbb{D}$ be the unit disc in $\mathbb{R}^2$, let $E \subset \partial\mathbb{D}$ be a nonempty open subset, and let $f : \overline{\mathbb{D}} \to \mathbb{R}$ be a Hölder continuous function. Can f be determined from the knowledge of its integrals over all broken lines which start and end on E and reflect on $\partial\mathbb{D} \setminus E$ in the standard way (angle of incidence = angle of reflection)?

Ilmavirta showed in [Il13] that the answer is positive if f is additionally quasianalytic in the angular variable. The question remains open without this additional assumption.

The next two papers of Ilmavirta were related to boundary determination [Il14] and reflection methods [Il15a] in broken ray tomography. Also these works were completed quickly and independently, and they contain original insight coming from his physics background. The fourth paper [Il15b] gives a new and very elegant Fourier approach to inverting the X-ray transform on the torus, also in the distributional and tensor valued cases. The fifth paper in the thesis [IS16] involves another different approach to broken ray tomography, based on PDE methods and energy estimates, and shows that broken ray tomography is possible in the exterior of a convex obstacle in a Riemann surface with nonpositive curvature. The methods are extensions of earlier arguments used for the case of non-reflected rays.

The PhD thesis of Ilmavirta consists of an introductory part and five articles, of which four are single-authored and the fifth is joint work of Ilmavirta and myself. The introductory part of the thesis is an excellent overview of different aspects of broken ray tomography. It is notable that the thesis contains as an appendix an introduction to inverse problems directed toward the general audience – this expository text is written in English, Finnish and Latin!

It is remarkable that within a period of two years, and with no prior exposure to the research field, Ilmavirta completed five articles with five different approaches to broken ray tomography. Problems of this type appear to be difficult, and not so much was known about the topic before, but the situation has changed with the PhD thesis of Ilmavirta which is now the basic reference for broken ray tomography. For his work, Ilmavirta has also received the Finnish Inverse Prize awarded by the Finnish Inverse Problems Society in 2014 (this prize is awarded for an outstanding PhD thesis in the field of inverse problems broadly defined).

References

[Ca80] A.P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matemática, Río de Janeiro, 1980.

[Co63] A.M. Cormack, *Representation of a function by its line integrals, with some radiological applications*, J. Appl. Phys. **34** (1963), 2722–2727.

[Il13] J. Ilmavirta, *Broken ray tomography in the disc*, Inverse Problems **29** (2013), 035008.

[Il14] J. Ilmavirta, *Boundary reconstruction for the broken ray transform*, Ann. Acad. Scient. Fenn. Math. **39** (2014), 485–502.

[Il15a] J. Ilmavirta, *A reflection approach to the broken ray transform*, Math. Scand. **117** (2015), 231–257.

[Il15b] J. Ilmavirta, *On Radon transforms on tori*, J. Fourier Anal. Appl. **21** (2015), 370–382.

[IS16] J. Ilmavirta, M. Salo, *Broken ray transform on a Riemann surface with a convex obstacle*, Comm. Anal. Geom. (to appear).

[Is06] V. Isakov, Inverse problems for partial differential equations. Applied Mathematical Sciences **127**, Springer-Verlag, 2006.

[KS13] C.E. Kenig, M. Salo, *The Calderón problem with partial data on manifolds and applications*, Analysis & PDE **6** (2013), 2003–2048.

[Ki11] A. Kirsch, An introduction to the mathematical theory of inverse problems. Applied Mathematical Sciences **120**, Springer-Verlag, 2011.

[Mu88] R. G. Mukhometov, *Problems of integral geometry in a domain with a reflecting part of the boundary* (in Russian, translation in Soviet Math. Dokl. **36** (1988), 260–264). Dokl. Akad. Nauk SSSR **296** (1987), 279–283.

[Na01] F. Natterer, The mathematics of computerized tomography. Classics in Applied Mathematics **32**, SIAM 2001.

[Uh14] G. Uhlmann, *Inverse problems: seeing the unseen*, Bull. Math. Sci. **4** (2014), 209–279.