NOWHERE CONFORMALLY HOMOGENEOUS MANIFOLDS AND LIMITING CARLEMAN WEIGHTS

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ABSTRACT. In this note we prove that a generic Riemannian manifold of dimension ≥ 3 does not admit any nontrivial local conformal diffeomorphisms. This is a conformal analog of a result of Sunada concerning local isometries, and makes precise the principle that generic manifolds in high dimensions do not have conformal symmetries. Consequently, generic manifolds of dimension ≥ 3 do not admit nontrivial conformal Killing vector fields near any point. As an application to the inverse problem of Calderón on manifolds, this implies that generic manifolds of dimension ≥ 3 do not admit limiting Carleman weights near any point.

1. INTRODUCTION

It is a well-known fact in geometry that generic Riemannian manifolds should not admit symmetries or other special structure. Various precise statements which formalize this principle may be found in the literature [1, 2, 3, 4, 14, 25]. We mention here the result concerning nonexistence of local isometries proved by Sunada [22] in his work on isospectral manifolds.

Theorem. Let M be a compact C^∞ manifold without boundary. There is a residual set of Riemannian metrics on M for which there are no isometries between any distinct open subsets of the manifold.

Recall that a subset of a topological space is called *residual* if it contains a countable intersection of open dense sets. A property is called *generic* if it holds for all elements in some residual set. We equip the set of all Riemannian metrics on M with the C^∞ topology as described in [3], and this space then becomes a complete metric space. By Baire's theorem, any residual set in a complete metric space is dense. Thus, in particular, the set of metrics admitting no local isometries is dense in the set of all Riemannian metrics on M.

We are interested in a conformal version of Sunada's result. Here the two-dimensional case is special, since any Riemann surface is locally conformal to a subset of Euclidean space by the existence of isothermal coordinates and thus admits many nontrivial local conformal diffeomorphisms. It is an accepted fact in conformal geometry that generic Riemannian manifolds of dimension $n \geq 3$ should not admit conformal symmetries. However, we were not able to find a precise proof for such a statement in the literature. In this note we prove the following theorem which makes this statement rigorous.

Theorem 1.1. Let M be a compact boundaryless C^∞ manifold having dimension $n \geq 3$. There is a residual set of Riemannian metrics on M for which there are no conformal diffeomorphisms between any distinct open subsets of the manifold.

Recall that a diffeomorphism ϕ from (U, g) onto (V, h) is called conformal if $\phi^* h = cg$ in U for some smooth positive function c on U. The function c is called the conformal factor, and two manifolds are called conformal if there is a conformal diffeomorphism between them. The result states that for a generic Riemannian metric g on M, there are no distinct open subsets U and V such that (U, g) and (V, g) would be conformal.

We now state some corollaries of the main theorem. A smooth vector field X on a Riemannian manifold (U, g) is called *Killing* if the local flows generated by X are isometries, and *conformal Killing* if X is Killing in (U, cg) for some positive function c. Equivalently, X is Killing (resp. conformal Killing) if and only if $\mathcal{L}_X g = 0$ (resp. $\mathcal{L}_X g = \lambda g$ for some function λ) where $\mathcal{L}_X g$ is the Lie derivative of g. If g is given in local coordinates by the matrix (g_{jk}), then $\mathcal{L}_X g$ is the symmetric 2-tensor with local coordinate expression

$$\mathcal{L}_X g = \sum_{i,j,k=1}^{n} (g_{jk} X^k_{;i} + g_{ik} X^k_{;j}) \, dx^i \otimes \, dx^j.$$

Here $X^k_{;i} = \partial_{x_i} X^k + \Gamma^k_{il} X^l$ is the covariant derivative, where Γ^c_{ab} are the Christoffel symbols for g.

It follows that local flows generated by conformal Killing vector fields are conformal diffeomorphisms, and the existence of a nontrivial conformal Killing field near a point implies the existence of a conformal diffeomorphism from some open set (U, g) onto a disjoint open set (V, g). Thus we have the following consequence of Theorem 1.1.

Corollary 1.2. Let M be a compact boundaryless C^∞ manifold having dimension $n \geq 3$. There is a residual set of Riemannian metrics on M which do not admit a nontrivial conformal Killing vector field near any point.

Following [18], we say that a Riemannian manifold is *locally conformally homogeneous* if for any point p and for any tangent vector v at p, there is a conformal Killing vector field X in some neighborhood of p such that $X(p) = v$. The last result implies that generic metrics in dimensions $n \geq 3$ are nowhere conformally homogeneous.

It is easy to convince oneself why Corollary 1.2 could be true. If $\mathcal{L}_X g = \lambda g$, it follows by taking traces that $\lambda = \frac{2}{n} \text{div}(X)$ where $\text{div}(X) = \sum_{k=1}^{n} X^k_{;k}$ is the metric divergence. Thus conformal Killing vector fields are solutions of the equation $T(X) = 0$, where T is the trace-free deformation tensor

$$T(X) = \mathcal{L}_X g - \frac{2}{n} \text{div}(X) g.$$

Since $T(X)$ is trace-free and symmetric, in local coordinates $T(X) = 0$ is a system of $\frac{n(n+1)}{2} - 1$ equations in n unknowns. Therefore, if $n \geq 3$, it is not surprising that for generic metrics the only solution is $X = 0$.

A more sophisticated approach, described in [5] in a general setup, is to observe that the overdetermined system $T(X) = 0$ is of finite type. By the method of prolongation, one can construct a vector bundle V together with a connection $\tilde{\nabla}$ such that solutions of $T(X) = 0$ correspond to parallel sections of V. However, the curvature of $\tilde{\nabla}$ imposes restrictions on the existence of parallel sections, and again for generic g one expects to have no nontrivial solutions. Related results are given in [9, 10, 21].

The final corollary concerns recent results on Calderón's inverse problem on manifolds. This was the original motivation for the present study, and we explain the setup in some detail. Calderón's inverse problem was introduced in the Euclidean case by Calderón [6] as the question of determining the conductivity function of a medium from voltage and current measurements made on its boundary, and it has been intensively studied (see the recent survey [24]). The anisotropic case corresponds to determining a conductivity matrix up to change of coordinates instead of a scalar function, and it was observed in [17] that this is very closely related to a geometric problem which we proceed to describe.

Let (M, g) be a compact oriented Riemannian manifold with smooth boundary and let Δ_g be the Laplace-Beltrami operator on M. The boundary data of harmonic functions on (M, g) is given by the Cauchy data set

$$C_g = \{(u|_{\partial M}, \partial_\nu u|_{\partial M})\,;\, \Delta_g u = 0 \text{ in } M, u \in C^\infty(M)\}.$$

Equivalently, the Cauchy data set is the graph of the Dirichlet-to-Neumann map

$$\Lambda_g : C^\infty(\partial M) \to C^\infty(\partial M),\ f \mapsto \partial_\nu u|_{\partial M} \text{ where } \Delta_g u = 0 \text{ in } M,\ u|_{\partial M} = f.$$

Here $\partial_\nu u|_{\partial M} = g(du, \nu)|_{\partial M}$ where ν is the 1-form corresponding to the outer unit normal vector on ∂M.

It follows immediately that $C_{\psi^* g} = C_g$ whenever $\psi : M \to M$ is a diffeomorphism satisfying $\psi|_{\partial M} = \mathrm{Id}$. The Calderón problem on manifolds asks to prove that the Riemannian metric on a manifold is determined up to diffeomorphism by the Cauchy data of harmonic functions (or, equivalently, by the Dirichlet-to-Neumann map).

Conjecture. Let (M, g_1) and (M, g_2) be two compact oriented Riemannian manifolds with smooth boundary having dimension $n \geq 3$. If $C_{g_1} = C_{g_2}$, then $g_2 = \psi^* g_1$ for some diffeomorphism $\psi : M \to M$ satisfying $\psi|_{\partial M} = \mathrm{Id}$.

This has been proved for real-analytic manifolds [15, 16, 17] and also for Einstein manifolds [11], and the corresponding two-dimensional result is known for any smooth Riemann surface [16]. The general case in dimensions $n \geq 3$ is open. Recently in [8, 12] progress was made on this problem for a certain class of conformal smooth manifolds. The results were based on a construction of special harmonic functions involving *limiting Carleman weights*, introduced in [13] in the Euclidean case. We refer to [8] for the precise definition. For the purposes of this paper, we only mention that a limiting Carleman weight is essentially a smooth real-valued function φ with

nonvanishing gradient for which one expects to have weighted estimates

$$\|u\|_{L^2} \leq \frac{C}{|\tau|}\|e^{\tau\varphi}\Delta_g e^{-\tau\varphi}u\|_{L^2}, \quad u \in C_c^\infty,$$

which hold for all real τ with $|\tau|$ sufficiently large and where C is independent of τ and u.

Let (U, g) be an open Riemannian manifold (that is, U has no boundary and no component is compact). It was proved in [8, Theorem 1.2] that a simply connected open manifold (U, g) admits a limiting Carleman weight if and only if (U, cg) admits a parallel unit vector field for some positive function c. Since a parallel vector field is Killing, the existence of a limiting Carleman weight implies the existence of a nontrivial conformal Killing vector field. Corollary 1.2 then implies the following result, which was stated informally in [8] without proof.

Corollary 1.3. Let (U, g) be an open submanifold of some compact manifold (M, g) without boundary, having dimension $n \geq 3$. There is a residual set of Riemannian metrics on M which do not admit limiting Carleman weights near any point of U.

2. Proof of Theorem 1.1

We now prove Theorem 1.1. Denote by $\mathcal{G}$ the set of all Riemannian metrics on M for which there are no conformal diffeomorphisms between any distinct open subsets of M. The statement of Theorem 1.1 is that $\mathcal{G}$ is a countable intersection of sets which are open and dense in the C^∞ topology of Riemannian metrics on M.

We divide the proof in parts. First we show that, by choosing a suitable basis of open sets in M, the problem can be localized and reduces to finding suitable metrics for open balls in $\mathbb{R}^n$. Subsequently, we argue that the open balls admit a residual set of metrics for which there are no conformal diffeomorphisms between the balls. This involves using compactness and regularity results for conformal diffeomorphisms, and a dimension count argument based on jet spaces. The proof is a conformal analogue of the proof of Sunada's result [22, Proposition 1] mentioned in the introduction.

Proof of Theorem 1.1. I. *Basis*: We choose a basis $\{U_i\}_{i=1}^\infty$ for the topology of M such that the closure $\overline{U}_i$ of each U_i is diffeomorphic to a closed ball in $\mathbb{R}^n$. Let us denote by $\mathcal{S}_{ik}$ the set of metrics g on M for which there exists a conformal diffeomorphism $\phi : (\overline{U}_i, g) \to (\overline{U}_k, g)$. We will show that

$$\mathcal{G} = \bigcap_{i,k \, ; \, \overline{U}_i \cap \overline{U}_k = \emptyset} \mathcal{C}\mathcal{S}_{ik}.$$

It is clear that the set $\mathcal{G}$ is contained in the complement of each $\mathcal{S}_{ik}$, $i \neq k$. For the other direction, let a metric g belong to the complement of $\mathcal{G}$. Then there exists a conformal diffeomorphism $\phi \neq \mathrm{Id}$ between some distinct open sets U and V in M. By changing ϕ to its (conformal) inverse if necessary, we can find a point $x \in U \setminus V$ with $\phi(x) \neq x$. Restricting ϕ to sufficiently small neighbourhoods of x and $\phi(x)$ we have $g \in \mathcal{S}_{ik}$ where $\overline{U}_i \cap \overline{U}_k = \emptyset$. Thus $\mathcal{C}\mathcal{G} \subset \bigcup_{i,k \, ; \, \overline{U}_i \cap \overline{U}_k = \emptyset} \mathcal{S}_{ik}$.

II. $\mathcal{S}_{ik}$ *is a countable union of closed sets*: We fix open sets U_i, U_k with $\overline{U}_i \cap \overline{U}_k = \emptyset$, and write $U = U_i$, $V = U_k$ and $\mathcal{S} = \mathcal{S}_{ik}$. If a metric belongs to $\mathcal{S}$, there is at least one conformal mapping between U and V and a corresponding conformal factor. We represent the set $\mathcal{S}$ as a countable union

$$\mathcal{S} = \bigcup_{N=1}^{\infty} \mathcal{S}^N,$$

where, for some fixed integer $s > \frac{n}{2} + 2$,

$$\mathcal{S}^N := \{g \in \mathcal{S} : \exists \, \phi : \overline{U} \to \overline{V}, \ cg = \phi^* g, \ \|c\|_{W^{s,2}(U)} + \|1/c\|_{W^{s,2}(U)} \le N\}.$$

Here, since $\overline{U}$ is diffeomorphic to the closed unit ball in $\mathbb{R}^n$, we can use the standard Sobolev norms in Euclidean space,

$$\|f\|_{W^{s,2}(U)} = \sum_{|\alpha| \le s} \|\partial^\alpha f\|_{L^2(U)}.$$

Suppose (g_n) is a sequence of metrics in $\mathcal{S}^N$ converging in the C^∞ topology to some metric g_0. We will prove that the limit metric g_0 belongs to $\mathcal{S}^N$, showing that $\mathcal{S}^N$ is closed as required.

Let ϕ_n be a conformal diffeomorphism from $\overline{U}$ onto $\overline{V}$ as in the definition of $\mathcal{S}^N$, with a conformal factor c_n so that $\phi_n^* g_n = c_n g_n$, $\|c_n\|_{W^{s,2}(U)} \le N$. The bounded sequence (c_n) has a subsequence (also denoted by (c_n)) converging weakly to some c_0 in $W^{s,2}(U)$. By compact Sobolev embedding $c_n \to c_0$ strongly in $C^2(\overline{U})$. We define new metrics h_n in $\overline{U} \cup \overline{V}$ by

$$h_n := \begin{cases} c_n g_n, & x \in \overline{U} \\ g_n, & x \in \overline{V} \end{cases}$$

and use a similar definition for h_0. The metrics h_n and h_0 are C^∞ and C^2 smooth, respectively. Moreover, for each $n \in \mathbb{N}$, ϕ_n is an isometry from $(\overline{U}, h_n)$ onto $(\overline{V}, h_n)$.

Consider the distance functions $d_n = d_{h_n}$ and $d_0 = d_{h_0}$. The definition of the distance functions and a computation in local coordinates show that for $x, y \in \overline{U}$ or for $x, y \in \overline{V}$,

$$C_n^{-1} d_n(x, y) \le d_0(x, y) \le C_n d_n(x, y)$$

where $C_n \ge 1$, and the convergence $h_n \to h_0$ in C^2 implies that $C_n \to 1$. Moreover, the isometry $\phi_n : (\overline{U}, h_n) \to (\overline{V}, h_n)$ is distance preserving, and we obtain

$$d_0(\phi_n(x), \phi_n(y)) \le C_n d_n(\phi_n(x), \phi_n(y)) = C_n d_n(x, y) \le C_n^2 d_0(x, y).$$

That is, the sequence (ϕ_n) is equicontinuous with respect to d_0. The Arzela-Ascoli theorem yields a uniformly converging subsequence, $\phi_n \to \phi_0$.

Similar considerations for ϕ_n^{-1}, using now the bound $\|1/c_n\|_{W^{s,2}(U)} \le N$, yield that ϕ_0 is a homeomorphism. It is also distance preserving, since

$$\frac{1}{C_n^2} d_0(x, y) = \frac{1}{C_n^2} d_0(\phi_n^{-1}\phi_n(x), \phi_n^{-1}\phi_n(y))$$

$$\le d_0(\phi_n(x), \phi_n(y)) \le C_n^2 d_0(x, y)$$

where $C_n \to 1$. A distance preserving homeomorphism ϕ_0 from U onto V, which have C^2 smooth metric, is itself C^3 smooth by Theorem 2.1 of [23]. By the same theorem, ϕ_0 is in fact an isometry in the sense that

$$h_0 = \phi_0^* h_0.$$

That is,

$$c_0 g_0 = \phi_0^* g_0.$$

This implies that ϕ_0 is a C^3 conformal diffeomorphism between two C^∞ manifolds. Now, a theorem of Ferrand [7] shows that ϕ_0 and also c_0 are actually infinitely smooth. Thus the metric g_0 belongs to $\mathcal{S}^N$, so $\mathcal{S}^N = \mathcal{S}_{ik}^N$ is closed for all i, k with $\overline{U}_i \cap \overline{U}_k = \emptyset$ and for all N.

III. $\mathcal{CS}_{ik}^N$ *is dense*: So far we have proved that

$$\mathcal{G} = \bigcap_{i,k,N \, ; \, \overline{U}_i \cap \overline{U}_k = \emptyset} \mathcal{CS}_{ik}^N$$

where each $\mathcal{CS}_{ik}^N$ is open. In the final step of the proof we show that $\mathcal{CS}_{ik}^N$ is dense for each i, k, N. In fact, it is sufficient to prove that any metric in $\mathcal{S}_{ik}$ can be approximated arbitrarily well by metrics which are not in $\mathcal{S}_{ik}$. This step concludes the proof.

We write $\mathcal{S} = \mathcal{S}_{ik}$, $U = U_i$ and $V = U_k$. We may assume $\overline{U}$ and $\overline{V}$ are disjoint closed balls in $\mathbb{R}^n$, with $0 \in U$. Let $g \in \mathcal{S}$ and let ϕ be a corresponding conformal diffeomorphism $(\overline{U}, g_1) \to (\overline{V}, g)$, where the restrictions of the metric g to $\overline{U}$ and $\overline{V}$ are denoted by g_1 and g, respectively. Let us also write $x = \phi(0) \in V$.

If τ_x is the translation $y \mapsto y + x$, then the mapping $\psi := \tau_x^{-1} \circ \phi$ is a diffeomorphism defined in the neighborhood $\overline{U}$ of the origin and it preserves the origin, $\psi(0) = 0$. By the conformality of ϕ, the metric in $\overline{U}$ is of the form

$$g_1 = c\phi^*(g) = c\psi^*(\tau_x^*(g)), \tag{2.1}$$

for some $c \in C^\infty(\overline{U})$.

The previous equation shows that the form of a metric in $\mathcal{S}$ is restricted near the origin. To study the nature of the restriction (2.1), we calculate upper bounds for the dimensions of the jet spaces [20] of metrics of the form (2.1). We will see that, for large enough k, the dimension of the kth jet space of all local metrics is larger than the dimension of the kth jet space of metrics of the form (2.1).

Metrics on a neighbourhood of the origin are mappings from U to the space of symmetric positive definite bilinear forms on $\mathbb{R}^n$, which has dimension $n(n+1)/2$. We denote by $\mathcal{M}^k$ the kth jet space of such mappings at the origin. Accordingly, the dimension of $\mathcal{M}^k$ is

$$\frac{n(n+1)}{2} \binom{n+k}{k}.$$

Let us then consider the jet spaces of metrics of the form (2.1). Let $\mathcal{D}^k$ be the kth jet space of local diffeomorphisms ψ at the origin with $\psi(0) = 0$. Its dimension is

$$n \binom{n+k}{k} - n.$$

By the chain rule, we see that the k-jet of ψ^*g at the origin is determined by the $(k+1)$-jet of ψ and the k-jet of g at the origin. In this way, $\mathcal{D}^{k+1}$ defines a smooth action

$$G_1 : \mathcal{D}^{k+1} \times \mathcal{M}^k \to \mathcal{M}^k.$$

Similarly $\mathcal{C}^k$, the kth jet space of (positive) functions at the origin, defines a smooth action in $\mathcal{M}^k$,

$$G_2 : \mathcal{C}^k \times \mathcal{M}^k \to \mathcal{M}^k.$$

Moreover, we define a mapping $\gamma : V \to \mathcal{M}^k$ as the k-jet at the origin of the pullback of g by τ_y:

$$y \mapsto (\tau_y^* g)^{(k)}. \qquad (2.2)$$

Combining the above defined mappings naturally, we get a smooth mapping $G : \mathcal{C}^k \times \mathcal{D}^{k+1} \times V \to \mathcal{M}^k$,

$$(c^{(k)}, \psi^{(k+1)}, y) \mapsto G_2\left(c^{(k)}, G_1(\psi^{(k+1)}, \gamma(y))\right).$$

By (2.1), the kth jet of g_1 at the origin is given by $G(c^{(k)}, \psi^{(k+1)}, x)$ where $c^{(k)}$ and $\psi^{(k+1)}$ are the corresponding jets of c and ψ at the origin.

By Sard's theorem [19], the image of the mapping G has Lebesgue measure zero if the rank of G is smaller than the dimension of the target space. The rank of the mapping G is at most the dimension of its domain, which is

$$\binom{n+k}{k} + n\binom{n+k+1}{k+1} - n + n = \binom{n+k}{k} + n\binom{n+k+1}{k+1}.$$

This is smaller than the dimension of the target space $\mathcal{M}^k$, when $k \geq 4$ and $n \geq 3$. Since we assume $n \geq 3$, the image of G is of measure zero in $\mathcal{M}^k$, and thus its complement is dense, when $k \geq 4$. Thus, using a C^∞ bump function we can deform the original metric near the origin such that the modified metric satisfies the following conditions:

- it is arbitrarily close to the original metric in the C^∞ topology
- it is unaltered in V
- the k-jets of the modified metric at the origin are not in image of G, for some $k \geq 4$.

We have shown that the complement of $\mathcal{S}$ is dense in the C^∞ topology of Riemannian metrics of M. This concludes the proof. $\qquad \square$

References

[1] R. Abraham, *Bumpy metrics*, in Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, 1968), AMS, Providence, 1970, p. 1–3.

[2] D.V. Anosov, *Generic properties of closed geodesics*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 657–709.

[3] S. Bando, H. Urakawa, *Generic properties of the eigenvalue of the Laplacian for compact Riemannian manifolds*, Tôhoku Math. J. **35** (1983), 155–172.

[4] M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330.

[5] T. Branson, A. Čap, M. Eastwood, A.R. Gover, *Prolongations of geometric overdetermined systems*, Internat. J. Math. **17** (2006), 641–664.

[6] A. P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matemática, Río de Janeiro, 1980.

[7] J. Lelong-Ferrand, *Geometrical interpretation of scalar curvature and regularity of conformal homeomorphisms*, Differential Geometry and Relativity (M. Caheu and M. Flato, eds.), D. Reidel, Dordrecht, 1976, pp. 91–105.

[8] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, G. Uhlmann, *Limiting Carleman weights and anisotropic inverse problems*, Invent. Math. **178** (2009), 119-171.

[9] A.R. Gover, *Laplacian operators and Q-curvature on conformally Einstein manifolds*, Math. Ann. **336** (2006), 311–334.

[10] A.R. Gover, J. Šilhan, *The conformal Killing equation on forms –prolongations and applications*, Differential Geom. Appl. **26** (2008), 244–266.

[11] C. Guillarmou, A. Sa Barreto, *Inverse problems for Einstein manifolds*, Inverse Probl. Imaging **3** (2009), 1-15.

[12] C.E. Kenig, M. Salo, G. Uhlmann, *Inverse problems for the anisotropic Maxwell equations*, Duke Math. J. (to appear), arXiv:0905.3275.

[13] C.E. Kenig, J. Sjöstrand, G. Uhlmann, *The Calderón problem with partial data*, Ann. of Math. **165** (2007), 567–591.

[14] W. Klingenberg, F. Takens, *Generic properties of geodesic flows*, Math. Ann. **197** (1972), 323–334.

[15] M. Lassas, M. Taylor, G. Uhlmann, *The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary*, Comm. Anal. Geom.,**11** (2003), 207–221.

[16] M. Lassas, G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, Ann. Sci. École Norm. Sup. **34** (2001), 771–787.

[17] J. Lee, G. Uhlmann, *Determining anisotropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math. **42** (1989), 1097–1112.

[18] Y.G. Nikonorov, E.D. Rodionov, V.V. Slavskii, *Geometry of homogeneous Riemannian manifolds*, J. Math. Sci. (N. Y.) **146** (2007), 6313–6390.

[19] A. Sard, *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc. **48** (1942), 883–890.

[20] D.J. Saunders, The geometry of jet bundles. Cambridge University Press, 1989.

[21] U. Semmelmann, Conformal Killing forms on Riemannian manifolds. Habilitation, Ludwig Maximilians University in Munchen, 2001.

[22] T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. **121** (1985), 169–186.

[23] M.E. Taylor, *Existence and regularity of isometries*, Trans. Amer. Math. Soc. **358** (2006), 2414–2423.

[24] G. Uhlmann, *Electrical impedance tomography and Calderón's problem*, Inverse Problems **25** (2009), 123011.

[25] K. Uhlenbeck, *Generic properties of eigenfunctions*, Amer. Math. J. **98** (1976), 1059–1078.

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