CARLEMAN ESTIMATES FOR GEODESIC X-RAY TRANSFORMS

GABRIEL P. PATERNAIN AND MIKKO SALO

ABSTRACT. In this article we introduce an approach for studying the geodesic X-ray transform and related geometric inverse problems by using Carleman estimates. The main result states that on compact negatively curved manifolds (resp. nonpositively curved simple or Anosov manifolds), the geodesic vector field satisfies a Carleman estimate with logarithmic weights (resp. linear weights) on the frequency side. As a particular consequence, on negatively curved simple manifolds the geodesic X-ray transform with attenuation given by a general connection and Higgs field is invertible modulo natural obstructions. The proof is based on showing that the Pestov energy identity for the geodesic vector field completely localizes in frequency. Our approach works in all dimensions ≥ 2, on negatively curved manifolds with or without boundary, and for tensor fields of any order.

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1. INTRODUCTION

Motivation. The geodesic X-ray transform is a central object in geometric inverse problems. In Euclidean space it reduces to the standard X-ray transform, which encodes the integrals of a function over straight lines and provides the mathematical model for imaging methods such as X-ray CT and PET [Na01]. If the underlying medium is not Euclidean, one needs to consider more general curve families. On a Riemannian manifold the geodesic curves provide a natural candidate, and the geodesic X-ray transform encodes the integrals of a function over geodesics. This

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transform plays an important role (often via linearization or pseudo-linearization arguments) in inverse problems such as

- boundary and lens rigidity on manifolds with boundary [PU05, Gu17, SUV17]
- spectral rigidity on closed manifolds with Anosov geodesic flow [PSU14b]
- inverse boundary value problems for hyperbolic equations [SY16]
- inverse boundary value problems for elliptic equations (Calderón problem) [DKSU09, DKLS16]

Several approaches have been introduced for studying the X-ray transform:

- direct methods such as Fourier analysis in symmetric geometries [He99]
- microlocal methods based on interpreting the transform or its normal operator as a Fourier integral or pseudodifferential operator [SU05, SU09, UV16]
- reductions to PDE and energy methods [Sh94, PSU13, GPSU16]

In particular, the works [PSU13, GPSU16, UV16] involve various L^2 estimates with exponential weights. This suggests that the Carleman estimate methodology, which relies on exponentially weighted L^2 estimates and provides a very powerful general approach to uniqueness results for linear PDE [Hö85, Chapter XXVIII], could have consequences for the geodesic X-ray transform as well. However, as explained in Section 3, the PDE related to the X-ray transform are nonstandard and thus the existing theory of Carleman estimates cannot be directly applied.

This paper presents the first steps toward a Carleman estimate approach for the geodesic X-ray transform. We prove two Carleman estimates for the geodesic vector field, one with logarithmic and another with linear Carleman weights, that have the right form in order to be applied to the geodesic X-ray transform. As a consequence we obtain invertibility results for certain weighted X-ray transforms that correspond to large lower order perturbations in the related PDE, and uniqueness results in related problems such as scattering rigidity and transparent connections. Our approach works in all dimensions ≥ 2, on manifolds with or without boundary, and for tensor fields of any order. However, the main Carleman estimate requires that the manifold has negative curvature, and the Carleman weights are purely on the frequency side (one can think of them as pseudodifferential weights) and thus the estimates do not currently lead to local results for the X-ray transform on the space side as in [UV16].

Main estimates. Let (M, g) be a compact oriented Riemannian manifold with or without boundary, and let X be the geodesic vector field regarded as a first order differential operator $X : C^\infty(SM) \to C^\infty(SM)$ acting on functions on the unit sphere bundle SM. Our first result is a new energy estimate for X when g is negatively curved. The new estimate involves polynomial weights on the Fourier side and it can be regarded as a *Carleman estimate* for the transport equation $Xu = f$ (see below for analogies with elliptic operators). To describe it we recall some basic harmonic analysis on SM. By considering the vertical Laplacian Δ on each fibre S_xM of SM we have a natural L^2-decomposition $L^2(SM) = \oplus_{m \geq 0} H_m(SM)$ into vertical spherical harmonics. We set $\Omega_m := H_m(SM) \cap C^\infty(SM)$. Then a function u belongs to Ω_m if and only if $-\Delta u = m(m+d-2)u$ where $d = \dim(M)$ (for details see [GK80b, DS11]).

If $u \in L^2(SM)$, this decomposition will be written as

$$u = \sum_{m=0}^{\infty} u_m, \qquad u_m \in H_m(SM).$$

We say that u has *finite degree* if the above sum is finite.

Here is our first main result (the norms are $L^2(SM)$ norms).

Theorem 1.1. *Let (M, g) be compact with sectional curvature $\leq -\kappa$ where $\kappa > 0$. Let also $\phi_l = \log(l)$. For any $\tau \geq 1$ and $m \geq 1$, one has*

$$\sum_{l=m}^{\infty} e^{2\tau\phi_l}\|u_l\|^2 \leq \frac{(d+4)^2}{\kappa\tau} \sum_{l=m+1}^{\infty} e^{2\tau\phi_l}\|(Xu)_l\|^2$$

whenever $u \in C^\infty(SM)$ (with $u|_{\partial(SM)} = 0$ in the boundary case).

We have a similar estimate for compact manifolds with nonpositive curvature, provided that they are *simple* (simply connected with strictly convex boundary and no conjugate points) or *Anosov* (no boundary, and the geodesic flow satisfies the Anosov property). Both the simple and Anosov conditions can be seen as slightly strengthened forms of the condition that no geodesic has conjugate points (see [PSU14b]). However, if the curvature is only nonpositive, the logarithmic weights above need to be replaced by stronger linear weights. We remark that both logarithmic and linear weights have been prominent in the theory of Carleman estimates, see e.g. [JK85, KRS87].

Theorem 1.2. *Let (M, g) be a simple/Anosov manifold having nonpositive sectional curvature. There exist $m_0, \tau_0, \kappa > 0$ such that for any $\tau \geq \tau_0$, for any $m \geq m_0$ and for any $u \in C^\infty(SM)$ of finite degree (with $u|_{\partial(SM)} = 0$ in the boundary case) one has*

$$\sum_{l=m}^{\infty} e^{2\tau\phi_l}\|u_l\|^2 \leq \frac{24}{\kappa e^{2\tau}} \sum_{l=m+1}^{\infty} e^{2\tau\phi_l}\|(Xu)_l\|^2$$

where $\phi_l = l$.

Both theorems above are based on weighted frequency localized versions of the *Pestov energy identity*, which has been the main tool in energy methods for X-ray transforms (see e.g. [Sh94, PSU13, PSU15]). The Pestov identity has been used in many forms. A powerful recent variant is the inequality proved in [PSU15],

$$(1.1) \qquad \qquad \|\nabla_{SM}u\| \leq C\|\overset{\mathrm{v}}{\nabla}Xu\|,$$

which is valid on any compact simple/Anosov manifold for any $u \in C^\infty(SM)$ (with $u|_{\partial(SM)} = 0$ in the boundary case). Here $\overset{\mathrm{v}}{\nabla}$ is the vertical gradient (see Section 2). So far the Pestov identity has been expressed in terms of L^2 norms, and it has not been known if it is possible to shift the related estimates to different Sobolev scales. However, Theorem 1.1 is actually a shifted version of the Pestov identity with respect to the mixed norms

$$\|u\|_{L_x^2 H_v^s} = \left(\sum_{l=0}^{\infty} \langle l \rangle^{2s}\|u_l\|^2\right)^{1/2}$$

where $s \in \mathbb{R}$ and $\langle l \rangle = (1 + l^2)^{1/2}$. Similar mixed norms for $\nabla_{SM} u$ and $\overset{v}{\nabla} X u$ are defined in Section 6. Then, the inequality (1.1) can be expressed as

$$\|\nabla_{SM} u\|_{L_x^2 H_v^0} \leq C \|\overset{v}{\nabla} X u\|_{L_x^2 H_v^0}.$$

The next result is a shifted version of this inequality.

Theorem 1.3. *Let (M, g) be compact with sectional curvature $\leq -\kappa$ where $\kappa > 0$. For any $s > -1/2$ there is $C = C_{d,s,\kappa} > 0$ such that one has*

$$\|\nabla_{SM} u\|_{L_x^2 H_v^s} \leq C \|\overset{v}{\nabla} X u\|_{L_x^2 H_v^s}$$

for any $u \in C^\infty(SM)$ (with $u|_{\partial(SM)} = 0$ in the boundary case).

Applications.

Uniqueness results for the transport equation. The significance of Theorem 1.1 is best illustrated by considering the transport equation

$$Xu = -f \text{ in } SM, \qquad u|_{\partial(SM)} = 0,$$

where f has *finite degree* m (i.e. $f_l = 0$ for $l \geq m+1$). Then Theorem 1.1 implies that any solution $u \in C^\infty(SM)$ must satisfy $u_l = 0$ for $l \geq m$ (u has finite degree $m - 1$) at least when $m \geq 1$. This fact provides a full solution to the tensor tomography problem in negative curvature (see [PS88, CS98] and [PSU14a] for a recent survey). The case $m = 2$ is at the heart of the proof that closed negatively curved manifolds are spectrally rigid [CS98, GK80a]. Given an isospectral deformation g_ε of the metric g it was shown in [GK80a] that the symmetric 2-tensor $f := \frac{\partial g_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}$ must have the property that it integrates to zero along every closed geodesic of g. Since the geodesic flow of g is Anosov an application of the Livsic theorem [dMM86] shows that there is a smooth function u such that $Xu = \tilde{f} \in \Omega_0 \oplus \Omega_2$, where $\tilde{f}$ is the natural function induced on SM by the symmetric 2-tensor f. The Carleman estimate in Theorem 1.1 implies that $u = u_1$ and this uniquely determines a vector field V on M. Doing this for every ε produces a parameter dependent vector field V_ε whose flow makes the deformation g_ε trivial, thus showing spectral rigidity.

However, the estimate in Theorem 1.1 is considerably more powerful since we can in principle adjust the parameter τ at will as long as we know that $u \in C^\infty(SM)$. Here is the typical application we have in mind. Suppose that $\Phi \in C^\infty(M, \mathbb{C}^{n \times n})$ is a smooth $n \times n$ matrix-valued potential in M (we call Φ a *general Higgs field*). Consider the attenuated transport equation

(1.2) $(X + \Phi)u = -f_0 \in \Omega_0, \qquad u|_{\partial(SM)} = 0$

where $u \in C^\infty(SM, \mathbb{C}^n)$. The objective is to show that $f_0 = 0$; this is precisely the uniqueness problem that arises when trying to prove injectivity of the attenuated geodesic X-ray transform where the attenuation is given by the matrix Φ (see below). Since $(\Phi u)_l = \Phi u_l$, we see using (1.2) that

$$\|(Xu)_l\| \leq C \|u_l\|, \qquad l \geq 1,$$

where $C = \|\Phi\|_{L^\infty(M)}$. If we input this information into Theorem 1.1 and we take τ sufficiently large (depending on $\|\Phi\|_{L^\infty(M)}$) we deduce that $u = u_0 \in \Omega_0$. Since $Xu_0 \in \Omega_1$, using (1.2) we obtain $f_0 = 0$ as desired.

However, the Carleman estimate can also deal with other matrix attenuations like *connections*. For our purposes these are simply $n \times n$ matrices A of complex valued 1-forms, i.e. $A \in C^\infty(M, (\Lambda^1 M)^{n \times n})$. We call such geometric objects *general connections* on M, and they can be naturally considered as smooth functions $A : SM \to \mathbb{C}^{n \times n}$ such that $A \in \Omega_1$. More generally, the Carleman estimate even makes it possible to include nonlinear attenuation terms in the transport equation. Here is a general result in this direction.

Theorem 1.4. *Let (M, g) be compact with sectional curvature $\leq -\kappa$ where $\kappa > 0$. Let $l_0 \geq 2$, and suppose that $\mathcal{A} : C^\infty(SM, \mathbb{C}^n) \to C^\infty(SM, \mathbb{C}^n)$ is a map such that*

$$\|(\mathcal{A}(u))_l\| \leq C(\|u_{l-1}\| + \|u_l\| + \|u_{l+1}\|), \qquad l \geq l_0.$$

If $f \in C^\infty(SM, \mathbb{C}^n)$ has finite degree and if $u \in C^\infty(SM, \mathbb{C}^n)$ satisfies

$$Xu + \mathcal{A}(u) = -f \ in \ SM, \qquad u|_{\partial(SM)} = 0,$$

then u has finite degree.

The previous result can be interpreted as a fundamental uniqueness, or vanishing, theorem for the attenuated transport equation. In [GPSU16], such a result was proved on negatively curved manifolds in the special (linear) case

$$\mathcal{A}(u) = (A + \Phi)u$$

where A and Φ are a *skew-Hermitian* connection and Higgs field (when A is skew-Hermitian, the connection is also referred to as *unitary* because the underlying structure group is the unitary group). The skew-Hermitian condition implies that the transport operator $X + A + \Phi$ is skew-symmetric, which led to improved energy estimates in [GPSU16]. In our case, where A and Φ can be general matrices, there is a loss of symmetry and large error terms appear in the energy estimates. The Carleman estimate in Theorem 1.1 is required to deal with these large errors (see Section 3 for a more detailed discussion).

In the more general case where (M, g) is a compact simple/Anosov manifold with nonpositive sectional curvature, one could use Theorem 1.2 to prove an analogue of Theorem 1.4 with two additional restrictions: the map $\mathcal{A}$ should satisfy

$$\|(\mathcal{A}(u))_l\| \leq C\|u_l\|, \qquad l \geq l_0,$$

and the solution $u \in C^\infty(SM, \mathbb{C}^n)$ should additionally satisfy

(1.3) $$\lim_{l \to \infty} e^{\tau l} \|X_- u_l\| = 0, \qquad \tau > 0,$$

where X_- is the part of X that maps Ω_m to Ω_{m-1} (see Section 2). The condition for $\mathcal{A}$ is satisfied if $\mathcal{A}$ is a general Higgs field, but (1.3) should be viewed as an additional regularity condition (it states that the Fourier coefficients of $X_- u$ decay exponentially, which corresponds to real-analyticity of $X_- u$ in the v variable). Thus, the Carleman estimate in Theorem 1.2 shows that the injectivity result for a general

Higgs field reduces to the regularity result (1.3). We do not have a proof for (1.3), but in Section 9 we will give an alternative argument to deal with general connections in the Euclidean disc.

Theorem 1.4 easily implies uniqueness results for X-ray transforms, whenever one has a regularity result showing that the vanishing of the X-ray transform yields a C^∞ solution u of a transport equation. Let us next discuss this in more detail.

The attenuated X-ray transform. Let (M, g) be a compact manifold with smooth boundary. We assume that (M, g) is *nontrapping*, i.e. for any $(x, v) \in SM$ the geodesic $\gamma_{x,v}(t)$ through (x, v) exits M at some finite time $\tau(x, v) \geq 0$. Given $a \in C^\infty(M)$ (scalar attenuation), the attenuated geodesic X-ray transform of f is the map

$$I_a f : \partial_+(SM) \to \mathbb{C}, \quad I_a f(x, v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t)) \, e^{\int_0^{\tau(x,v)} a(\gamma_{x,v}(s)) \, ds} \, dt,$$

where $\partial_\pm(SM) = \{(x, v) \in SM \,;\, x \in \partial M, \pm\langle v, \nu \rangle \leq 0\}$ are the inward and outward pointing boundaries of SM (with ν being the outer unit normal to ∂M). It is easy to see that $I_a f$ can be equivalently defined as

$$I_a f = u|_{\partial_+(SM)}$$

where u is the solution of

$$(X + a)u = -f \text{ in } SM, \qquad u|_{\partial_-(SM)} = 0.$$

More generally, we can consider the vector-valued case where $f \in C^\infty(SM, \mathbb{C}^n)$, and the attenuation is given by a general connection $A \in C^\infty(M, (\Lambda^1 M)^{n \times n})$ and a general Higgs field $\Phi \in C^\infty(M, \mathbb{C}^{n \times n})$. The matrix of 1-forms A can be identified with the connection $\nabla = d + A$ on the trivial bundle $\mathcal{E} = M \times \mathbb{C}^n$. In this case, the attenuated X-ray transform of f is defined as

$$I_{A+\Phi} f = u|_{\partial_+(SM)},$$

where u is the solution of

$$(X + A + \Phi)u = -f \text{ in } SM, \qquad u|_{\partial_-(SM)} = 0.$$

The basic inverse problem for 0-tensors (resp. m-tensors) is the following: given $f \in C^\infty(M, \mathbb{C}^n)$ (resp. $f \in C^\infty(SM, \mathbb{C}^n)$ with degree m) such that $I_{A,\Phi} f = 0$, is it true that $f \equiv 0$ (resp. $f = -(X + A + \Phi)u$ for some $u \in C^\infty(SM, \mathbb{C}^n)$ of degree $m-1$ with $u|_{\partial(SM)} = 0$)? The following theorem gives an affirmative answer on nontrapping negatively curved manifolds.

Theorem 1.5. *Let (M, g) be a compact nontrapping negatively curved manifold with C^∞ boundary, let A be a general connection, and let Φ be a general Higgs field in M. Let $f \in C^\infty(SM, \mathbb{C}^n)$, and assume either that M has strictly convex boundary, or that f is supported in the interior of SM. If f has degree $m \geq 0$ and if the attenuated X-ray transform of f vanishes (i.e. $I_{A+\Phi} f = 0$), then*

$$f = -(X + A + \Phi)u$$

for some $u \in C^\infty(SM, \mathbb{C}^n)$ with degree $m - 1$ such that $u|_{\partial(SM)} = 0$.

This kind of result was proved for skew-Hermitian A and Φ in [GPSU16] when M is negatively curved with strictly convex boundary (in this case there is a stronger regularity result based on the work [Gu17], showing that one can drop the nontrapping assumption). Also, for strictly convex manifolds with $\dim(M) \geq 3$ a stronger result was proved in [PSUZ16], based on the method of [UV16]. Genericity results for simple manifolds are established in [Z17] and reconstruction formulas for simple surfaces (up to a Fredholm error) are given in [MP18]. The result for nonconvex boundaries is similar to [Da06, GMT17] which consider the unattenuated case. Theorem 1.5 is new when $\dim(M) = 2$, and also when $\dim(M) \geq 3$ if the boundary is nonconvex.

Scattering rigidity and transparent pairs. Finally, we briefly discuss results for non-linear geometric inverse problems that can be reduced to Theorems 1.4 and 1.5 via pseudo-linearization arguments. Given a general connection A and a Higgs field Φ on the trivial bundle $M \times \mathbb{C}^n$, the *scattering data* for (A, Φ) is the map

$$C_{A,\Phi} : \partial_-(SM) \to GL(n, \mathbb{C}), \quad C_{A,\Phi} = U|_{\partial_-(SM)},$$

where $U : SM \to GL(n, \mathbb{C})$ is the unique matrix solution of the transport equation

$$(X + A + \Phi)U = 0 \text{ in } SM, \quad U|_{\partial_+(SM)} = \mathrm{Id}.$$

The data $C_{A,\Phi}$ corresponds to boundary measurements for parallel transport if (M, g) is known (it measures how vectors are transformed when they are parallel transported with respect to (A, Φ) along geodesics between two boundary points). The map $(A, \Phi) \mapsto C_{A,\Phi}$ is sometimes called the non-abelian Radon transform, or the X-ray transform for a non-abelian connection and Higgs field (see [PSU12, No18]). The data $C_{A,\Phi}$ has a natural gauge invariance under a change of basis, i.e.,

$$C_{Q^{-1}(X+A)Q, Q^{-1}\Phi Q} = C_{A,\Phi} \quad \text{if } Q \in C^\infty(M, GL(n, \mathbb{C})) \text{ satisfies } Q|_{\partial M} = \mathrm{Id}.$$

It follows that from the knowledge of $C_{A,\Phi}$ one can only expect to recover A and Φ up to a gauge transformation via Q which satisfies $Q|_{\partial M} = \mathrm{Id}$. Our next result gives the following positive answer:

Theorem 1.6. *Let (M, g) be a compact negatively curved nontrapping manifold, let A and B be two general connections, and let Φ and Ψ be two general Higgs fields. Assume either that M has strictly convex boundary, or that the pairs (A, Φ), (B, Ψ) are supported in the interior of M. Then $C_{A,\Phi} = C_{B,\Psi}$ implies that there exists $Q \in C^\infty(M, GL(n, \mathbb{C}))$ such that $Q|_{\partial M} = \mathrm{Id}$ and $B = Q^{-1}dQ + Q^{-1}AQ$, $\Psi = Q^{-1}\Phi Q$.*

Again, when M has strictly convex boundary, this was proved for skew-Hermitian connections and Higgs fields in [GPSU16] (also without the non-trapping assumption, based on the regularity theory of [Gu17]), and for a more general class of manifolds with $\dim(M) \geq 3$ in [PSUZ16]. See [GPSU16] for further references.

An analogous result can be formulated on closed negatively curved manifolds without boundary. In this case, we show that if A is a general connection and Φ is a general Higgs field in M, and if the parallel transport for (A, Φ) along periodic geodesics is the identity map, then (A, Φ) is gauge equivalent to the trivial pair $(0, 0)$ (i.e. there

are no *transparent pairs*) unless there is an obstruction given by twisted conformal Killing tensors. We refer to Section 9 for more details.

Methods and analogies. The starting point of the Carleman estimate is a well-known L^2 energy identity on SM called *Pestov identity*. If $u \in C^\infty(SM)$, there is a splitting (induced by the Sasaki metric and the Levi-Civita connection)

$$\nabla_{SM}u = \underbrace{(Xu)X + \overset{\text{h}}{\nabla}u}_{x-\text{derivatives}} + \underbrace{\overset{\text{v}}{\nabla}u}_{v-\text{derivatives}}.$$

The basic energy identity for $P := \overset{\text{v}}{\nabla}X$ reads [PSU15, Proposition 2.2]

$$(1.4) \qquad \|Pu\|^2 = ((-X^2 - R)\overset{\text{v}}{\nabla}u, \overset{\text{v}}{\nabla}u) + (d-1)\|Xu\|^2$$

where $R(x,v) : \{v\}^\perp \to \{v\}^\perp$ is the operator determined by the Riemann curvature tensor R of (M,g) by $R(x,v)w = R_x(w,v)v$. An important observation of the present paper is that (1.4) *localizes in frequency*. By this we mean that it is possible to recover (1.4) by summing in l the identity (1.4) applied to functions $u \in \Omega_l$. This was observed in [PSU15] for $d = 2$, but here we prove it for all dimensions. This fact paves the way for the proof of the Carleman estimates: by multiplying the frequency localized estimates by suitable weights, adding them up and using negative curvature to absorb errors we are able to derive the desired inequality.

There are several interesting analogies between the methods in this paper and classical methods for elliptic operators. In particular, in some of our applications the geodesic vector field X behaves in a somewhat similar fashion as the standard Laplacian in $\mathbb{R}^n$. For instance, we can consider the operator $P = -\Delta + V$ in a bounded domain $\Omega \subset \mathbb{R}^n$, with potential $V \in L^\infty(\Omega)$. As explained in Section 3, in standard energy methods for $P = -\Delta + V$ the positivity conditions $V \geq 0$ or $V > -\lambda_1$ (where λ_1 is the first Dirichlet eigenvalue for $-\Delta$ in Ω) appear. Moreover, in the absence of such positivity conditions, one can employ the method of Carleman estimates as a replacement for standard energy methods to obtain uniqueness results.

It turns out that for the very different operator $P = \overset{\text{v}}{\nabla}X$ on SM that appears in the Pestov identity and (implicitly) in our Carleman estimates, there are direct analogues of the above properties. These are summarized in the following table, where K denotes sectional curvature:

$P = -\Delta + V$	$P = \overset{\text{v}}{\nabla}X$
potential V	curvature $-R$
$V \geq 0$	$K \leq 0$
$V > -\lambda_1$	simple/Anosov
Carleman estimates	?

In particular, the curvature $-R$ somehow plays the role of the potential V. An analogue of the Carleman estimates approach for uniqueness in elliptic equations has so far been missing in the case of X-ray transforms (the reasons why this is a difficult problem and why the PDE related to X-ray transforms are not covered by the standard theory of Carleman estimates are explained in detail in Section 3). The results in this paper represent the first progress in this direction.

There is another interesting analogy between the proof of Theorem 1.5, which is related to the X-ray transform, and the *absence of embedded eigenvalues* for Schrödinger operators. If $H = -\Delta + V - \lambda$ is a Schrödinger operator in $\mathbb{R}^d$ where V is a short range potential (i.e. $|V(x)| \leq C(1 + |x|)^{-1-\varepsilon}$ for some $\varepsilon > 0$ and for a.e. $x \in \mathbb{R}^d$) and $\lambda > 0$ is a positive energy, the absence of embedded eigenvalues states that any solution $u \in L^2(\mathbb{R}^d)$ of $Hu = 0$ must satisfy $u \equiv 0$. A standard proof of this fact proceeds in three steps:

1. *Rapid decay*: any solution u decays rapidly at infinity.
2. *Unique continuation at infinity*: any solution u that decays rapidly at infinity must vanish outside a compact set.
3. *Weak unique continuation*: any solution u that vanishes outside a compact set must be identically zero.

Our proof of Theorem 1.5 follows the exact same pattern, even though the equation is very different. The rapid decay result is replaced by a regularity result from [PSU12], unique continuation at infinity is provided by Theorem 1.1, and weak unique continuation is replaced by the absence of twisted conformal Killing tensors proved in [GPSU16]. See Section 3 for more details.

As a final remark, the uniqueness problems studied in this article are actually closer to long range scattering theory for the Laplacian rather than short range scattering. For instance, a Higgs field Φ can be interpreted as a "potential" $\mathcal{V} : u \mapsto \Phi u$ in the equation $(X + \Phi)u = -f$. In terms of Fourier coefficients, one has

$$\|(\mathcal{V}(u))_l\| \leq C\|u_l\|.$$

This means that the potential $\mathcal{V}$ is bounded but has no decay as $l \to \infty$. The analogue of a short range condition would be $\|(\mathcal{V}(u))_l\| \leq \langle l \rangle^{-1-\varepsilon}\|u_l\|$, and an analogue of a long range condition would be $\|(\mathcal{V}(u))_l\| \leq C\langle l \rangle^{-\varepsilon}\|u_l\|$ for some $\varepsilon > 0$ (possibly with a condition for the derivatives, see [Hö85, Chapter XXX]). The fact that general connections and Higgs fields lead to potentials having no decay as $l \to \infty$ makes the uniqueness questions studied in this article challenging.

Organization of the paper. Section 2 contains geometric preliminaries on the unit sphere bundle and vertical spherical harmonics. Section 3 discusses in more detail certain useful analogies between the geodesic vector field X and the Laplace operator that have in part motivated this paper. Section 4 describes a general Pestov identity with connection and Section 5 contains the key result on frequency localization mentioned above. Section 6 establishes the Carleman estimate in Theorem 1.1, and contains a proof Theorem 1.3 which interprets the Carleman estimate as a version of the Pestov identity (1.4) that has been shifted to a different regularity scale.

Section 7 proves the Carleman estimate in nonpositive curvature, Theorem 1.2. Section 8 contains a regularity result for the transport equation in the case when the boundary is not strictly convex. Finally, Section 9 contains all the applications and the proofs of Theorems 1.4–1.6. We also give an additional application using the notion of projective equivalence; it turns out that for connections one can use the Carleman estimate in negative curvature to derive a result in Euclidean space. In this fashion we give a completely new proof of a result of Eskin in [Es04] on the non-abelian Radon transform. Eskin's result uses a delicate theorem about existence of holomorphic integrating factors established in [ER04, Theorem 5]; we avoid this by making use of our Carleman estimate for hyperbolic space. Appendix A contains a basic positivity result for simple/Anosov manifolds, and Appendix B shows how the results in Section 7 can be improved on simple surfaces with nonpositive curvature.

2. Geometric preliminaries

Unit sphere bundle. To begin, we need to recall certain notions related to the geometry of the unit sphere bundle. We follow the setup and notation of [PSU15]; for other approaches and background information see [GK80b, Sh94, Pa99, Kn02, DS11].

Let (M, g) be a d-dimensional compact Riemannian manifold with or without boundary, having unit sphere bundle $\pi : SM \to M$, and let X be the geodesic vector field. We equip SM with the Sasaki metric. If $\mathcal{V}$ denotes the vertical subbundle given by $\mathcal{V} = \operatorname{Ker} d\pi$, then there is an orthogonal splitting with respect to the Sasaki metric:

$$(2.1) \qquad\qquad TSM = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}.$$

The subbundle $\mathcal{H}$ is called the horizontal subbundle. Elements in $\mathcal{H}(x, v)$ and $\mathcal{V}(x, v)$ are canonically identified with elements in the codimension one subspace $\{v\}^{\perp} \subset T_x M$ by the isomorphisms

$$d\pi_{x,v} : \mathcal{V}(x, v) \to \{v\}^{\perp}, \quad \mathcal{K}_{x,v} : \mathcal{H}(x, v) \to \{v\}^{\perp},$$

here $\mathcal{K}_{(x,v)}$ is the connection map coming from Levi-Civita connection. We will use these identifications freely below.

We shall denote by $\mathcal{Z}$ the set of smooth functions $Z : SM \to TM$ such that $Z(x, v) \in T_x M$ and $\langle Z(x, v), v \rangle = 0$ for all $(x, v) \in SM$. Another way to describe the elements of $\mathcal{Z}$ is a follows. Consider the pull-back bundle $\pi^* TM$ over SM. Let N denote the subbundle of $\pi^* TM$ whose fiber over (x, v) is given by $N_{(x,v)} = \{v\}^{\perp}$. Then $\mathcal{Z}$ coincides with the smooth sections of the bundle N. Notice that N carries a

natural scalar product and thus an L^2-inner product (using the Liouville measure on SM for integration).

Given a smooth function $u \in C^\infty(SM)$ we can consider its gradient ∇u with respect to the Sasaki metric. Using the splitting above we may write uniquely in the decomposition (2.1)

$$\nabla u = ((Xu)X, \overset{\text{h}}{\nabla} u, \overset{\text{v}}{\nabla} u).$$

The derivatives $\overset{\text{h}}{\nabla} u \in \mathcal{Z}$ and $\overset{\text{v}}{\nabla} u \in \mathcal{Z}$ are called horizontal and vertical derivatives respectively. Note that this differs from the definitions in [Kn02, Sh94] since here all objects are defined on SM as opposed to TM.

Observe that X acts on $\mathcal{Z}$ as follows:

$$(2.2) \qquad\qquad XZ(x,v) := \frac{DZ(\varphi_t(x,v))}{dt}\Big|_{t=0}$$

where D/dt is the covariant derivative with respect to Levi-Civita connection and φ_t is the geodesic flow. With respect to the L^2-product on N, the formal adjoints of $\overset{\text{v}}{\nabla} : C^\infty(SM) \to \mathcal{Z}$ and $\overset{\text{h}}{\nabla} : C^\infty(SM) \to \mathcal{Z}$ are denoted by $-\overset{\text{v}}{\text{div}}$ and $-\overset{\text{h}}{\text{div}}$ respectively. Note that since X leaves invariant the volume form of the Sasaki metric we have $X^* = -X$ for both actions of X on $C^\infty(SM)$ and $\mathcal{Z}$. In what follows, we will need to work with the complexified version of N with its natural inherited Hermitian product. This will be clear from the context and we shall employ the same letter N to denote the complexified bundle and also $\mathcal{Z}$ for its sections.

Let $R(x,v) : \{v\}^\perp \to \{v\}^\perp$ be the operator determined by the Riemann curvature tensor by $R(x,v)w = R(w,v)v$, and let $d = \dim(M)$.

Spherical harmonics decomposition. Recall the spherical harmonics decomposition with respect to the vertical Laplacian $\Delta = \overset{\text{v}}{\text{div}}\overset{\text{v}}{\nabla}$ (cf. [PSU15, Section 3]):

$$L^2(SM, \mathbb{C}^n) = \bigoplus_{m=0}^\infty H_m(SM, \mathbb{C}^n),$$

so that any $f \in L^2(SM, \mathbb{C}^n)$ has the orthogonal decomposition

$$f = \sum_{m=0}^\infty f_m.$$

We write $\Omega_m = H_m(SM, \mathbb{C}^n) \cap C^\infty(SM, \mathbb{C}^n)$. Then $-\Delta u = \lambda_m u$ for $u \in \Omega_m$, where we set

$$\lambda_m := m(m + d - 2).$$

Decomposition of X. The geodesic vector field behaves nicely with respect to the decomposition into fibrewise spherical harmonics: it maps Ω_m into $\Omega_{m-1} \oplus \Omega_{m+1}$ [GK80b, Proposition 3.2]. Hence on Ω_m we can write

$$X = X_- + X_+$$

where $X_- : \Omega_m \to \Omega_{m-1}$ and $X_+ : \Omega_m \to \Omega_{m+1}$. By [GK80b, Proposition 3.7] the operator X_+ is overdetermined elliptic (i.e. it has injective principal symbol). One

can gain insight into the decomposition $X = X_- + X_+$ as follows. Fix $x \in M$ and consider local coordinates which are geodesic at x (i.e. all Christoffel symbols vanish at x). Then $Xu(x, v) = v^i \frac{\partial u}{\partial x^i}$. We now use the following basic fact about spherical harmonics: the product of a spherical harmonic of degree m with a spherical harmonic of degree one decomposes as the sum of spherical harmonics of degree $m-1$ and $m+1$.

3. Useful analogies

Analogies with elliptic operators. It is instructive to compare our approach to X-ray transforms based on energy methods and related well known energy methods for elliptic operators in $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $P = -\Delta + V$ be the Schrödinger operator in Ω where $V \in L^\infty(\Omega)$. We consider the uniqueness problem for solutions of the equation

$$(3.1) \qquad\qquad Pu = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = 0.$$

It is well known that under a positivity condition for the potential V, any solution u of (3.1) in $C^2(\overline{\Omega})$ (say) must be zero. In fact, if we assume that $V \geq 0$, this follows from the simple energy estimate where we integrate the equation $Pu = 0$ against the test function u. This implies that, in terms of $L^2(\Omega)$ inner products,

$$0 = (Pu, u) = (-\Delta u, u) + (Vu, u) \geq \|\nabla u\|^2.$$

Here we integrated by parts using that $u|_{\partial\Omega} = 0$, and used that $V \geq 0$. Thus $\nabla u = 0$, showing that u is constant, and the boundary condition $u|_{\partial\Omega} = 0$ implies that $u \equiv 0$.

Uniqueness for solutions of (3.1) still holds under the weaker condition $V > -\lambda_1$, where $\lambda_1 > 0$ is the first Dirichlet eigenvalue of $-\Delta$ in Ω. Then the Poincaré inequality implies that $\|\nabla w\|^2 \geq \lambda_1 \|w\|^2$ whenever $w|_{\partial\Omega} = 0$. The same argument as above shows that

$$0 = (Pu, u) = (-\Delta u, u) + (Vu, u) = \|\nabla u\|^2 + (Vu, u) \geq ((\lambda_1 + V)u, u).$$

If $\lambda_1 + V \geq c$ a.e. in Ω for some $c > 0$, it follows that $u \equiv 0$. (Combining this argument with the unique continuation principle, it would be enough to assume that $\lambda_1 + V|_U > 0$ in some set U of positive measure.)

If the potential V is very negative, uniqueness for solutions of (3.1) may fail. For example, if $V = -\lambda_1$ where λ_1 is the first Dirichlet eigenvalue, then the corresponding Dirichlet eigenfunction is a nontrivial solution of (3.1). However, uniqueness will be true if we assume more vanishing. For instance, any $u \in C^2(\overline{\Omega})$ satisfying

$$(3.2) \qquad\qquad Pu = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = \partial_\nu u|_\Omega = 0$$

must be identically zero. This follows from the unique continuation principle, which is typically established by using Carleman estimates (i.e. exponentially weighted L^2 estimates for P).

Carleman estimates themselves correspond to a version of energy methods (more precisely, positive commutator methods), which makes use of the gauge invariance of the problem: writing $u = e^\varphi w$ for some $\varphi \in C^2(\overline{\Omega})$, the problem (3.2) is equivalent with

$$P_\varphi w = 0 \text{ in } \Omega, \qquad w|_{\partial\Omega} = \partial_\nu w|_\Omega = 0$$

where $P_\varphi = e^{-\varphi} P e^{\varphi}$ is the operator P conjugated with an exponential weight. For some choices of φ (for which the commutator $[P_\varphi^*, P_\varphi]$ is positive), the operator P_φ is "more positive" than P, and an energy estimate for $P_\varphi w = 0$ (where one integrates against the test function $P_\varphi w$) implies that $w \equiv 0$.

Let us now return from the case of the Schrödinger operator back to X-ray transforms. As explained in [PSU13, PSU14b], in this case we are considering the operator $P = \overset{\text{v}}{\nabla} X$ on SM, and we wish to show (say) that any $u \in C^\infty(SM)$ satisfying

$$Pu = 0 \text{ in } SM, \qquad u|_{\partial(SM)} = 0$$

must be identically zero. The basic energy identity for the operator P is (1.4), where the operator $-X^2 - R$ formally corresponds to the operator $-\Delta + V$ in the above examples. In particular, $-R$ (curvature) plays a similar role as V (potential). For instance, the condition $K \leq 0$ where K denotes sectional curvature corresponds to $V \geq 0$. Moreover, the simple/Anosov condition for (M, g) ensures that $-X^2 - R > 0$ which corresponds to $V > -\lambda_1$ (i.e. $-\Delta + V > 0$).

Let us formulate these analogies in the following table:

$P = -\Delta + V$	$P = \overset{\text{v}}{\nabla} X$
potential V	curvature $-R$
$V \geq 0$	$K \leq 0$
$V > -\lambda_1$	simple/Anosov
Carleman estimates	?

An analogue of the Carleman estimates approach for uniqueness in elliptic equations has so far been missing in the case of X-ray transforms. Theorem 1.1 represents the first progress in this direction.

Properties of P. Let us consider in more detail the operator $P = \overset{\text{v}}{\nabla} X$ on SM that is fundamental in the energy method for geodesic X-ray transforms. The operator P is a second order differential operator, scalar if $d = 2$ and vector-valued if $d \geq 3$, on the compact $(2d - 1)$-dimensional manifold SM with boundary. However, even in the case $d = 2$ where $P = VX$ is the product of two vector fields, the equation $Pu = 0$ does not seem to fall into any known class of PDEs for which there would be a uniqueness theory.

One major issue is that P is not of principal type. To see this, let $d = 2$ and write $P = P_1 P_2$ where $P_1 = V$ and $P_2 = X$. Then P_1 and P_2 are smooth vector fields on SM. Following [PSU13, Section 2] (and writing $x_3 = \theta$) we consider local coordinates (x_1, x_2, x_3) on SM so that $g = e^{2\lambda(x_1, x_2)}(dx_1^2 + dx_2^2)$. In this notation, the principal symbols of P_1 and P_2 are

$$p_1 = \xi_3, \qquad p_2 = e^{-\lambda}(\cos(x_3)\xi_1 + \sin(x_3)\xi_2 + [-\partial_{x_1}\lambda \sin(x_3) + \partial_{x_2}\lambda \cos(x_3)]\xi_3).$$

Thus P has real principal symbol $p = p_1 p_2$, and the characteristic set $p^{-1}(0)$ is the union of $p_1^{-1}(0)$ and $p_2^{-1}(0)$. However, the intersection

$$p_1^{-1}(0) \cap p_2^{-1}(0) = \{(x, \xi) \in T^*(SM) \, ; \, \xi_3 = 0, \ \cos(x_3)\xi_1 + \sin(x_3)\xi_2 = 0\}$$

is nontrivial and in this set $dp = p_1 \, dp_2 + p_2 \, dp_1$ vanishes. This means that any null bicharacteristic curve through $p_1^{-1}(0) \cap p_2^{-1}(0)$ reduces to a point, and in particular P is not of principal type (see [Hö85, Chapter XXVI] for more on principal type operators).

The fact that P is not of principal type means that its properties may depend on lower order terms. Indeed, if $d = 2$ one can find first order operators W on SM so that $P + W$ has nontrivial compactly supported solutions. To see this, note that if $W = X_\perp$ then $P + W = XV$ (see [PSU13]), and any $u \in C_c^\infty(M^{\mathrm{int}})$ satisfies $XVu = 0$ in SM. Moreover, the counterexample in [Bo10] implies that one may even find such an operator W arising from a weighted X-ray transform: when (M, g) is the Euclidean unit disc, there is $\mathcal{A} \in C^\infty(SM)$ such that $V(X + \mathcal{A})u = 0$ for some nontrivial $u \in C_c^\infty(SM^{\mathrm{int}})$. These observations indicate that the structure of lower order terms is crucial for the uniqueness problem, and principal symbol computations will not be sufficient.

However, in spite of the above issues it is possible to obtain uniqueness results for P via energy methods. Based on the Pestov energy identity, in [PSU15] it was proved that if (M, g) is a compact simple/Anosov manifold, one has the inequality

$$\|u - (u)_{SM}\|_{H^1(SM)} \leq C\|Pu\|_{L^2(SM)}$$

valid for all $u \in H^2(SM)$ (with $u|_{\partial(SM)} = 0$ in the boundary case). Here $(u)_{SM} = \frac{1}{\mathrm{Vol}(SM)} \int_{SM} u$. Such an inequality immediately implies that if $Pu = 0$ in SM and $u|_{\partial(SM)} = 0$, then $u \equiv 0$.

Various generalizations of the geodesic X-ray transform lead to estimates of the form

(3.3) $$\|u - (u)_{SM}\|_{H^1(SM)} \leq C\|Pu\|_{L^2(SM)} + \text{error term}$$

where, roughly,

$$\left\{ \begin{array}{l} \text{unitary connections} \\ \text{higher order tensors} \end{array} \right. \rightsquigarrow H^{1/2} \text{ error,}$$

$$\left\{ \begin{array}{c} \text{general connections} \\ \text{curvature} \\ \text{spatial localization} \end{array} \right. \rightsquigarrow H^1 \text{ error.}$$

To justify these statements, note that a unitary connection A introduces the term $-(F_A u, \overset{\mathtt{v}}{\nabla} u)$ in the Pestov identity [GPSU16], and that when $d = 2$ the argument leading to [PSU13, equation (6)] shows that higher order tensors introduce a term of the form $\sum_{k=-\infty}^\infty |k|(Kv_k, v_k)$ (this extends to any dimension). Both cases correspond to $H^{1/2}$ error terms in (3.3). However, a general (non-unitary) connection A can be introduced to the Pestov identity by writing P as $P + \overset{\mathtt{v}}{\nabla} A - \overset{\mathtt{v}}{\nabla} A$ and using the triangle

inequality, leading to an H^1 error term $\|\overset{\text{v}}{\nabla}(Au)\|$. The curvature term $-(R\overset{\text{v}}{\nabla}u, \overset{\text{v}}{\nabla}u)$ in the Pestov identity is also of H^1 type, and spatial localization corresponds to commuting P with a spatial cutoff function which again introduces an H^1 error.

The above statements indicate that unitary connections and higher order tensors lead to $H^{1/2}$ errors in the energy estimate (3.3). This is a weaker norm than the H^1 norm appearing on the left hand side of (3.3), which suggests that such perturbations might be manageable (indeed, the high frequency components in the error term can be absorbed, which in particular implies that there is a finite dimensional kernel). However, general connections, curvature or spatial localization correspond to H^1 error terms which are as strong as the H^1 term on the left, suggesting that these perturbations may be challenging to handle by energy methods (it seems that so far only the method in [UV16] can really deal with such perturbations). In this article, we introduce energy methods that are able to deal with H^1 perturbations arising from general connections.

Analogies with scattering theory. There is another potentially useful analogy between the geodesic vector field X on SM and the Laplacian $-\Delta$ in $\mathbb{R}^d$. The recent work [Gu17] connects tensor tomography to the study of resolvents $R_{\pm}(0)$ of X and earlier microlocal work on the spectral properties of X. Similarly, spectral and scattering theory for $-\Delta$ is a very classical topic and involves studying the resolvents $R_{\pm}(\lambda) = (-\Delta - (\lambda \pm i0))^{-1}$ on $\mathbb{R}^d$. One potentially relevant result is the absence of embedded eigenvalues for short range potentials (see [Hö85, Section 14.7]):

Theorem 3.1. *Let $V \in L^\infty(\mathbb{R}^d)$ be a real valued function such that*
$$|V(x)| \leq C(1 + |x|)^{-1-\varepsilon}$$
for some $C, \varepsilon > 0$. Given any $\lambda > 0$, the only solution of $(-\Delta + V - \lambda)u = 0$ in $\mathbb{R}^d$ with $u \in L^2(\mathbb{R}^d)$ is $u \equiv 0$.

There are different proofs, but one of them proceeds as follows:
1. (Rapid decay) One has
$$\begin{cases} (-\Delta + V - \lambda)u = 0 \text{ in } \mathbb{R}^d \\ u \in L^2(\mathbb{R}^d) \end{cases} \implies (1 + |x|)^\tau u \in L^2(\mathbb{R}^d) \text{ for all } \tau > 0.$$

This follows by using properties of the equation.
2. (Unique continuation at infinity) One has
$$\begin{cases} (-\Delta + V - \lambda)u = 0 \text{ in } \mathbb{R}^d \\ (1 + |x|)^\tau u \in L^2(\mathbb{R}^d) \text{ for } \tau > 0 \end{cases} \implies u = 0 \text{ outside some ball.}$$

This can be done by Carleman estimates with weights $|x|^\tau$.
3. (Unique continuation) One has
$$\begin{cases} (-\Delta + V - \lambda)u = 0 \text{ in } \mathbb{R}^d \\ u = 0 \text{ outside some ball} \end{cases} \implies u \equiv 0.$$

This can be done by standard Carleman estimates.

Compare this with the proof of one of our main new applications (this can be viewed as a special case of Theorem 1.5).

Theorem 3.2. *Let (M, g) be a compact simple manifold with negative sectional curvature. If A is a general connection and if $u \in C^\infty(SM \setminus \partial(SM)) \cap C(SM)$ satisfies*

$$(X + A)u = f_0 \text{ in } SM, \qquad u|_{\partial(SM)} = 0,$$

then $u = 0$.

This proof can also be split as follows where the main novelty is in Step 2.

1. (Rapid decay) One has

$$\begin{cases} (X + A)u = f_0 \\ u|_{\partial(SM)} = 0 \end{cases} \implies u \in C^\infty(SM), \text{ so } \sum l^{2\tau} \|u_l\|^2 < \infty \text{ for all } \tau > 0.$$

   This is a regularity result from the [PSU12], based on an earlier regularity result in [PU05].

2. (Unique continuation at infinity) One has

$$\begin{cases} (X + A)u = f_0, \ u|_{\partial(SM)} = 0 \\ u \in C^\infty(SM) \end{cases} \implies u \text{ has finite degree.}$$

   This is done by the new Carleman estimate with polynomial given in Theorem 1.1.

3. (Unique continuation) One has

$$\begin{cases} (X + A)u = f_0, \ u|_{\partial(SM)} = 0 \\ u \text{ has finite degree} \end{cases} \implies u \equiv 0.$$

   This follows from the injectivity of X_+^A on each Ω_m (the absence of twisted conformal Killing tensor fields (CKTs)), see [GPSU16]. The operator X_+^A is defined in Section 4 below.

4. PESTOV IDENTITY WITH GENERAL CONNECTION

In this section we will prove a version of the Pestov identity that involves a general connection, extending the version for unitary connections proved in [GPSU16]. We also give an equivalent version stated in terms of the $X_\pm$ operators. For the applications we will only use the identity with unitary connection. However, the general setup here highlights the fact that moving from unitary to general connections results in a lack of symmetry in the energy estimates. This indicates that general connections are indeed stronger perturbations than unitary connections, as explained in Section 3. For simplicity of presentation we will work on the trivial bundle $M \times \mathbb{C}^n$, but straightforward modifications would lead to analogous results on general Hermitian bundles as in [GPSU16]. We will use the notation from Section 2.

Let (M, g) be a compact manifold with or without boundary, with $d = \dim(M)$. Let A be an $n \times n$ matrix of smooth complex 1-forms on M, or equivalently a smooth

function $A : SM \to \mathbb{C}^{n \times n}$ so that $A(x, v)$ is linear in v. Then A defines a connection on the trivial bundle $M \times \mathbb{C}^n$. We define the following operators on $C^\infty(SM, \mathbb{C}^n)$,

$$X^A := X + A,$$

$$\overset{h}{\nabla}_A := \overset{h}{\nabla} + (\overset{v}{\nabla}A).$$

Here A and $(\overset{v}{\nabla}A)$ act by multiplication. The horizontal divergence $\overset{h}{\operatorname{div}}_A$ is defined for $Z \in \mathcal{Z}^n$ by

$$\overset{h}{\operatorname{div}}_A Z := \overset{h}{\operatorname{div}} Z + \langle \overset{v}{\nabla}A, Z \rangle.$$

Finally, we define the curvature type operator $F_A : C^\infty(SM, \mathbb{C}^{n \times n}) \to \mathcal{Z}^{n \times n}$ by

$$F_A := X(\overset{v}{\nabla}A) - \overset{h}{\nabla}A + [A, \overset{v}{\nabla}A].$$

In the natural L^2 inner product, one has

$$(X^A)^* = -X^{-A^*}, \qquad (\overset{h}{\nabla}_A)^* = -\overset{h}{\operatorname{div}}_{-A^*}.$$

Note that if A is a unitary connection, meaning that $A^* = -A$, then of course $(X^A)^* = -X^A$, $(\overset{h}{\nabla}_A)^* = -\overset{h}{\operatorname{div}}_A$, and F_A is the curvature operator in [GPSU16].

With the above conventions, one can check by direct computations that the basic commutator formulas given in [PSU15] for $A = 0$ and in [GPSU16] for unitary A remain true for a general connection A.

Lemma 4.1. *One has the following commutator identities on* $C^\infty(SM, \mathbb{C}^n)$,

$$[X^A, \overset{v}{\nabla}] = -\overset{h}{\nabla}_A,$$

$$[X^A, \overset{h}{\nabla}_A] = R\overset{v}{\nabla} + F_A,$$

$$\overset{h}{\operatorname{div}}_A\overset{v}{\nabla} - \overset{v}{\operatorname{div}}\overset{h}{\nabla}_A = (d - 1)X^A,$$

and by duality one gets the following identity on $\mathcal{Z}^n$,

$$[X^A, \overset{v}{\operatorname{div}}] = -\overset{h}{\operatorname{div}}_A.$$

The paper [GPSU16] gave a version of the Pestov identity with unitary connection in any dimension. If the connection is not unitary, one loses symmetry in the Pestov identity but the following form of this identity remains valid.

Lemma 4.2. *If A is a general connection, one has the identity*

$$(\overset{v}{\nabla}X^A u, \overset{v}{\nabla}X^{-A^*}u)$$

$$= (X^A\overset{v}{\nabla}u, X^{-A^*}\overset{v}{\nabla}u) - (R\overset{v}{\nabla}u, \overset{v}{\nabla}u) - (F_A u, \overset{v}{\nabla}u) + (d - 1)(X^A u, X^{-A^*}u)$$

for any $u \in C^\infty(SM, \mathbb{C}^n)$ with $u|_{\partial(SM)} = 0$ in the boundary case.

Proof. Observe that

$$(\overset{\mathrm{v}}{\nabla}X^A u, \overset{\mathrm{v}}{\nabla}X^{-A^*}u) - (X^A\overset{\mathrm{v}}{\nabla}u, X^{-A^*}\overset{\mathrm{v}}{\nabla}u) = ((X^A\mathrm{div}\overset{\mathrm{v}}{\nabla}X^A - \mathrm{div}X^A X^A\overset{\mathrm{v}}{\nabla})u, u).$$

The commutator identities above imply that

$$X^A\mathrm{div}\overset{\mathrm{v}}{\nabla}X^A - \mathrm{div}X^A X^A\overset{\mathrm{v}}{\nabla} = -(d-1)X^A X^A + \mathrm{div}(R\overset{\mathrm{v}}{\nabla} + F_A).$$

The result follows. $\qquad\square$

If A is a general connection, the same argument as in Section 2 shows that X maps Ω_m to $\Omega_{m-1} \oplus \Omega_{m+1}$. Thus we have the decomposition

$$X^A = X^A_+ + X^A_-, \qquad X^A_\pm : \Omega_m \to \Omega_{m\pm 1}.$$

We will next rewrite the Pestov identity in terms of X^A_+ and X^A_-. To do this, we need some notation: for a polynomially bounded sequence $\alpha = (\alpha_l)_{l=0}^\infty$ of real numbers, we define a corresponding "inner product"

$$(u, w)_\alpha = \sum_{l=0}^\infty \alpha_l (u_l, w_l)_{L^2(SM)}, \qquad u, w \in C^\infty(SM, \mathbb{C}^n).$$

We also write $\|u\|_\alpha^2 = \sum_{l=0}^\infty \alpha_l \|u_l\|^2$. (If each α_l is positive one gets an actual inner product and norm, but it is notationally convenient to allow zero or negative α_l.)

The Pestov identity can then be written in the following form.

Lemma 4.3. *If* $u \in C^\infty(SM, \mathbb{C}^n)$ *with* $u|_{\partial(SM)} = 0$ *in the boundary case, then*

$$(X^A_- u, X^{-A^*}_- u)_\alpha - (R\overset{\mathrm{v}}{\nabla}u, \overset{\mathrm{v}}{\nabla}u) - (F_A u, \overset{\mathrm{v}}{\nabla}u) + (Z_A(u), Z_{-A^*}(u)) = (X^A_+ u, X^{-A^*}_+ u)_\beta$$

where $Z_A(u)$ *is the* $\overset{\mathrm{v}}{\mathrm{div}}$-*free part of* $\overset{\mathrm{h}}{\nabla}_A u$, *and*

$$\alpha_l = \begin{cases} d-1, & l = 0, \\ (2l + d - 2)\left(1 + \frac{1}{l+d-2}\right), & l \geq 1, \end{cases}$$

$$\beta_l = \begin{cases} 0, & l = 0, 1, \\ (2l + d - 2)\left(1 - \frac{1}{l}\right), & l \geq 2. \end{cases}$$

Proof. We will use Lemma 4.2 in the form

$$(\overset{\mathrm{v}}{\nabla}X^A u, \overset{\mathrm{v}}{\nabla}X^{-A^*}u) - (d-1)(X^A u, X^{-A^*}u) = (X^A\overset{\mathrm{v}}{\nabla}u, X^{-A^*}\overset{\mathrm{v}}{\nabla}u) - (R\overset{\mathrm{v}}{\nabla}u, \overset{\mathrm{v}}{\nabla}u) - (F_A u, \overset{\mathrm{v}}{\nabla}u).$$

Note that Lemma 4.1 gives

$$(4.1) \qquad\qquad X^A\overset{\mathrm{v}}{\nabla}u = \overset{\mathrm{v}}{\nabla}X^A u - \overset{\mathrm{h}}{\nabla}_A u.$$

We also have the commutator formula $[X^A, -\mathrm{div}\overset{\mathrm{v}}{\nabla}] = 2\mathrm{div}\overset{\mathrm{v}}{\nabla}_A + (d-1)X^A$ by the identities in Lemma 4.1 (compare with [PSU15, Lemma 3.5]). Thus we obtain as in [PSU15, Lemma 4.4] that

$$(4.2) \qquad \overset{\mathrm{h}}{\nabla}_A u = \overset{\mathrm{v}}{\nabla}\left[\sum_{l=1}^\infty \left(\frac{1}{l}X^A_+ u_{l-1} - \frac{1}{l+d-2}X^A_- u_{l+1}\right)\right] + Z_A(u)$$

where $Z_A(u) \in \mathcal{Z}^n$ satisfies $\overset{\mathrm{v}}{\mathrm{div}}\, Z_A(u) = 0$. Thus (4.1) and (4.2) yield that

$$X^A \overset{\mathrm{v}}{\nabla} u = \overset{\mathrm{v}}{\nabla} \sum_{l=1}^{\infty} \left[\left(1 - \frac{1}{l}\right) X_+^A u_{l-1} + \left(1 + \frac{1}{l+d-2}\right) X_-^A u_{l+1} \right] - Z_A(u).$$

Applying this for A and $-A^*$ gives

$$(X^A \overset{\mathrm{v}}{\nabla} u, X^{-A^*} \overset{\mathrm{v}}{\nabla} u)$$

$$= \sum_{l=1}^{\infty} \lambda_l \left(\left(1 - \frac{1}{l}\right) X_+^A u_{l-1} + \left(1 + \frac{1}{l+d-2}\right) X_-^A u_{l+1}, \left(1 - \frac{1}{l}\right) X_+^{-A^*} u_{l-1} + \left(1 + \frac{1}{l+d-2}\right) X_-^{-A^*} u_{l+1} \right)$$

$$\quad + (Z_A(u), Z_{-A^*}(u))$$

$$= \sum_{l=1}^{\infty} \lambda_l \left[\left(1 - \frac{1}{l}\right)^2 (X_+^A u_{l-1}, X_+^{-A^*} u_{l-1}) + \left(1 + \frac{1}{l+d-2}\right)^2 (X_-^A u_{l+1}, X_-^{-A^*} u_{l+1}) \right]$$

$$\quad + \sum_{l=1}^{\infty} \lambda_l \left(1 - \frac{1}{l}\right) \left(1 + \frac{1}{l+d-2}\right) \left[(X_+^A u_{l-1}, X_-^{-A^*} u_{l+1}) + (X_-^A u_{l+1}, X_+^{-A^*} u_{l-1}) \right] + (Z_A(u), Z_{-A^*}(u)).$$

On the other hand, one has

$$(\overset{\mathrm{v}}{\nabla} X^A u, \overset{\mathrm{v}}{\nabla} X^{-A^*} u) - (d-1)(X^A u, X^{-A^*} u)$$

$$= -(d-1)(X_-^A u_1, X_-^{-A^*} u_1) + \sum_{l=1}^{\infty} (\lambda_l - (d-1))(X_+^A u_{l-1} + X_-^A u_{l+1}, X_+^{-A^*} u_{l-1} + X_-^{-A^*} u_{l+1})$$

$$= -(d-1)(X_-^A u_1, X_-^{-A^*} u_1) + \sum_{l=1}^{\infty} (\lambda_l - (d-1)) \left[(X_+^A u_{l-1}, X_+^{-A^*} u_{l-1}) + (X_-^A u_{l+1}, X_-^{-A^*} u_{l+1}) \right]$$

$$\quad + \sum_{l=1}^{\infty} (\lambda_l - (d-1)) \left[(X_+^A u_{l-1}, X_-^{-A^*} u_{l+1}) + (X_-^A u_{l+1}, X_+^{-A^*} u_{l-1}) \right].$$

Somewhat miraculously, we observe that

$$\lambda_l \left(1 - \frac{1}{l}\right) \left(1 + \frac{1}{l+d-2}\right) = \lambda_l - (d-1).$$

This means that the two sums above involving $\left[(X_+^A u_{l-1}, X_-^{-A^*} u_{l+1}) + (X_-^A u_{l+1}, X_+^{-A^*} u_{l-1}) \right]$ terms are equal. The Pestov identity in the beginning of the proof now yields

$$-(d-1)(X_-^A u_1, X_-^{-A^*} u_1) + \sum_{l=1}^{\infty} (\lambda_l - (d-1)) \left[(X_+^A u_{l-1}, X_+^{-A^*} u_{l-1}) + (X_-^A u_{l+1}, X_-^{-A^*} u_{l+1}) \right]$$

$$= \sum_{l=1}^{\infty} \lambda_l \left[\left(1 - \frac{1}{l}\right)^2 (X_+^A u_{l-1}, X_+^{-A^*} u_{l-1}) + \left(1 + \frac{1}{l+d-2}\right)^2 (X_-^A u_{l+1}, X_-^{-A^*} u_{l+1}) \right]$$

$$\quad - (R \overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u) - (F_A u, \overset{\mathrm{v}}{\nabla} u) + (Z_A(u), Z_{-A^*}(u)).$$

We rewrite this as

$$\sum_{l=0}^{\infty} \alpha_l(X_-^A u_{l+1}, X_-^{-A^*} u_{l+1}) - (R\overset{v}{\nabla}u, \overset{v}{\nabla}u) - (F_A u, \overset{v}{\nabla}u) + (Z_A(u), Z_{-A^*}(u))$$

$$= \sum_{l=1}^{\infty} \beta_l(X_+^A u_{l-1}, X_+^{-A^*} u_{l-1})$$

where

$$\alpha_l = \lambda_l \left[\left(1 + \frac{1}{l+d-2}\right)^2 - 1 \right] + (d-1),$$

$$\beta_l = \lambda_l \left[1 - \left(1 - \frac{1}{l}\right)^2 \right] - (d-1).$$

The result follows after simplifying the expressions for α_l and β_l. $\qquad \square$

We will use the previous identity only in the case where the connection A is unitary, or when $A = 0$. In these cases the inner products become squares of L^2 norms, and we obtain the following energy identity which is equivalent with the Pestov identity with unitary connection given in [GPSU16].

Proposition 4.4 (Pestov identity in terms of $X_\pm^A$). *Let (M, g) be a compact manifold with or without boundary, and let A be a unitary connection. Then*

$$\|X_-^A u\|_\alpha^2 - (R\overset{v}{\nabla}u, \overset{v}{\nabla}u) - (F_A u, \overset{v}{\nabla}u) + \|Z_A(u)\|^2 = \|X_+^A u\|_\beta^2$$

for any $u \in C^\infty(SM, \mathbb{C}^n)$ with $u|_{\partial(SM)} = 0$ in the boundary case.

5. FREQUENCY LOCALIZATION

Recall that any $u \in C^\infty(SM, \mathbb{C}^n)$ admits an L^2-orthogonal decomposition

$$u = \sum_{l=0}^{\infty} u_l, \qquad u_l \in \Omega_l,$$

where Ω_l corresponds to the set of vertical spherical harmonics of degree l. Since $X_\pm^A$ maps Ω_l to $\Omega_{l\pm 1}$, it is immediate that the Pestov identity with unitary connection (Proposition 4.4) reduces to the following identity when applied to functions in Ω_l.

Proposition 5.1 (Pestov identity on Ω_l). *Let (M, g) be a compact manifold with or without boundary, let A be a unitary connection, and let $l \geq 0$. One has*

$$\alpha_{l-1}\|X_-^A u\|^2 - (R\overset{v}{\nabla}u, \overset{v}{\nabla}u) - (F_A u, \overset{v}{\nabla}u) + \|Z_A(u)\|^2 = \beta_{l+1}\|X_+^A u\|^2, \qquad u \in \Omega_l,$$

if additionally $u|_{\partial(SM)} = 0$ in the boundary case. (We define $\alpha_{-1} = 0$.)

If $\dim(M) = 2$, the Pestov identity on Ω_l is the same as the Guillemin-Kazhdan energy identity [GK80a]. In [GPSU16, Appendix B] it was observed that in two dimensions the Guillemin-Kazhdan identity is actually equivalent with the Pestov

identity, in the sense that summing that Guillemin-Kazhdan identity over all l gives back the Pestov identity.

We will now show that the same is true in any dimension: the Pestov identity is equivalent with the frequency localized identities of Proposition 5.1. This means that the Pestov identity localizes completely with respect to vertical Fourier decompositions. This will be a very important observation in what follows.

Lemma 5.2. *The Pestov identity on Ω_l is equivalent with the Pestov identity with unitary connection in the following sense: for any $u \in C^\infty(SM, \mathbb{C}^n)$ with $u|_{\partial(SM)} = 0$ in the boundary case, one has*

$$\sum_{l=0}^{\infty}\left[\alpha_{l-1}\|X_-^A u_l\|^2 - (R\overset{\text{v}}{\nabla}u_l, \overset{\text{v}}{\nabla}u_l) - (F_A u_l, \overset{\text{v}}{\nabla}u_l) + \|Z_A(u_l)\|^2 - \beta_{l+1}\|X_+^A u_l\|^2\right]$$

$$= \|X_-^A u\|_\alpha^2 - (R\overset{\text{v}}{\nabla}u, \overset{\text{v}}{\nabla}u) - (F_A u, \overset{\text{v}}{\nabla}u) + \|Z_A(u)\|^2 - \|X_+^A u\|_\beta^2.$$

The result will follow if we can show that the curvature and Z_A terms localise. Thus Lemma 5.2 is a corollary of the next result.

Lemma 5.3. *If (M, g) is a Riemannian manifold and A is a unitary connection, then*

$$(R\overset{\text{v}}{\nabla}u, \overset{\text{v}}{\nabla}w) = 0, \qquad (F_A u, \overset{\text{v}}{\nabla}w) = 0, \qquad (Z_A(u), Z_A(w)) = 0,$$

whenever $u \in \Omega_m$, $w \in \Omega_l$ and $m \neq l$.

The proof reduces to a statement for fixed x and can be carried out in local coordinates. Let x be local coordinates in M, let (x, y) be associated local coordinates on TM and let (x, v) be corresponding local coordinates in SM as in [PSU15, Appendix A]. Define the vector fields

$$\partial_j u = \partial_{y_j}(u(x, y/|y|_g))|_{SM}.$$

We will need an auxiliary lemma.

Lemma 5.4. *If $u \in \Omega_m$, then $\partial_j u \in \Omega_{m+1} \oplus \Omega_{m-1}$ and one has*

$$\partial_j u = -m v_j u + h_j$$

for some $h_j \in \Omega_{m-1}$. Moreover, for any j, k one has $v_j h_k \in \Omega_m \oplus \Omega_{m-2}$ and

$$v_j h_k = h_{jk} + f_{jk}$$

for some $h_{jk} \in \Omega_{m-2}$ with $h_{jk} = h_{kj}$, and for some $f_{jk} \in \Omega_m$.

Proof. It is enough to work with fixed x, which we will suppress from the notation. In $\mathbb{R}^d$ with the Euclidean metric, the first result follows since $u = U|_{S^{d-1}}$ where U is a harmonic homogeneous polynomial of degree m, and one has

$$\partial_j u = \partial_{y_j}(u(y/|y|)|_{S^{d-1}} = \partial_{y_j}(|y|^{-m}U(y))|_{S^{d-1}} = (-m y_j U + \partial_{y_j}U)|_{S^{d-1}}$$

where $\partial_{y_j}U|_{S^{d-1}}$ is a spherical harmonic of degree $m - 1$. Similarly, for some constant $c = c(d, m)$, the fact that $\Delta_y^2(y_j \partial_{y_k}U) = 0$ and [DX13, Lemma 1.2.1] imply that

$$v_j h_k = y_j \partial_{y_k}U|_{S^{d-1}} = \Omega_m + c\Delta_y(y_j \partial_{y_k}U)|_{S^{d-1}} = \Omega_m + 2c\partial_{y_j y_k}U|_{S^{d-1}}$$

showing that $h_{jk} = 2c\partial_{y_j y_k} U|_{S^{d-1}} = h_{kj}$. Analogous results on $S_x M$ follow from the Euclidean results since $S_x M$ and S^{d-1} are isometric (see [PSU15, Appendix A]).

Alternatively, the first result on $S_x M$ follows from the formula

$$(\Delta - \lambda_{m-1})(\partial_j u + m v_j u) = 0, \qquad u \in \Omega_m,$$

where $\Delta = -\mathrm{div} \overset{\mathrm{v}}{\nabla}$. This in turn follows from the commutator formulas

$$[\Delta, \partial_j] = -(d-3)\partial_j - 2v_j \Delta,$$
$$[\Delta, v_j] = -2\partial_j + (d-1)v_j,$$

where $d = \dim(M)$. The second result on $S_x M$ follows since

$$v_j h_k - v_k h_j = v_j(\partial_k u + m v_k u) - v_k(\partial_j u + m v_j u) = (v_j \partial_k - v_k \partial_j)u \in \Omega_m$$

using that $[\Delta, v_j \partial_k - v_k \partial_j] = 0$. This proves that $h_{jk} = h_{kj}$. $\qquad\square$

Proof of Lemma 5.3. Working in local coordinates as in [PSU15, Appendix A], we have $v = v^j \partial_{x_j}$ and $\overset{\mathrm{v}}{\nabla} u(x,v) = \partial^j u(x,v)\partial_{x_j}$. Thus we have

$$\int_{S_x M} R(\overset{\mathrm{v}}{\nabla} u, v, v, \overset{\mathrm{v}}{\nabla} w)\, d(S_x M) = \int_{S_x M} R_{abcd}\partial^a u v^b v^c \partial^d w\, d(S_x M).$$

Use Lemma 5.4 to write

$$\partial^a u = -m v^a u + h^a, \qquad \partial^d w = -l v^d w + q^d$$

where each h^a is in Ω_{m-1} and each q^d is in Ω_{l-1}. By the symmetries of the curvature tensor, we get

$$\int_{S_x M} R(\overset{\mathrm{v}}{\nabla} u, v, v, \overset{\mathrm{v}}{\nabla} w)\, d(S_x M) = \int_{S_x M} R_{abcd} h^a v^b v^c q^d\, d(S_x M).$$

Since $h^a v^b \in \Omega_m \oplus \Omega_{m-2}$ and $v^c q^d \in \Omega_l \oplus \Omega_{l-2}$, the last integral can only be nonzero if $l \in \{m-2, m, m+2\}$. Assume first that $l = m-2$. Lemma 5.4 shows that

$$h^a v^b = h^{ba} + \Omega_m$$

where $h^{ba} \in \Omega_{m-2}$ and $h^{ba} = h^{ab}$. The integral involving the Ω_m part is zero since $v^c q^d \in \Omega_{m-2} \oplus \Omega_{m-4}$, and the integral involving the h^{ba} part is zero using that $h^{ab} = h^{ba}$ and the symmetries of R. The case $l = m+2$ follows similarly. We have proved that $(R\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} w) = 0$ whenever $m \neq l$.

For the result involving F_A, we first use [GPSU16, Lemma 3.1] and observe that given $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$ and $Z = (Z_1, \ldots, Z_n)$ with $\langle Z_\alpha, v \rangle = 0$ for $1 \leq \alpha \leq n$, one has

$$\langle F_A u, Z \rangle = \sum_{\alpha,\beta} \langle (F_A)_{\alpha\beta} u_\alpha, Z_\beta \rangle = \sum_{\alpha,\beta} (f_A)_{\alpha\beta}(v, u_\alpha Z_\beta)$$

where $f_A = dA + A \wedge A$ is an $n \times n$ matrix of complex 2-forms. It follows that

$$(F_A u, \overset{\mathrm{v}}{\nabla} w) = \int_{S_x M} (f_A)_{\alpha\beta}(v, u_\alpha \overset{\mathrm{v}}{\nabla} \bar{w}_\beta)\, d(S_x M).$$

Writing $v = v^j \partial_{x_j}$ and $\overset{v}{\nabla} w_\beta = \partial^k w_\beta \partial_{x_k}$, we have

$$(F_A u, \overset{v}{\nabla} w) = \int_{S_x M} u_\alpha v^j \partial^k \bar{w}_\beta (f_A)_{\alpha\beta}(\partial_{x_j}, \partial_{x_k}) \, d(S_x M).$$

By Lemma 5.4, if $w \in \Omega_l$ one has

$$\partial^k \bar{w}_\beta = -lv^k \bar{w}_\beta + q_\beta^k, \qquad \text{where } q_\beta^k \in \Omega_{l-1}.$$

The integral involving the $-lv^k \bar{w}_\beta$ part is zero since f_A is a 2-form. Using the fact that $u_\alpha v^j \in \Omega_{m+1} \oplus \Omega_{m-1}$, the integral involving the q_β^k part is zero unless $l \in \{m, m+2\}$. Next we invoke the fact that A is unitary: the definition of F_A and Lemma 4.1 imply

$$\overset{v}{\mathrm{div}}(F_A) = 0$$

and consequently, since A is unitary,

(5.1) $$(F_A u, \overset{v}{\nabla} w) = \overline{(F_A w, \overset{v}{\nabla} u)}.$$

By (5.1) we can change the roles of u and w, and we see that $(F_A u, \overset{v}{\nabla} w)$ is also zero unless $m \in \{l, l+2\}$. Thus $(F_A u, \overset{v}{\nabla} w) = 0$ whenever $m \neq l$.

Finally, to prove to statement involving Z_A we observe that Proposition 4.4, the formula (5.1), and the polarization identity imply that

$$(Z_A(u), Z_A(w)) = (X_+^A u, X_+^A w)_\beta - (X_-^A u, X_-^A w)_\alpha + (R \overset{v}{\nabla} u, \overset{v}{\nabla} w) + (F_A u, \overset{v}{\nabla} w).$$

The statements proved above imply that $(Z_A(u), Z_A(w)) = 0$ when $m \neq l$. $\qquad \square$

6. Carleman estimates in negative curvature

We will now give the first Carleman estimate for X-ray transforms, valid for negative sectional curvature. Throughout this section we will assume that (M, g) is a compact Riemannian manifold with or without boundary, with $d = \dim(M) \geq 2$. The following theorem is the main result of this section.

Theorem 6.1. *Assume that that (M, g) has sectional curvature $\leq -\kappa$ where $\kappa > 0$. For any $s > -1/2$, for any $m \geq 1$ and for any $u \in C^\infty(SM, \mathbb{C}^n)$ (with $u|_{\partial(SM)} = 0$ in the boundary case) one has*

$$2 \sum_{l=m}^{m+1} l^{2s+1} \|X_- u_l\|^2 + (2s+1) \sum_{l=m+2}^{\infty} (l-1)^{2s} \|X_- u_l\|^2 + \kappa \sum_{l=m}^{\infty} l^{2s+2} \|u_l\|^2$$

$$+ \sum_{l=m}^{\infty} l^{2s} \|Z(u_l)\|^2 \leq C \sum_{l=m+1}^{\infty} l^{2s+2} \|(Xu)_l\|^2$$

where $C = C(d, s) > 0$. If $d = 2$, or if $d \geq 3$ and $s \geq 0$, one can take

$$C = \frac{(d+4)^2}{2s+1}.$$

If s is large, the previous result easily implies the Carleman estimate stated in the introduction.

Theorem 6.2. *Assume that that (M, g) has sectional curvature $\leq -\kappa$ where $\kappa > 0$. For any $\tau \geq 1$ and $m \geq 1$, one has*

$$\sum_{l=m}^{\infty} e^{2\tau \log(l)} \|u_l\|^2 \leq \frac{(d+4)^2}{\kappa \tau} \sum_{l=m+1}^{\infty} e^{2\tau \log(l)} \|(Xu)_l\|^2$$

whenever $u \in C^{\infty}(SM, \mathbb{C}^n)$ (with $u|_{\partial(SM)} = 0$ in the boundary case).

Proof. It is enough to choose $\tau = s + 1$ with $s \geq 0$ in Theorem 6.1. $\square$

As discussed in the introduction, the Carleman estimate in Theorem 6.1 can also be understood as a version of the Pestov identity that has been shifted to a different regularity scale. To explain this, introduce the mixed norms

$$\|u\|_{L_x^2 H_v^s}^2 = \sum_{l=0}^{\infty} \langle l \rangle^{2s} \|u_l\|^2,$$

$$\|\overset{\text{v}}{\nabla} u\|_{L_x^2 H_v^s}^2 = \sum_{l=0}^{\infty} \langle l \rangle^{2s} \|\overset{\text{v}}{\nabla} u_l\|^2,$$

$$\|\nabla_{SM} u\|_{L_x^2 H_v^s}^2 = \sum_{l=0}^{\infty} \langle l \rangle^{2s} (\|X_- u_{l+1}\|^2 + \|X_+ u_{l-1}\|^2 + \|Z(u_l)\|^2 + \|\overset{\text{v}}{\nabla} u_l\|^2)$$

where $\langle l \rangle = (1 + l^2)^{1/2}$, $\|\overset{\text{v}}{\nabla} u_l\|^2 = \lambda_l \|u_l\|^2$, and $u_{-1} = 0$.

Clearly $\|u\|_{L_x^2 H_v^0} = \|u\|_{L^2(SM)}$ and $\|\overset{\text{v}}{\nabla} u\|_{L_x^2 H_v^0} = \|\overset{\text{v}}{\nabla} u\|_{L^2(SM)}$. We also have that $\|\nabla_{SM} u\|_{L_x^2 H_v^0} \sim \|\nabla_{SM} u\|_{L^2(SM)}$ by the formula (see [LRS18, Lemma 5.1])

$$\|\nabla_{SM} u\|^2 = \|Xu\|^2 + \|\overset{\text{h}}{\nabla} u\|^2 + \|\overset{\text{v}}{\nabla} u\|^2 \sim \|X_- u\|^2 + \|X_+ u\|^2 + \|Z(u)\|^2 + \|\overset{\text{v}}{\nabla} u\|^2,$$

and using that $\|Z(u)\|^2 = \sum_{l=0}^{\infty} \|Z(u_l)\|^2$ by Lemma 5.3.

Theorem 6.1 can now be restated as a shifted Pestov identity.

Theorem 6.3. *Assume that that (M, g) has sectional curvature $\leq -\kappa$ where $\kappa > 0$. For any $s > -1/2$ there is $C = C_{d,s,\kappa} > 0$ such that one has*

$$\|\nabla_{SM} u\|_{L_x^2 H_v^s} \leq C \|\overset{\text{v}}{\nabla} Xu\|_{L_x^2 H_v^s}$$

for any $u \in C^{\infty}(SM, \mathbb{C}^n)$ (with $u|_{\partial(SM)} = 0$ in the boundary case).

Proof. Theorem 6.1 with the choice $m = 1$ yields that

$$2\|X_- u_1\|^2 + 2\|X_- u_2\|^2 + (2s+1) \sum_{l=2}^{\infty} l^{2s} \|(X_- u)_l\|^2 + \kappa \sum_{l=1}^{\infty} l^{2s+2} \|u_l\|^2$$

$$+ \sum_{l=1}^{\infty} l^{2s} \|Z(u_l)\|^2 \leq C \sum_{l=2}^{\infty} l^{2s+2} \|(Xu)_l\|^2 \leq C \|\overset{\text{v}}{\nabla} Xu\|_{L_x^2 H_v^s}^2.$$

Note also that $Z(u_0) = 0$, which follows from Proposition 5.1 with $l = 0$ (recall that $\beta_1 = 0$). Thus we have

$$\sum_{l=0}^{\infty} \langle l \rangle^{2s} (\|X_- u_{l+1}\|^2 + \|Z(u_l)\|^2 + \|\overset{\text{v}}{\nabla} u_l\|^2) \leq C \|\overset{\text{v}}{\nabla} X u\|_{L_x^2 H_v^s}^2.$$

Finally, note that

$$\|X_+ u\|_{L_x^2 H_v^s}^2 = \sum_{l=1}^{\infty} \langle l \rangle^{2s} \|(Xu)_l - X_- u_{l+1}\|^2 \leq C(\|\overset{\text{v}}{\nabla} X u\|_{L_x^2 H_v^s}^2 + \|X_- u\|_{L_x^2 H_v^s}^2).$$

The result follows upon combining the last two inequalities. $\qquad\square$

We now begin the proof of Theorem 6.1. The first step is to observe that the localized Pestov identity in Proposition 5.1 gains a positive term in negative sectional curvature.

Lemma 6.4. *Assume that (M, g) has sectional curvature $\leq -\kappa$ where $\kappa > 0$, and let $l \geq 0$. One has*

$$\alpha_{l-1}\|X_- u\|^2 + \kappa \lambda_l \|u\|^2 + \|Z(u)\|^2 \leq \beta_{l+1}\|X_+ u\|^2, \qquad u \in \Omega_l,$$

if additionally $u|_{\partial(SM)} = 0$ in the boundary case. (We define $\alpha_{-1} = 0$.)

Proof. This follows from Proposition 5.1 upon choosing $A = 0$ and noting that

$$-(R\overset{\text{v}}{\nabla} u, \overset{\text{v}}{\nabla} u) \geq \kappa \|\overset{\text{v}}{\nabla} u\|^2 = \kappa(-\overset{\text{v}}{\text{div}}\overset{\text{v}}{\nabla} u, u) = \kappa \lambda_l \|u\|^2. \qquad\square$$

Remark 6.5. The $\|Z(u)\|^2$ term can be simplified when $d = 2$. In this case, one has in the notation of [PSU15, Appendix B]

$$Z(u) = -(X_\perp u)_0 iv = (i(\eta_+ u_{-1} - \eta_- u_1))iv,$$

which shows in particular that $Z(u_k) = 0$ unless $k = \pm 1$. To prove the above claim, assume that $d = 2$ and note that, in the notation of [PSU15, Appendix B],

$$\overset{\text{h}}{\nabla} u = -(X_\perp u)iv, \qquad \overset{\text{v}}{\nabla} a = (Va)iv.$$

We may write $X_\perp u$ as

$$X_\perp u = (X_\perp u)_0 + Va, \qquad a = \sum_{k \neq 0} \frac{1}{ik}(X_\perp u)_k.$$

It follows that

$$\overset{\text{h}}{\nabla} u = \overset{\text{v}}{\nabla}(-a) - (X_\perp u)_0 iv.$$

Since $Z(u)$ was defined as the div-free part of $\overset{\text{v}}{\nabla} u$, we must have $Z(u) = -(X_\perp u)_0 iv$.

The Carleman estimate in Theorem 6.1 will follow after multiplying the localized estimates by suitable weights and adding them together. We first give an estimate with rather general weights. In what follows, $C_F^\infty(SM)$ denotes the set of smooth functions in SM with finite degree.

Proposition 6.6. *Assume that (M,g) has sectional curvature $\leq -\kappa$ where $\kappa > 0$. If $m \geq 1$ and if $(\gamma_l)_{l=0}^\infty$ is any sequence of positive real numbers satisfying*

$$(6.1) \qquad\qquad \alpha_l \gamma_{l+1}^2 > \beta_l \gamma_{l-1}^2, \qquad l \geq m+1,$$

and if $(\delta_l)_{l=0}^\infty$ is any sequence where $\delta_l \in (0,1]$, then one has

$$\sum_{l=m}^{m+1} \alpha_{l-1} \gamma_l^2 \|X_- u_l\|^2 + \sum_{l=m+2}^\infty (1-\delta_{l-1})(\alpha_{l-1}\gamma_l^2 - \beta_{l-1}\gamma_{l-2}^2)\|X_- u_l\|^2 + \kappa \sum_{l=m}^\infty \lambda_l \gamma_l^2 \|u_l\|^2$$

$$+ \sum_{l=m}^\infty \gamma_l^2 \|Z(u_l)\|^2 \leq \sum_{l=m+1}^\infty \left[1 + \frac{1-\delta_l}{\delta_l} \frac{\beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2}\right] \frac{\alpha_l \gamma_{l+1}^2 \beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2} \|(Xu)_l\|^2.$$

whenever $u \in C_F^\infty(SM, \mathbb{C}^n)$ (with $u|_{\partial(SM)} = 0$ in the boundary case).

Remark 6.7. The sequence (γ_l) corresponds to the weights in the Carleman estimate, and the parameters (δ_l) fine-tune the weights for the X_- terms on the left. Later we will essentially choose $\gamma_l = l^s$ for $s > -1/2$ and $\delta_l \equiv 1/2$. An even simpler choice, which would also be sufficient for most of our purposes, would be to take $\delta_l \equiv 1$. Then the above estimate becomes (after dropping the $X_- u_l$ and $Z(u_l)$ terms)

$$\kappa \sum_{l=m}^\infty \lambda_l \gamma_l^2 \|u_l\|^2 \leq \sum_{l=m+1}^\infty \frac{\alpha_l \gamma_{l+1}^2 \beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2} \|(Xu)_l\|^2$$

for any $u \in C_F^\infty(SM)$ (with $u|_{\partial(SM)} = 0$ in the boundary case)

Proof of Proposition 6.6. Let $u \in C_F^\infty(SM, \mathbb{C}^n)$, and let $(\gamma_l)_{l=0}^\infty$ be a sequence of positive real numbers satisfying (6.1). Lemma 6.4 implies that for each $l \geq 0$,

$$\alpha_{l-1} \gamma_l^2 \|X_- u_l\|^2 + \kappa \lambda_l \gamma_l^2 \|u_l\|^2 + \gamma_l^2 \|Z(u_l)\|^2 \leq \beta_{l+1} \gamma_l^2 \|X_+ u_l\|^2.$$

Let now $\varepsilon_l > 0$ be positive numbers, and observe that

$$\|X_+ u_l\|^2 = \|(Xu)_{l+1} - X_- u_{l+2}\|^2 = \|(Xu)_{l+1}\|^2 - 2\operatorname{Re}((Xu)_{l+1}, X_- u_{l+2}) + \|X_- u_{l+2}\|^2$$

$$(6.2) \qquad \leq (1 + \frac{1}{\varepsilon_l})\|(Xu)_{l+1}\|^2 + (1+\varepsilon_l)\|X_- u_{l+2}\|^2.$$

Combining the last two inequalities gives the estimate

$$\alpha_{l-1} \gamma_l^2 \|X_- u_l\|^2 + \kappa \lambda_l \gamma_l^2 \|u_l\|^2 + \gamma_l^2 \|Z(u_l)\|^2$$

$$\leq \beta_{l+1} \gamma_l^2 (1 + \frac{1}{\varepsilon_l})\|(Xu)_{l+1}\|^2 + \beta_{l+1} \gamma_l^2 (1+\varepsilon_l)\|X_- u_{l+2}\|^2.$$

Adding up these estimates for $m \leq l \leq N$, where $m \geq 1$, yields

$$(6.3) \quad \sum_{l=m}^N \alpha_{l-1} \gamma_l^2 \|X_- u_l\|^2 + \sum_{l=m}^N \kappa \lambda_l \gamma_l^2 \|u_l\|^2 + \sum_{l=m}^N \gamma_l^2 \|Z(u_l)\|^2$$

$$\leq \sum_{l=m+1}^{N+1} \beta_{l-1} \gamma_{l-1}^2 (1 + \frac{1}{\varepsilon_{l-1}})\|(Xu)_l\|^2 + \sum_{l=m+2}^{N+2} \beta_{l-1} \gamma_{l-2}^2 (1+\varepsilon_{l-2})\|X_- u_l\|^2.$$

We would like to choose (γ_l) and (ε_l) so that a large part of the last term on the right can be absorbed in the first term on the left. The minimal requirement is that

$$\beta_{l-1}\gamma_{l-2}^2(1+\varepsilon_{l-2}) \le \alpha_{l-1}\gamma_l^2, \qquad m+2 \le l \le N.$$

We will choose (ε_l) so that

(6.4) $$\varepsilon_{l-2} = \delta_{l-1}\left[\frac{\alpha_{l-1}}{\beta_{l-1}}\frac{\gamma_l^2}{\gamma_{l-2}^2} - 1\right], \qquad l \ge m+2,$$

where $\delta_{l-1} \in (0,1]$. In order for ε_l to be positive, we need to have $\alpha_{l-1}\gamma_l^2 > \beta_{l-1}\gamma_{l-2}^2$ for $l \ge m+2$, which follows from the assumption (6.1).

Using (6.4), we may express the weights on the right hand side of (6.3) as

$$\beta_{l-1}\gamma_{l-2}^2(1+\varepsilon_{l-2}) = \beta_{l-1}\gamma_{l-2}^2 + \delta_{l-1}(\alpha_{l-1}\gamma_l^2 - \beta_{l-1}\gamma_{l-2}^2)$$

and

$$\beta_l\gamma_{l-1}^2\left(1 + \frac{1}{\varepsilon_{l-1}}\right) = \beta_l\gamma_{l-1}^2\left[1 + \frac{1}{\delta_l}\frac{\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2 - \beta_l\gamma_{l-1}^2}\right]$$

$$= \frac{\alpha_l\gamma_{l+1}^2\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2 - \beta_l\gamma_{l-1}^2}\left[1 + \frac{1-\delta_l}{\delta_l}\frac{\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2}\right].$$

Inserting these expressions in (6.3), it follows that

$$\sum_{l=m}^{m+1}\alpha_{l-1}\gamma_l^2\|X_-u_l\|^2 + \sum_{l=m+2}^{N}(1-\delta_{l-1})(\alpha_{l-1}\gamma_l^2 - \beta_{l-1}\gamma_{l-2}^2)\|X_-u_l\|^2 + \kappa\sum_{l=m}^{N}\lambda_l\gamma_l^2\|u_l\|^2$$

$$+ \sum_{l=m}^{N}\gamma_l^2\|Z(u_l)\|^2 \le \sum_{l=m+1}^{N+1}\left[1 + \frac{1-\delta_l}{\delta_l}\frac{\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2}\right]\frac{\alpha_l\gamma_{l+1}^2\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2 - \beta_l\gamma_{l-1}^2}\|(Xu)_l\|^2$$

$$+ \sum_{l=N+1}^{N+2}\left[\beta_{l-1}\gamma_{l-2}^2 + \delta_{l-1}(\alpha_{l-1}\gamma_l^2 - \beta_{l-1}\gamma_{l-2}^2)\right]\|X_-u_l\|^2.$$

Since u has finite degree we can take the limit as $N \to \infty$, which proves the result. $\square$

We now wish to fix a suitable choice of weights in Proposition 6.6. To motivate this choice, it is instructive to first consider the case $d = 2$. In this case $\alpha_l = 2l+2$, $\beta_l = 2l-2$ (for $l \ge 2$), and $\lambda_l = l^2$. Thus, writing

$$r_l = l\gamma_l^2,$$

the estimate in Proposition 6.6, with the choice $\delta_l = 1$ which keeps the weights on the right as small as possible, implies that

(6.5) $$\kappa\sum_{l=m}^{\infty}lr_l\|u_l\|^2 \le 2\sum_{l=m+1}^{\infty}\frac{r_{l+1}r_{l-1}}{r_{l+1} - r_{l-1}}\|(Xu)_l\|^2, \qquad u \in C_F^{\infty}(SM, \mathbb{C}^n).$$

In fact, the estimate (6.5) is valid for any strictly increasing sequence (r_l) of positive real numbers since (6.1) is valid then.

We will eventually apply (6.5) to functions $u \in C^\infty(SM, \mathbb{C}^n)$ with $(Xu + Au)_l = 0$ for $l \geq m + 1$ where A is a general connection, since such functions arise from the attenuated geodesic X-ray transform on m-tensors. Thus, writing $R = 4\|A\|_{L^\infty(M)}$, one has

$$\|(Xu)_l\|^2 \leq R(\|u_{l-1}\|^2 + \|u_{l+1}\|^2), \qquad l \geq m + 1.$$

Inserting this estimate in (6.5), one can absorb the resulting terms to the left hand side provided that the weights (r_l) satisfy

$$2R \left[\frac{r_{l-2}r_l}{r_l - r_{l-2}} + \frac{r_l r_{l+2}}{r_{l+2} - r_l} \right] \leq \kappa l r_l, \qquad l \geq m.$$

The last condition implies in particular that

$$2S r_{l-2} \leq l(r_l - r_{l-2}), \qquad l \geq m,$$

whenever $S \leq \frac{R}{\kappa}$. A simple inductive argument then shows that

$$r_{m+2k} \geq \left[\prod_{j=1}^{k} \frac{m + 2j + 2S}{m + 2j} \right] r_m, \qquad k \geq 1.$$

Choosing S to be an integer $\leq \frac{R}{\kappa}$, it follows that

$$r_{m+2k} \geq \frac{1}{(m + 2S)^S}(m + 2k)^S.$$

Finally, this implies that $r_l \geq c l^{\frac{R}{2\kappa}}$ for l large, so that $\gamma_l \geq c l^s$ for some sufficiently large $s > 0$.

The previous discussion implies that the weights γ_l have to grow at least like l^s for some large s in order to absorb errors coming from large general connections. On the other hand, γ_l should grow at most polynomially since one needs the sums in the estimate to be finite for $u \in C^\infty(SM, \mathbb{C}^n)$. We will now prove Theorem 6.1 in any dimension $d \geq 2$ by choosing $\gamma_l = l^s$ in Proposition 6.6 at least for large l. To deal with the cases where s is small, we will use the next elementary lemma.

Lemma 6.8. *If $s \geq 0$, then*

$$(l + 1)^s - (l - 1)^s \geq s l^{s-1}, \qquad l \geq 1.$$

Moreover, if $-1 < s < 0$, then

$$(l + 1)^s - (l - 1)^s \geq -\eta_s(l_0) l^{s-1}, \qquad l \geq l_0 \geq 2,$$

where $\eta_s(l_0)$ is positive and decreases to $-2s$ as $l_0 \to \infty$.

Proof. Let $s \in \mathbb{R}$, and consider the map

$$f_s : (0, 1) \to \mathbb{R}, \quad f_s(t) = \frac{(1 + t)^s - (1 - t)^s}{t}.$$

By the generalized binomial theorem, one has

(6.6)
$$f_s(t) = 2 \sum_{k=0}^{\infty} \binom{s}{1 + 2k} t^{2k}.$$

If $s = 2N + \delta$ for some integer $N \geq 0$ and $\delta \in [0, 1]$, then $\binom{s}{1+2k} \geq 0$ for any integer $k \geq 0$, which implies that $f_s(t) \geq 2s$.

On the other hand, let $s = 2N + 1 + \delta$ for some integer $N \geq 0$ and $\delta \in (0, 1)$. In this case $\binom{s}{1+2k} \geq 0$ for $k \leq N$ and $\binom{s}{1+2k} \leq 0$ for $k \geq N + 1$, and one does not immediately get a lower bound for $f_s(t)$. However, differentiating the series (6.6), the signs of the binomial coefficients yield that

$$(6.7) \qquad f_s(0) = 2s, \qquad f_s^{(j)}(0) \geq 0 \text{ for } 1 \leq j \leq 2N, \qquad f_s^{(2N+1)}(t) \leq 0.$$

If $N = 0$, the third condition in (6.7) gives $f_s'(t) \leq 0$, and thus in this case

$$f_s(t) \geq \lim_{r \to 1^-} f_s(r) = 2^s.$$

If $N \geq 1$, the third condition $f_s^{(2N+1)}(t) \leq 0$ similarly implies that

$$f_s^{(2N)}(t) \geq \lim_{r \to 1^-} f_s^{(2N)}(r).$$

To compute the last limit, write $f_s(t) = g(t)h(t)$ where $g(t) = (1 + t)^s - (1 - t)^s$ and $h(t) = t^{-1}$. Then

$$f_s^{(2N)}(1^-) = \sum_{j=0}^{2N} \binom{2N}{j} g^{(j)}(1^-) h^{(2N-j)}(1)$$

$$= \sum_{j=0}^{2N} \binom{2N}{j} s(s-1) \cdots (s - (j-1)) 2^{s-j} (-1)^j (2N - j)!$$

$$= (2N)!(2^s) \sum_{j=0}^{2N} \binom{s}{j} \left[-\frac{1}{2} \right]^j$$

$$\geq (2N)!(2^s) \sum_{j=0}^{2N} \binom{2N}{j} \left[-\frac{1}{2} \right]^j = (2N)!(2^s) 2^{-2N} \geq 0.$$

This shows that $f_s^{(2N)}(t) \geq 0$. Now, developing the Taylor expansion of $f_s(t)$ at the origin and using (6.7) yields, for some $c = c(t) \in [0, 1)$,

$$f_s(t) = \sum_{j=0}^{2N-1} \frac{f_s^{(j)}(0)}{j!} t^j + \frac{1}{(2N)!} f_s^{(2N)}(c) t^{2N} \geq 2s.$$

We have proved that

$$(1 + t)^s - (1 - t)^s \geq \min\{2s, 2^s\} t, \qquad s \geq 0, \ \ 0 \leq t \leq 1.$$

It follows that for $s \geq 0$ and $l \geq 1$, one has

$$(l + 1)^s - (l - 1)^s = l^s[(1 + 1/l)^s - (1 - 1/l)^s] \geq \min\{2s, 2^s\} l^{s-1}.$$

One has $2^s \geq (e \log 2)s$ for $1 < s < 2$ where $e \log 2 \approx 1.88 \geq 1$. This shows in particular that $\min\{2s, 2^s\} \geq s$ for $s \geq 0$. The first statement in the lemma follows.

Finally, assume that $-1 < s < 0$. In this case the formula (6.6) is still valid when $0 < t < 1$, and each term in the series is negative. Thus $f_s(t)$ is negative and decreases from $2s$ to $-\infty$ as $t \to 1-$. It follows that for $l \geq l_0 \geq 2$ one has

$$(l+1)^s - (l-1)^s = l^s[(1+1/l)^s - (1-1/l)^s] = f_s(1/l)l^{s-1} \geq f_s(1/l_0)l^{s-1}$$

for $l \geq l_0$. It is enough to choose $\eta_s(l_0) = -f_s(1/l_0)$. $\square$

Proof of Theorem 6.1. Choose $\gamma_l = l^s$ in Proposition 6.6, where $s \in \mathbb{R}$. We note that the numbers α_l and β_l in Lemma 4.3 satisfy

$$\alpha_l = 2(l+1) + (d-2)(1 - \frac{1}{l+d-2})$$
$$= 2(l+1) + (d-2)(1 - \frac{1}{l}) + \frac{(d-2)^2}{l(l+d-2)}$$

and

$$\beta_l = 2(l-1) + (d-2)(1 - \frac{1}{l}), \qquad l \geq 2.$$

It follows that for $l \geq 2$ and for any $s \in \mathbb{R}$,

$$\alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2 = 2[(l+1)^{2s+1} - (l-1)^{2s+1}] + (d-2)(1 - \frac{1}{l})[(l+1)^{2s} - (l-1)^{2s}]$$

$$(6.8) \qquad\qquad + \frac{(d-2)^2}{l(l+d-2)}(l+1)^{2s}.$$

We now make the assumption $s > -1/2$, ensuring that $(l+1)^{2s+1} - (l-1)^{2s+1} > 0$ for $l \geq 1$. If $d = 2$, then the last two terms of (6.8) are $= 0$ and Lemma 6.8 yields that

$$(6.9) \qquad\qquad \alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2 \geq 2(2s+1)l^{2s}.$$

If $d \geq 3$ and $s \geq 0$, the last two terms of (6.8) are ≥ 0 and (6.9) also holds. The remaining case where $d \geq 3$ and $-1/2 < s < 0$ will be considered separately below. In particular, assuming that $d = 2$ or $s \geq 0$, our choice of γ_l satisfies (6.1). The bound (6.9) and the formulas for α_l and β_l imply that for $l \geq 2$

$$\frac{\alpha_l \gamma_{l+1}^2 \beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2} \leq \frac{\alpha_l \beta_l (l^2 - 1)^{2s}}{2(2s+1)l^{2s}}$$

$$\leq \max\{1, (\frac{3}{4})^{2s}\} \frac{(2l+d)(2l+d-4)l^{2s}}{2(2s+1)}$$

$$(6.10) \qquad\qquad \leq \frac{(2+\frac{d}{l})^2}{2s+1}l^{2s+2} \leq \frac{(2+\frac{d}{2})^2}{2s+1}l^{2s+2}.$$

We additionally choose

$$\delta_l \equiv \frac{1}{2}.$$

Proposition 6.6 together with the estimates (6.9), (6.10), $2l \leq \alpha_{l-1}$, $l^2 \leq \lambda_l$, and

$$\frac{\beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2} \leq \frac{(2 + (d-2)\frac{1}{l})(l-1)^{2s+1}}{(2 + (d-2)\frac{l-1}{l(l+1)})(l+1)^{2s+1}} \leq 3\left(\frac{l-1}{l+1}\right)^{2s+1} \leq 3, \quad l \geq 2,$$

imply that when $d = 2$ or $s \geq 0$, when $m \geq 1$, and for $u \in C_F^\infty(SM, \mathbb{C}^n)$, one has

$$2\sum_{l=m}^{m+1} l^{2s+1} \|X_- u_l\|^2 + (2s+1)\sum_{l=m+2}^\infty (l-1)^{2s}\|X_- u_l\|^2 + \kappa \sum_{l=m}^\infty l^{2s+2}\|u_l\|^2$$

$$+ \sum_{l=m}^\infty l^{2s}\|Z(u_l)\|^2 \leq \frac{(d+4)^2}{2s+1}\sum_{l=m+1}^\infty l^{2s+2}\|(Xu)_l\|^2.$$

Since the weights grow polynomially, this still holds true for $u \in C^\infty(SM, \mathbb{C}^n)$.

Finally, assume that $d \geq 3$ and $-1/2 < s < 0$. In this case we define

$$\gamma_l = l^s, \qquad l \geq l_0$$

where $l_0 \geq 2$ is to be determined. Then for $l \geq l_0 + 1$ the formula (6.8) together with Lemma 6.8 gives that

$$\alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2 \geq 2[(l+1)^{2s+1} - (l-1)^{2s+1}] + (d-2)(1 - \frac{1}{l})[(l+1)^{2s} - (l-1)^{2s}]$$

$$\geq \left[2(2s+1) - (d-2)\eta_{2s}(l_0)\frac{1}{l_0}\right]l^{2s}.$$

We now choose $l_0 = l_0(d, s)$ so large that $(d-2)\eta_{2s}(l_0)\frac{1}{l_0} \leq 2s+1$. Then it follows that

$$\alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2 \geq (2s+1)l^{2s}, \qquad l \geq l_0 + 1,$$

and consequently by the same argument as above

$$\frac{\alpha_l \gamma_{l+1}^2 \beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2} \leq \frac{1}{2}\frac{(2 + \frac{d}{2})^2}{2s+1}l^{2s+2}, \qquad l \geq l_0 + 1.$$

We still need to define γ_l for $l \leq l_0 - 1$ so that one at least should have

$$0 < \gamma_{l-1}^2 < \frac{\alpha_l}{\beta_l}\gamma_{l+1}^2, \qquad 1 \leq l \leq l_0.$$

The second inequality ensures that (6.1) holds. We make the rather arbitrary choice

$$\gamma_{l-1}^2 = \frac{1}{2}\frac{\alpha_l}{\beta_l}\gamma_{l+1}^2, \qquad 1 \leq l \leq l_0.$$

This determines γ_l for $l \leq l_0 - 1$ recursively from the knowledge of γ_{l_0} and γ_{l_0+1}, and γ_l for $l \leq l_0 - 1$ will then depend on d, s, and $l_0 = l_0(d, s)$.

To relate the weights in Proposition 6.6 to the weights appearing in the statement of the theorem, we next choose $\delta_l \equiv 1/2$ and define

$$c = \min_{1 \le l < l_0} \min \left\{ \frac{\alpha_{l-1}\gamma_l^2}{2l^{2s+1}}, \frac{\frac{1}{2}(\alpha_{l-1}\gamma_l^2 - \beta_{l-1}\gamma_{l-2}^2)}{(2s+1)(l-1)^{2s}}, \frac{\lambda_l\gamma_l^2}{l^{2s+2}}, \frac{\gamma_l^2}{l^{2s}} \right\},$$

$$C = \max_{1 \le l \le l_0} \left[1 + \frac{\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2} \right] \frac{\alpha_l\gamma_{l+1}^2\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2 - \beta_l\gamma_{l-1}^2} \frac{1}{l^{2s+2}}.$$

Then c and C are positive constants depending on d and s. It follows by combining the bounds above that for any $l \ge 2$, one has

$$\alpha_{l-1}\gamma_l^2 \ge \min\{c, 1\}2l^{2s+1},$$

$$\frac{1}{2}(\alpha_{l-1}\gamma_l^2 - \beta_{l-1}\gamma_{l-2}^2) \ge \min\{c, 1\}(2s+1)(l-1)^{2s},$$

$$\lambda_l\gamma_l^2 \ge \min\{c, 1\}l^{2s+2},$$

$$\gamma_l^2 \ge \min\{c, 1\}l^{2s},$$

$$\left[1 + \frac{\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2} \right] \frac{\alpha_l\gamma_{l+1}^2\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2 - \beta_l\gamma_{l-1}^2} \le \max\left\{ C, \frac{1}{2}\frac{(d+4)^2}{2s+1} \right\}l^{2s+2}.$$

Now Proposition 6.6 implies that

$$\min\{c, 1\} \left[2\sum_{l=m}^{m+1} l^{2s+1}\|X_-u_l\|^2 + (2s+1)\sum_{l=m+2}^{\infty} (l-1)^{2s}\|X_-u_l\|^2 + \kappa\sum_{l=m}^{\infty} l^{2s+2}\|u_l\|^2 \right]$$

$$+ \min\{c, 1\}\sum_{l=m}^{\infty} l^{2s}\|Z(u_l)\|^2 \le \max\left\{ C, \frac{1}{2}\frac{(d+4)^2}{2s+1} \right\} \sum_{l=m+1}^{\infty} l^{2s+2}\|(Xu)_l\|^2$$

where c and C depend only on d and s. This is true for all $m \ge 1$ and for all $u \in C^\infty(SM, \mathbb{C}^n)$ (with $u|_{\partial(SM)} = 0$ in the boundary case). The theorem follows. $\square$

Remark 6.9. For any $s \ge 0$ and $m \ge 1$, the choice $\gamma_l = l^s$ satisfies (6.1). Thus Proposition 6.6 with the choice $\delta_l \equiv 1$ implies in particular that

$$\kappa\sum_{l=m}^{\infty} l^{2s+2}\|u_l\|^2 \le C_{d,s,m}\sum_{l=m+1}^{\infty} l^{2s+2}\|(Xu)_l\|^2, \qquad u \in C^\infty(SM, \mathbb{C}^n),$$

where $C_{d,s,m}$ is given by

$$C_{d,s,m} = \sup_{l \ge m+1} \frac{\alpha_l\beta_l(l^2-1)^{2s}}{l^{2s+2}(\alpha_l(l+1)^{2s} - \beta_l(l-1)^{2s})}.$$

The proof of Theorem 6.1 shows that $C_{d,s,m}$ is indeed finite and that

$$C_{d,s,m} \le \frac{(d+4)^2}{2s+1}.$$

On the other hand, since $\alpha_l = 2(l + 1) + O(1)$ and $\beta_l = 2(l - 1) + O(1)$ as $l \to \infty$,

$$
\begin{aligned}
C_{d,s,m} &\geq \lim_{l \to \infty} \frac{\alpha_l \beta_l (l^2 - 1)^{2s}}{l^{2s+2}(\alpha_l(l+1)^{2s} - \beta_l(l-1)^{2s})} \\
&= \lim_{l \to \infty} \frac{4(l^2 - 1)^{2s+1}}{2l^{2s+2}[(l+1)^{2s+1} - (l-1)^{2s+1}]} \\
&= \lim_{l \to \infty} \frac{2l^{2s}}{(l+1)^{2s+1} - (l-1)^{2s+1}} \\
&= \lim_{l \to \infty} \frac{2l^{2s}}{2(2s+1)l^{2s}} = \frac{1}{2s+1}.
\end{aligned}
$$

This shows that while the constant in Theorem 6.1 is certainly not optimal, the optimal constant is $\sim (2s + 1)^{-1}$ as $s \to \infty$.

7. CARLEMAN ESTIMATES IN NONPOSITIVE CURVATURE

In this section we discuss Carleman estimates in the case of nonpositive sectional curvature instead of strictly negative sectional curvature. The extra positive term in Lemma 6.4 becomes zero in the case, but one gains some positivity from the fact that there are no conjugate points (in a suitable strong sense). Thus, in this section (M, g) will be a compact Riemannian manifold with or without boundary, and we will assume that (M, g) is simple/Anosov with nonpositive sectional curvature.

The main result is the following Carleman estimate with linear instead of logarithmic weights.

Theorem 7.1. *Let (M, g) be a simple/Anosov manifold having nonpositive sectional curvature. There exist $m_0, \tau_0, \kappa > 0$ such that for any $\tau \geq \tau_0$, for any $m \geq m_0$ and for any $u \in C_F^\infty(SM, \mathbb{C}^n)$ (with $u|_{\partial(SM)} = 0$ in the boundary case) one has*

$$
\sum_{l=m}^{\infty} e^{2\tau\phi_l} \|u_l\|^2 \leq \frac{24}{\kappa e^{2\tau}} \sum_{l=m+1}^{\infty} e^{2\tau\phi_l} \|(Xu)_l\|^2
$$

where $\phi_l = l$.

The constant $\kappa > 0$ in the above theorem comes from the fact that compact/simple manifolds with nonpositive curvature have no conjugate points. It is the same constant that appears in the following lemma:

Lemma 7.2. *If (M, g) is a compact simple/Anosov manifold, there exist $\alpha \in (0, 1]$ and $\kappa > 0$ so that*

$$
\|(X + A)Z\|^2 - (RZ, Z) \geq \alpha\|(X + A)Z\|^2 + \kappa\|Z\|^2
$$

for any unitary connection A and for any $Z \in \mathcal{Z}^n$ (with $Z|_{\partial(SM)} = 0$ in the boundary case).

Proof. If $A = 0$, this result is contained in [PSU15, Theorem 7.2 and Lemma 11.2] and this is the only case that we will use. The result for unitary A, and the fact

that the constants α and κ are indeed independent of A, is recorded here for later purposes and will be proved in Appendix A. $\qquad\square$

The Carleman estimate in Theorem 7.1 could be used to absorb a general Higgs field, but not a general connection. For simple two-dimensional manifolds there is an improved estimate where one can also include a general connection by using holomorphic integrating factors as in [PSU12], see Appendix B. However, the estimate does not immediately yield a uniqueness result for the X-ray transform for the following reason: if $u \in C^\infty(SM, \mathbb{C}^n)$ solves $(X + \Phi)u = f$ in SM where Φ is a general Higgs field and f has finite degree, it is not clear if the sums in the Carleman estimate is finite (the weights are exponential, whereas the assumption $u \in C^\infty(SM, \mathbb{C}^n)$ only ensures that $\sum l^{2N}\|u_l\|^2$ is finite for any $N \geq 0$).

We begin the proof of Theorem 7.1 with the following result, which is a counterpart of Lemma 6.4.

Lemma 7.3. *Let (M, g) be a simple/Anosov manifold having nonpositive sectional curvature. There is $\kappa > 0$ such that for any $l \geq 1$ and for any σ_l with $0 \leq \sigma_l \leq 1$ one has*

$$(\alpha_{l-1} - \sigma_l \lambda_{l-1}(1 + \frac{1}{l+d-3})^2)\|X_- u\|^2 + \sigma_l \kappa \lambda_l \|u\|^2$$

$$\leq (\beta_{l+1} + \sigma_l \lambda_{l+1}(1 - \frac{1}{l+1})^2)\|X_+ u\|^2, \qquad u \in \Omega_l,$$

if additionally $u|_{\partial(SM)} = 0$ in the boundary case.

Proof. We use the standard Pestov identity (Lemma 4.2) with $A = 0$:

$$\|X\overset{\text{v}}{\nabla} u\|^2 - (R\overset{\text{v}}{\nabla} u, \overset{\text{v}}{\nabla} u) + (d-1)\|Xu\|^2 = \|\overset{\text{v}}{\nabla} Xu\|^2.$$

The nonpositive curvature assumption implies that

$$\|X\overset{\text{v}}{\nabla} u\|^2 - (R\overset{\text{v}}{\nabla} u, \overset{\text{v}}{\nabla} u) \geq \|X\overset{\text{v}}{\nabla} u\|^2.$$

On the other hand, the simple/Anosov assumption together with Lemma 7.2 implies that there is $\kappa > 0$ (we omit the $\alpha\|X\overset{\text{v}}{\nabla} u\|^2$ term) such that

$$\|X\overset{\text{v}}{\nabla} u\|^2 - (R\overset{\text{v}}{\nabla} u, \overset{\text{v}}{\nabla} u) \geq \kappa\|\overset{\text{v}}{\nabla} u\|^2.$$

Interpolating the last two inequalities yields that for any σ_l with $0 \leq \sigma_l \leq 1$,

$$\|X\overset{\text{v}}{\nabla} u\|^2 - (R\overset{\text{v}}{\nabla} u, \overset{\text{v}}{\nabla} u) \geq (1 - \sigma_l)\|X\overset{\text{v}}{\nabla} u\|^2 + \sigma_l \kappa\|\overset{\text{v}}{\nabla} u\|^2.$$

Inserting this estimate in the Pestov identity implies that

$$(7.1) \qquad (1 - \sigma_l)\|X\overset{\text{v}}{\nabla} u\|^2 + \sigma_l \kappa\|\overset{\text{v}}{\nabla} u\|^2 \leq \|\overset{\text{v}}{\nabla} Xu\|^2 - (d-1)\|Xu\|^2.$$

We now assume that $u = u_l \in \Omega_l$ and simplify (7.1) as in the proof of Lemma 4.3. This gives

$$\alpha_{l-1}\|X_- u_l\|^2 + \|Z(u_l)\|^2 + \sigma_l \kappa\|\overset{\text{v}}{\nabla} u\|^2 \leq \beta_{l+1}\|X_+ u_l\|^2 + \sigma_l\|X\overset{\text{v}}{\nabla} u\|^2$$

Again as in the proof of Lemma 4.3, one has

$$\|X\overset{\vee}{\nabla} u_l\|^2 = \|\overset{\vee}{\nabla}[(1 - \frac{1}{l+1})X_+ u_l + (1 + \frac{1}{l+d-3})X_- u_l) - Z(u_l)\|^2$$

$$= \lambda_{l+1}(1 - \frac{1}{l+1})^2\|X_+ u_l\|^2 + \lambda_{l-1}(1 + \frac{1}{l+d-3})^2\|X_- u_l\|^2 + \|Z(u_l)\|^2.$$

The result follows by combining the last two formulas and using the trivial estimate $(1 - \sigma_l)\|Z(u_l)\|^2 \geq 0$. $\qquad\square$

The estimate in Lemma 7.3 is of course only useful when the coefficient in front of $\|X_- u\|^2$ is nonnegative. For large l one has

$$\alpha_{l-1} - \sigma_l \lambda_{l-1}(1 + \frac{1}{l+d-3})^2 \sim 2l - \sigma_l l^2,$$

so one needs to have $\sigma_l \leq \frac{2}{l}$ for l large. (If one keeps the $\alpha\|X\overset{\vee}{\nabla} u\|^2$ that was omitted in the proof of Lemma 7.3, the previous condition becomes $\sigma_l \leq \frac{2}{(1-\alpha)l}$ so one still needs $\sigma_l \lesssim l^{-1}$ for l large.)

We next give the counterpart of Proposition 6.6 (for $\delta_l \equiv 1$), which corresponds to a Carleman estimate with general weights.

Proposition 7.4. *Assume that (M, g) is a simple/Anosov manifold of nonpositive sectional curvature. If (γ_l) and (σ_l) are sequences satisfying $\gamma_l > 0$, $0 < \sigma_l < 1$, and*

$$(7.2) \qquad (\alpha_l - \sigma_{l+1}\lambda_l(1 + \frac{1}{l+d-2})^2)\gamma_{l+1}^2 > (\beta_l + \sigma_{l-1}\lambda_l(1 - \frac{1}{l})^2)\gamma_{l-1}^2$$

then one has

$$\kappa \sum_{l=m}^{\infty} \sigma_l \lambda_l \gamma_l^2 \|u_l\|^2$$

$$\leq \sum_{l=m+1}^{\infty} \frac{(\alpha_l - \sigma_{l+1}\lambda_l(1 + \frac{1}{l+d-2})^2)\gamma_{l+1}^2(\beta_l + \sigma_{l-1}\lambda_l(1 - \frac{1}{l})^2)\gamma_{l-1}^2}{(\alpha_l - \sigma_{l+1}\lambda_l(1 + \frac{1}{l+d-2})^2)\gamma_{l+1}^2 - (\beta_l + \sigma_{l-1}\lambda_l(1 - \frac{1}{l})^2)\gamma_{l-1}^2}\|(Xu)_l\|^2$$

whenever $m \geq 1$ and $u \in C_F^{\infty}(SM, \mathbb{C}^n)$ (with $u|_{\partial(SM)} = 0$ in the boundary case).

Proof. Multiplying the estimate in Lemma 7.3 by γ_l^2 and using (6.2), we obtain

$$(\alpha_{l-1} - \sigma_l \lambda_{l-1}(1 + \frac{1}{l+d-3})^2)\gamma_l^2\|X_- u_l\|^2 + \sigma_l \kappa \lambda_l \gamma_l^2 \|u_l\|^2$$

$$\leq (\beta_{l+1} + \sigma_l \lambda_{l+1}(1 - \frac{1}{l+1})^2)\gamma_l^2 \left[(1 + \frac{1}{\varepsilon_l})\|(Xu)_{l+1}\|^2 + (1 + \varepsilon_l)\|X_- u_{l+2}\|^2\right].$$

Summing up these estimates for $l \geq m$, where $m \geq 1$, yields

$$\sum_{l=m}^{\infty} (\alpha_{l-1} - \sigma_l \lambda_{l-1}(1 + \frac{1}{l+d-3})^2) \gamma_l^2 \|X_- u_l\|^2 + \sum_{l=m}^{\infty} \sigma_l \kappa \lambda_l \gamma_l^2 \|u_l\|^2$$

$$\leq \sum_{l=m+1}^{\infty} (\beta_l + \sigma_{l-1} \lambda_l (1 - \frac{1}{l})^2) \gamma_{l-1}^2 (1 + \frac{1}{\varepsilon_{l-1}}) \|(Xu)_l\|^2$$

$$+ \sum_{l=m+2}^{\infty} (\beta_{l-1} + \sigma_{l-2} \lambda_{l-1}(1 - \frac{1}{l-1})^2) \gamma_{l-2}^2 (1 + \varepsilon_{l-2}) \|X_- u_l\|^2.$$

We now choose ε_l so that for $l \geq m+2$,

$$1 + \varepsilon_{l-2} = \frac{(\alpha_{l-1} - \sigma_l \lambda_{l-1}(1 + \frac{1}{l+d-3})^2) \gamma_l^2}{(\beta_{l-1} + \sigma_{l-2} \lambda_{l-1}(1 - \frac{1}{l-1})^2) \gamma_{l-2}^2}.$$

It follows that

$$\frac{1}{\varepsilon_{l-1}} = \frac{(\beta_l + \sigma_{l-1} \lambda_l (1 - \frac{1}{l})^2) \gamma_{l-1}^2}{(\alpha_l - \sigma_{l+1} \lambda_l (1 + \frac{1}{l+d-2})^2) \gamma_{l+1}^2 - (\beta_l + \sigma_{l-1} \lambda_l (1 - \frac{1}{l})^2) \gamma_{l-1}^2}.$$

The result follows. $\square$

We now discuss some possible choices for σ_l and γ_l, following the argument before the proof of Theorem 6.1. For simplicity we assume $d = 2$ and write $r_l = l\gamma_l^2$ and $\delta_l = \frac{1}{2} l \sigma_l$. Then the estimate in Proposition 7.4 takes the form

$$(7.3) \qquad \kappa \sum_{l=m}^{\infty} \delta_l r_l \|u_l\|^2 \leq \sum_{l=m+1}^{\infty} \frac{(1 - \delta_{l+1})(1 + \delta_{l-1}) r_{l+1} r_{l-1}}{(1 - \delta_{l+1}) r_{l+1} - (1 + \delta_{l-1}) r_{l-1}} \|(Xu)_l\|^2.$$

Here of course $0 < \delta_l < 1$ for all l. We formally assume that u solves $(X + \Phi)u = f$ where Φ is a general Higgs field and f has degree $\leq m$. Then, for some (large) constant $R > 0$,

$$\|(Xu)_l\|^2 \leq R\|u_l\|^2, \qquad l \geq m+1.$$

In order to absorb the resulting error in the estimate (7.3), we need that

$$(7.4) \qquad R \frac{(1 + \delta_{l-1}) r_{l+1} r_{l-1}}{r_{l+1} - \frac{1 + \delta_{l-1}}{1 - \delta_{l+1}} r_{l-1}} \leq \kappa \delta_l r_l, \qquad l \geq m+1.$$

For this estimate to make sense, we need $r_{l+1} > \frac{1 + \delta_{l-1}}{1 - \delta_{l+1}} r_{l-1}$. Let $a_l > 1$ be such that

$$r_{l+1} = a_l \frac{1 + \delta_{l-1}}{1 - \delta_{l+1}} r_{l-1}.$$

The estimate (7.4) becomes

$$\frac{R r_{l-1}}{1 - \frac{1}{a_l}} \leq \kappa \frac{\delta_l}{1 + \delta_{l-1}} r_l,$$

which can be rewritten as

$$\frac{r_l}{r_{l-1}} \geq \frac{R}{\kappa} \frac{1}{1 - \frac{1}{a_l}} \frac{1 + \delta_{l-1}}{\delta_l}.$$

Here $a_l > 1$ are free parameters at this point, so $\frac{1}{1 - \frac{1}{a_l}} \in (1, \infty)$, and also $\frac{1 + \delta_{l-1}}{\delta_l} > 1$ for all l since $0 < \delta_j < 1$. Thus, in order to absorb a large Higgs field, we at the least need that

$$r_l \geq \frac{R}{\kappa} r_{l-1}, \qquad l \geq m + 1.$$

This implies that $r_{m+k} \geq \mu^k r_m$ where $\mu = \frac{R}{\kappa}$, which implies that the weights r_l need to grow exponentially (at least if the Higgs field is large).

Possible choices that would make the above argument work would be $\delta_l = \frac{1}{2}$ and $a_l = \mu$ for all l, where $\mu = \max\{2R/\kappa, 2\}$. Then we get $r_{l+1} = 3\mu r_{l-1}$, and choosing $r_0 = 1$ and $r_1 = 3\mu$ yields $r_l = (3\mu)^l$. Then

$$\frac{R}{\kappa} \frac{1}{1 - \frac{1}{a_l}} \frac{1 + \delta_{l-1}}{\delta_l} \leq \frac{\mu}{2} \frac{1}{1 - \frac{1}{2}} \frac{1 + \frac{1}{2}}{\frac{1}{2}} = 3\mu = \frac{r_l}{r_{l-1}}.$$

These choices translate to $\sigma_l = \frac{1}{l}$ and $\gamma_l = l^{-1/2}(\sqrt{3\mu})^l$.

We will now follow the above argument to prove the Carleman estimate in any dimension.

Proof of Theorem 7.1. Fix a constant $\mu > 1$, choose γ_l so that $l\gamma_l^2 = \mu^l$, and choose $\sigma_l = \frac{2\delta}{l}$ where $0 < \delta < 1$ is fixed. The estimate in Proposition 7.4 implies that

$$2\kappa\delta \sum_{l=m}^{\infty} \mu^l \|u_l\|^2 = \kappa \sum_{l=m}^{\infty} \sigma_l l \mu^l \|u_l\|^2 \leq \kappa \sum_{l=m}^{\infty} \sigma_l \frac{\lambda_l}{l} \mu^l \|u_l\|^2 = \kappa \sum_{l=m}^{\infty} \sigma_l \lambda_l \gamma_l^2 \|u_l\|^2$$

$$\leq \sum_{l=m+1}^{\infty} \frac{(\alpha_l - \sigma_{l+1}\lambda_l(1 + \frac{1}{l+d-2})^2)\gamma_{l+1}^2(\beta_l + \sigma_{l-1}\lambda_l(1 - \frac{1}{l})^2)\gamma_{l-1}^2}{(\alpha_l - \sigma_{l+1}\lambda_l(1 + \frac{1}{l+d-2})^2)\gamma_{l+1}^2 - (\beta_l + \sigma_{l-1}\lambda_l(1 - \frac{1}{l})^2)\gamma_{l-1}^2} \|(Xu)_l\|^2.$$

Using the formulas in Lemma 4.3 and the fact that $\sigma_l = \frac{2\delta}{l}$, we have

$$\alpha_l - \sigma_{l+1}\lambda_l(1 + \frac{1}{l+d-2})^2$$

$$= (2l + d - 2)\left(1 + \frac{1}{l+d-2}\right) - \frac{2\delta}{l+1}\lambda_l(1 + \frac{1}{l+d-2})^2$$

$$= 2(1 - \delta)l + O(1)$$

and

$$\beta_l + \sigma_{l-1}\lambda_l(1 - \frac{1}{l})^2$$

$$= (2l + d - 2)\left(1 - \frac{1}{l}\right) + \frac{2\delta}{l-1}\lambda_l(1 - \frac{1}{l})^2$$

$$= 2(1 + \delta)l + O(1)$$

when l is large. These estimates also imply that

$$(\alpha_l - \sigma_{l+1}\lambda_l(1 + \frac{1}{l+d-2})^2)\gamma_{l+1}^2 = (2(1-\delta) + O(l^{-1}))\mu^{l+1},$$

$$(\beta_l + \sigma_{l-1}\lambda_l(1 - \frac{1}{l})^2)\gamma_{l-1}^2 = (2(1+\delta) + O(l^{-1}))\mu^{l-1}$$

for l large.

Using the above bounds and choosing m large enough, we obtain the estimate

$$2\kappa\delta \sum_{l=m}^{\infty} \mu^l \|u_l\|^2$$

$$\leq \sum_{l=m+1}^{\infty} \frac{(2(1-\delta) + O(l^{-1}))(2(1+\delta) + O(l^{-1}))}{(2(1-\delta) + O(l^{-1})) - (2(1+\delta) + O(l^{-1}))\mu^{-2}}\mu^{l-1}\|(Xu)_l\|^2$$

$$\leq \frac{8(1-\delta)(1+\delta)}{(1-\delta) - 4(1+\delta)\mu^{-2}} \sum_{l=m+1}^{\infty} \mu^{l-1}\|(Xu)_l\|^2.$$

This estimate makes sense if $1 - \delta > 4(1+\delta)\mu^{-2}$, i.e. $\mu^2 > \frac{4(1+\delta)}{1-\delta}$.

Writing $\mu^2 = \frac{4(1+\delta)}{1-\delta}a$ where $a > 1$, the above estimate becomes

$$2\kappa\delta \sum_{l=m}^{\infty} \mu^l \|u_l\|^2 \leq \frac{8(1+\delta)}{1 - 1/a} \sum_{l=m+1}^{\infty} \mu^{l-1}\|(Xu)_l\|^2.$$

We will now make the choices

$$\delta = \frac{1}{2}, \qquad a = 2, \qquad \mu = e^{2\tau}$$

where $\tau \geq \tau_0$ with τ_0 chosen so that $e^{4\tau_0} > 12$. Then one has $\mu^2 > \frac{4(1+\delta)}{1-\delta}$, and the estimate takes the required form

$$\kappa \sum_{l=m}^{\infty} e^{2\tau l}\|u_l\|^2 \leq \frac{24}{e^{2\tau}} \sum_{l=m+1}^{\infty} e^{2\tau l}\|(Xu)_l\|^2. \qquad \square$$

Remark 7.5. So far, we have considered the case where (M, g) has nonpositive curvature. One could ask if it is possible to obtain weighted estimates whenever (M, g) is a general compact simple/Anosov manifold. In this case the sectional curvatures may be positive, and the most efficient way seems to be to write the Pestov identity as

$$\|X\overset{v}{\nabla}u\|^2 - (R\overset{v}{\nabla}u, \overset{v}{\nabla}u) + (d-1)\|Xu\|^2 = \|\overset{v}{\nabla}Xu\|^2$$

and use the fact that, by Lemma 7.2,

$$\|X\overset{v}{\nabla}u\|^2 - (R\overset{v}{\nabla}u, \overset{v}{\nabla}u) \geq \alpha\|X\overset{v}{\nabla}u\|^2 + \kappa\|\overset{v}{\nabla}u\|^2$$

for some $\alpha, \kappa > 0$. If one does this and runs the argument as in the negative curvature case, one arrives at the following estimate for any $u \in C^\infty(SM)$ (with $u|_{\partial(SM)} = 0$ in

the boundary case):

$$\alpha \sum_{l=0}^{\infty} \alpha_l \|X_- u_{l+1}\|^2 + \kappa \sum_{l=1}^{\infty} \lambda_l \|u_l\|^2$$

$$\leq (1 - \alpha) \sum_{l=2}^{\infty} (\lambda_l - (d - 1)) \|(Xu)_l\|^2 + \alpha \sum_{l=2}^{\infty} \beta_l \|X_+ u_{l-1}\|^2.$$

The X_+ terms on the right can be absorbed in the X_- terms on the left, even with weights, just as in the negative curvature case. However, the new $(Xu)_l$ terms on the right present problems. If one considers u with $u_l = 0$ for $l < m$, which is the case relevant for the X-ray transform on m-tensors, then the $(Xu)_{m-1} = X_- u_m$ and $(Xu)_m = X_- u_{m+1}$ terms on the right cannot be absorbed to the left if $m \geq 2$ is large. Moreover, adding weights (i.e. replacing u_l by $\gamma_l u_l$) does not help, since the $X_- u_m$ term on the right is multiplied by the same γ_m^2 as the $X_- u_m$ and u_m terms on the left. Thus this method does not seem to shed new light to the tensor tomography problem of simple/Anosov manifolds (which is still open when $\dim(M) \geq 3$ and $m \geq 2$).

One could still ask if one gets new weighted estimates in the cases $m = 0, 1$. In these cases the $(Xu)_0$ and $(Xu)_1$ terms pose no problem (they are not present on the right). However, replacing u by its weighted version $\sum \gamma_l u_l$ in the $(Xu)_l$ terms on the right will generate new X_- and X_+ terms, and it seems that absorbing the new terms from right to left requires weights that grow at most mildly. This is not sufficient for dealing with large general connections or Higgs fields, but could lead to a shifted version of the Pestov identity as in Theorem 6.3 but where s has to be in the range $-1/2 < s \leq 1/2$. We omit the details.

8. A REGULARITY RESULT FOR THE TRANSPORT EQUATION

Let (M, g) be a compact Riemannian manifold with boundary whose geodesic flow is *nontrapping*, and let $\mathcal{A} : SM \to \mathbb{C}^{n \times n}$ be an arbitrary smooth attenuation. If the boundary of M is strictly convex, it was proved in [PSU12, Proposition 5.2] that any solution to $Xu + \mathcal{A}u = -f$ in SM with $u|_{\partial(SM)} = 0$ is smooth in SM whenever f is smooth. We would like to prove an analogue of this statement, but without assuming that the boundary of M is strictly convex. We will do so at the cost of assuming that the function f is supported in the interior of SM.

Given a smooth $f : SM \to \mathbb{C}^n$, denote by u^f the unique solution to

$$\begin{cases} Xu + \mathcal{A}u = -f, \\ u|_{\partial_-(SM)} = 0. \end{cases}$$

The function u^f may fail to be differentiable due to the non-smoothness of τ. In fact, τ might not even be continuous.

Proposition 8.1. *Let $f : SM \to \mathbb{C}^n$ be smooth, supported in the interior of SM with $I_{\mathcal{A}}(f) = u^f|_{\partial_+(SM)} = 0$. Then $u^f : SM \to \mathbb{C}^n$ is smooth.*

Proof. We consider (M, g) isometrically embedded in a closed manifold (N, g) and we extend f by zero. We also extend $\mathcal{A}$ smoothly to N. The idea is to use the argument

in [Da06, Lemma 2.3] and replace τ locally by other suitable smooth functions. Let $\ell(x, v)$ be any time such that $\gamma_{x,v}(\ell(x, v)) \notin M$. Consider the function $w(t, x, v)$ defined by the ordinary differential equation

$$\begin{cases} \frac{d}{dt}w(t, x, v) + \mathcal{A}(\varphi_t(x, v))w(t, x, v) = -f(\varphi_t(x, v)), \\ w(\ell(x, v), x, v) = 0. \end{cases}$$

We claim that for $(x, v) \in SM$

(8.1) $$u^f(x, v) = w(0, x, v).$$

To prove (8.1) observe the function $u(t, x, v) := u^f(\varphi_t(x, v))$ solves the differential equation

$$\begin{cases} \frac{d}{dt}u(t, x, v) + \mathcal{A}(\varphi_t(x, v))u(t, x, v) = -f(\varphi_t(x, v)), \\ u(\tau(x, v), x, v) = 0. \end{cases}$$

Next note that the interval $[\tau, \ell]$ can be decomposed into two types of intervals: the first type consists of intervals where the geodesic $\gamma_{x,v}$ is a maximal geodesic segment in M with boundary points on ∂M, and the second type consists of intervals where $\gamma_{x,v}$ runs outside M. Recall that outside M, $f = 0$ and while inside M, $I_{\mathcal{A}}(f) = 0$. Thus $w(\tau(x, v), x, v) = 0$. Hence (8.1) follows from uniqueness of solutions of ordinary differential equations.

Since $\gamma_{x,v}(\ell(x, v)) \notin M$, we can take a small hypersurface Σ in N transversally intersecting $\gamma_{x,v}$ at the point $\gamma_{x,v}(\ell(x, v))$ and disjoint from M. Then there is a neighbourhood of (x, v) in SN such that for every θ in this neighbourhood the geodesic γ_θ will hit Σ at the time $\ell(\theta)$ smoothly depending on θ. If we now use this local function $\ell(\theta)$ to define w, using (8.1) we see that u^f is smooth around (x, v) since $w(0, \theta)$ is.   □

Remark 8.2. All that is needed is that f has vanishing jet at the boundary. It might be possible to prove that if $f = f(x)$ and $I_{\mathcal{A}}(f) = 0$, then f has vanishing jet at the boundary following the ideas in [GMT17] or [SU09], but we do not pursue this matter here.

9. Applications

Finally, we will employ the Carleman estimates proved in this article in various applications as described in the introduction.

Uniqueness results for the transport equation and X-ray transforms. The next result is a restatement of Theorem 1.4.

Theorem 9.1. *Let (M, g) be compact with sectional curvature $\leq -\kappa$ where $\kappa > 0$. Let $l_0 \geq 2$, and suppose that $\mathcal{A} : C^\infty(SM, \mathbb{C}^n) \to C^\infty(SM, \mathbb{C}^n)$ is a map such that*

$$\|(\mathcal{A}(u))_l\| \leq R(\|u_{l-1}\| + \|u_l\| + \|u_{l+1}\|), \qquad l \geq l_0.$$

If $f \in C^\infty(SM, \mathbb{C}^n)$ has finite degree and if $u \in C^\infty(SM, \mathbb{C}^n)$ satisfies

$$Xu + \mathcal{A}(u) = -f \text{ in } SM, \qquad u|_{\partial(SM)} = 0,$$

then u has finite degree.

Proof. Suppose that f has degree $m_0 \geq 0$, and let $m \geq \max\{l_0 - 1, m_0\}$. Then $(Xu)_l = -(\mathcal{A}(u))_l$ for $l \geq m + 1$ and

$$\|(Xu)_l\| \leq R(\|u_{l-1}\| + \|u_l\| + \|u_{l+1}\|), \qquad l \geq m + 1.$$

Inserting this estimate in Theorem 1.1 yields that, for $\tau \geq 1$,

$$\sum_{l=m}^{\infty} l^{2\tau} \|u_l\|^2 \leq \frac{CR}{\tau} \sum_{l=m+1}^{\infty} l^{2\tau} (\|u_{l-1}\|^2 + \|u_l\|^2 + \|u_{l+1}\|^2)$$

$$\leq \frac{CR}{\tau} \sum_{l=m}^{\infty} (l+1)^{2\tau} \|u_l\|^2$$

where $C = C_{d,\kappa}$. If we additionally assume that $m \geq 2\tau$, then

$$(l+1)^{2\tau} = \left(1 + \frac{1}{l}\right)^{2\tau} l^{2\tau} \leq e l^{2\tau}, \qquad l \geq m.$$

Thus if $m \geq \max\{l_0 - 1, m_0, 2\tau\}$ we have

$$\sum_{l=m}^{\infty} l^{2\tau} \|u_l\|^2 \leq \frac{CR}{\tau} \sum_{l=m}^{\infty} l^{2\tau} \|u_l\|^2.$$

Now fix $\tau = \tau_0 = \max\{2CR, 1\}$ and let $m = \max\{l_0 - 1, m_0, 2\tau_0\}$. Then the right hand side of the last inequality can be absorbed to the left, and we obtain

$$\sum_{l=m}^{\infty} l^{2\tau} \|u_l\|^2 \leq 0.$$

Thus $u_l = 0$ for $l \geq m$, so u has degree $\leq m - 1$ where $m = \max\{l_0 - 1, m_0, 2CR\}$ where $C = C_{d,\kappa}$. $\qquad\square$

We next show how the vanishing of an attenuated X-ray transform leads to smooth solutions u that may be used in the previous theorem. The following result is a restatement of Theorem 1.5.

Theorem 9.2. *Let (M, g) be a compact nontrapping negatively curved manifold with C^∞ boundary, let A be a general connection, and let Φ be a general Higgs field in M. Let $f \in C^\infty(SM, \mathbb{C}^n)$, and assume either that M has strictly convex boundary, or that f is supported in the interior of SM. If f has degree $m \geq 0$ and if the attenuated X-ray transform of f vanishes (i.e. $I_{A+\Phi} f = 0$), then*

$$f = -(X + A + \Phi)u$$

for some $u \in C^\infty(SM, \mathbb{C}^n)$ with degree $m - 1$ such that $u|_{\partial(SM)} = 0$.

Proof. Let $u = u^f$ be the solution of

$$(X + A + \Phi)u = -f \text{ in } SM, \qquad u|_{\partial_-(SM)} = 0.$$

Since the attenuated X-ray transform of f vanishes, we have $u|_{\partial(SM)} = 0$. This implies that $u \in C^\infty(SM, \mathbb{C}^n)$; if M has strictly convex boundary this follows from

[PSU12, Proposition 5.2], and alternatively if f is supported in the interior of SM we may use Proposition 8.1. We are now exactly in the setting of Theorem 9.1 with $\mathcal{A}(u) = Au + \Phi u$ and $l_0 = \max\{m-1, 2\}$, and since f has finite degree it follows that also u has finite degree, i.e. $u_l = 0$ for $l \geq m_0 + 1$ for some m_0.

We still need to show that $m_0 \leq m$. If this is not the case, we use the transport equation $Xu + Au + \Phi u = -f$ again to derive $X_+^A u_{m_0} = 0$. Since $u_{m_0}|_{\partial(SM)} = 0$, [GPSU16, Theorem 5.2] implies that $u_{m_0} = 0$ and arguing in the same way with u_{m_0-1} and so on, we deduce that all $u_l = 0$ for $l \geq m$. This shows that $f = -(X + A + \Phi)u$ where u has degree $m-1$ and $u|_{\partial(SM)} = 0$. $\square$

Scattering rigidity for general connections in negative curvature. Let us move to our next application, concerning an inverse problem with scattering data. On a nontrapping compact manifold (M, g) with strictly convex boundary, the scattering relation $\alpha = \alpha_g : \partial_+(SM) \to \partial_-(SM)$ maps a starting point and direction of a geodesic to the end point and direction. If (M, g) is simple, then knowing α_g is equivalent to knowing the boundary distance function d_g which encodes the distances between any pair of boundary points [Mi81]. On two dimensional simple manifolds, the boundary distance function d_g determines the metric g up to an isometry which fixes the boundary [PU05].

Given a connection A and a potential Φ on the bundle $M \times \mathbb{C}^n$, there is an additional piece of scattering data. Consider the unique matrix solution $U : SM \to GL(n, \mathbb{C})$ to the transport equation

$$XU + AU + \Phi U = 0 \text{ in } SM, \quad U|_{\partial_+(SM)} = \text{Id}.$$

As discussed in the introduction, the scattering data (or relation) corresponding to a matrix attenuation pair (A, Φ) in (M, g) is the map

$$C_{A,\Phi} : \partial_-(SM) \to GL(n, \mathbb{C}), \quad C_{A,\Phi} := U|_{\partial_-(SM)}.$$

Theorem 1.6 in the introduction proves that knowledge of $C_{A,\Phi}$ uniquely determines the pair (A, Φ) up to gauge. The theorem can be proved by introducing a *pseudo-linearization* that reduces the nonlinear problem to the linear one in Theorem 9.2. The argument is identical to the one used in [PSU12] and [PSUZ16], and we do not need to repeat it here.

Transparent pairs. We next discuss the problem of when the parallel transport associated with a pair (A, Φ) determines the pair up to gauge equivalence in the case of closed manifolds. This problem is discussed in detail in [GPSU16, Pa09, Pa11, Pa12, Pa13], but the results are only for pairs taking values in the Lie algebra of the unitary group. The extension to the group $GL(n, \mathbb{R})$ is non-trivial on two counts. It requires a recent extension of the Livsic theorem to arbitrary matrix groups [Ka11] and our new Carleman estimate for the geodesic vector field in negative curvature.

Since there is no boundary, we need to consider the parallel transport of a pair along closed geodesics. We shall consider a simplified version of the problem, which is interesting in its own right. We will attempt to understand those pairs (A, Φ) with the property that the parallel transport along closed geodesics is the identity. These

pairs will be called *transparent* as they are invisible from the point of view of the closed geodesics of the Riemannian metric.

Let (M, g) be a closed Riemannian manifold, A a connection taking values in the set of real $n \times n$ matrices and Φ a potential also taking values in $\mathbb{R}^{n \times n}$. The pair (A, Φ) naturally induces a $GL(n, \mathbb{R})$-cocycle over the geodesic flow φ_t of the metric g acting on the unit sphere bundle SM with projection $\pi : SM \to M$. The cocycle is defined by

$$\frac{d}{dt}C(x, v, t) = -(A(\varphi_t(x, v)) + \Phi(\pi \circ \varphi_t(x, v)))C(x, v, t), \quad C(x, v, 0) = \mathrm{Id}.$$

The cocycle C is said to be *cohomologically trivial* if there exists a smooth function $u : SM \to GL(n, \mathbb{R})$ such that

$$C(x, v, t) = u(\varphi_t(x, v))u^{-1}(x, v)$$

for all $(x, v) \in SM$ and $t \in \mathbb{R}$. We call u a trivializing function and note that two trivializing functions u_1 and u_2 (for the same cocycle) are related by $u_2 w = u_1$ where $w : SM \to GL(n, \mathbb{R})$ is constant along the orbits of the geodesic flow. In particular, if φ_t is transitive (i.e. there is a dense orbit) there is a unique trivializing function up to right multiplication by a constant matrix in $GL(n, \mathbb{R})$.

Definition 9.3. We will say that a pair (A, Φ) is cohomologically trivial if C is cohomologically trivial. The pair (A, Φ) is said to be *transparent* if $C(x, v, T) = \mathrm{Id}$ every time that $\varphi_T(x, v) = (x, v)$.

Observe that the gauge group given by the set of smooth maps $r : M \to GL(n, \mathbb{R})$ acts on pairs as follows:

$$(A, \Phi) \mapsto (r^{-1}dr + r^{-1}Ar, r^{-1}\Phi r).$$

This action leaves invariant the set of cohomologically trivial pairs: indeed, if u trivializes the cocycle C of a pair (A, Φ), then it is easy to check that $r^{-1}u$ trivializes the cocycle of the pair $(r^{-1}dr + r^{-1}Ar, r^{-1}\Phi r)$.

Obviously a cohomologically trivial pair is transparent. There is one important situation in which both notions agree. If φ_t is Anosov, then the Livsic theorem for $GL(n, \mathbb{R})$ cocycles due to Kalinin [Ka11] (extending the work of Livsic for the case of a cocycle taking values in a compact Lie group [Li71, Li72]) together with the regularity results in [NT98, Theorem 2.4] imply that a transparent pair is also cohomologically trivial. We already pointed out that the Anosov property is satisfied, if for example (M, g) has negative curvature.

Given a cohomologically trivial pair (A, Φ), a trivializing function u satisfies

(9.1) $(X + A + \Phi)u = 0.$

If we assume now that (M, g) is negatively curved and the kernel of X_+^A on Ω_m is trivial for $m \geq 1$ (i.e. there are no nontrivial twisted conformal Killing tensors (CKTs), see [GPSU16]), then the proof of Theorem 9.2 implies that $u = u_0$. If we split equation (9.1) in degrees zero and one we obtain $\Phi u_0 = 0$ and $du + Au = 0$. Equivalently, $\Phi = 0$ and A is gauge equivalent to the trivial connection. Hence we have proved

Theorem 9.4. *Let (M, g) be a closed negatively curved manifold and (A, Φ) a transparent pair. If there are no nontrivial twisted CKTs, then A is gauge equivalent to the trivial connection and $\Phi = 0$.*

Projective equivalence and scattering rigidity in Euclidean space. The objective of this subsection is to use Theorem 1.6, which holds for manifolds of negative curvature, for the case of a compactly supported connection A in $\mathbb{R}^2$. This recovers a result of Eskin in [Es04]. Eskin's result uses a delicate theorem about existence of holomorphic integrating factors in the matrix case established in [ER04, Theorem 5]. Our approach is completely different and uses the notion of projective equivalence between metrics; this is quite suitable to deal with connections but unfortunately not with Higgs fields. We begin with the following:

Definition 9.5. Two metrics g_1 and g_2 on a manifold M are said to be *geodesically equivalent* if they have the same *unparametrized* geodesics. Two Riemannian manifolds (M, g) and (N, h) are said to be *projectively equivalent* if there exists a diffeomorphism $\varphi : M \to N$ such that g and $\varphi^* h$ are geodesically equivalent.

It is obvious from the definition that projective equivalence preserves the property of being nontrapping. The next simple lemma is the key for our argument.

Lemma 9.6. *Let g_1 and g_2 be two geodesically equivalent nontrapping metrics on M and let f be a $\mathbb{C}^n$-valued 1-form. Then for $(x, v) \in \partial_+(SM_1)$:*

$$I_A^1(f)(x, v) = I_A^2(f)(x, v/\sqrt{g_2(v, v)}).$$

Moreover

$$C_A^1(x, v) = C_A^2(x, v/\sqrt{g_2(v, v)}).$$

Proof. Let $\gamma_1 : [0, l_1] \to M$ be the unique g_1-geodesic defined by $(x, v) \in \partial_+(SM_1)$. By the hypothesis of geodesic equivalence there exists a smooth function $\tau : [0, l_2] \to [0, l_1]$ such that $\gamma_2(t) = \gamma_1(\tau(t))$ is the unique g_2-geodesic defined by $(x, v/\sqrt{g_2(v, v)}) \in \partial_+(SM_2)$. Obviously $\dot{\gamma}_2(t) = \gamma_1'(\tau(t))\dot{\tau}(t)$.
Consider the differential equation:

$$\dot{u}(t) + A(\gamma_2(t), \dot{\gamma}_2(t))u(t) = -f(\gamma_2(t), \dot{\gamma}_2(t))$$

with initial condition $u(l_2) = 0$. Then by definition $I_A^1(f)(x, v) = u(0)$. Using that A and f are 1-forms, we see that $s(r) := u(\tau^{-1}(r))$ solves

$$s'(r) + A(\gamma_1(r), \dot{\gamma}_1(r))s(r) = -f(\gamma_1(r), \dot{\gamma}_1(r))$$

with initial condition $s(l_1) = 0$. Since $I_A^2(f)(x, v/\sqrt{g_2(v, v)}) = s(0)$ the first claim in the lemma follows. The proof of the second claim is quite similar. $\square$

Here is a restatement of a result from [Es04], and a new proof for this result.

Theorem 9.7. *Let A and B be two $GL(n, \mathbb{C})$-connections in $\mathbb{R}^2$ with compact support inside a disk M. If $C_A = C_B$, then there exists a smooth $u : M \to GL(n, \mathbb{C})$ such that $u|_{\partial M} = \text{Id}$ and $B = u^{-1}du + u^{-1}Au$.*

Proof. Consider the half-sphere $S_-^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z < 0\}$ and the plane $E = \{(x, y, z) \in \mathbb{R}^3 : z = -1\}$. It is a classical fact that E, S_-^2 and the hyperbolic plane $\mathbb{H}^2$ are all projectively equivalent. We quickly review the proof of this fact. Stereographic projection from the origin gives a projective equivalence $\varphi : S_-^2 \to E$ since it sends great circles to straight lines. Consider now the Beltrami-Klein model of the hyperbolic plane. This is a unit disk $\mathbb{K}$ in which the geodesics are straight segments connecting boundary points. To compare it with Poincaré's disk model, let us call the latter $\mathbb{D}$. The map $\varphi : \mathbb{D} \to \mathbb{K}$ given by $\varphi(z) = \frac{2z}{1+|z|^2}$ maps geodesics to geodesics. Another way to see this projective equivalence is to consider the hyperboloid model of the hyperbolic plane. This is $\mathbb{H}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1, z > 0\}$. The geodesics are the intersection of planes through the origin with $\mathbb{H}^2$. Once again stereographic projection $\varphi : \mathbb{H}^2 \to E$, where $E = \{(x, y, z) \in \mathbb{R}^3 : z = 1\}$ gives a projective equivalence. The metric of constant negative curvature on $\mathbb{H}^2$ is induced by the Lorentzian metric $dx^2 + dy^2 - dz^2$. Thus E, S_-^2 and $\mathbb{H}^2$ are all projectively equivalent.

To prove the theorem consider a projective equivalence $\varphi : \mathbb{H}^2 \to \mathbb{R}^2$. Since A and B are compactly supported, φ^*A and φ^*B are also compactly supported inside some disk N in $\mathbb{H}^2$. Since $C_A = C_B$, by Lemma 9.6 we deduce that $C_{\varphi^*A} = C_{\varphi^*B}$. Using Theorem 1.6 we derive that φ^*A and φ^*B are gauge equivalent and moving everything back to $\mathbb{R}^2$ via φ gives the result. $\qquad\square$

Remark 9.8. One can go further by making use of certain remarkable first integral. (We restrict this remark to the 2D case.) Suppose g_1 and g_2 are two metrics on M. Consider the following function $I : TM \to \mathbb{R}$:

$$H(x, v) = \left(\frac{\det(g_1)}{\det(g_2)}\right)^{2/3} g_2(v, v).$$

Then H is a first integral of the geodesic flow of g_1 if and only if g_1 and g_2 are geodesically equivalent [D1896, MT98].

Define $F : M \to \mathbb{R}$ by $F = \left(\frac{\det(g_1)}{\det(g_2)}\right)^{1/3}$ so that $H^{1/2}(x, v) = F(x)\sqrt{g_2(v, v)}$.

Lemma 9.9. *Let g_1 and g_2 be two geodesically equivalent non-trapping metrics on the surface M and let f be a $\mathbb{C}^n$-valued symmetric tensor of order k. Then for any $(x, v) \in \partial_+(SM_1)$ we have:*

$$I_A^1(f(g_2)^{(1-k)/2})(x, v) = I_A^2(f)(x, v/\sqrt{g_2(v, v)}).$$

Proof. The proof is completely analogous to the proof of Lemma 9.6. One only needs to observe that $1 = g_2(\gamma_1', \gamma_1')(\dot\tau)^2$ and use that f is homogeneous of degree k in the velocity variable. $\qquad\square$

Let us combine this lemma with the fact that H is a first integral of the geodesic flow of g_1. Introduce a smooth diffeomorphism: $\ell : SM_1 \to SM_2$ by setting $\ell(x, v) = (x, v/\sqrt{g_2(v, v)})$. Then the previous lemma gives:

$$I_A^1\left(\frac{f}{F^{1-k}}H^{(1-k)/2}\right) = I_A^2(f) \circ \ell.$$

Since H is a first integral, it follows from the integral expression of $I_A(f)$ that

$$(9.2) \qquad H^{(1-k)/2} I_A^1\left(\frac{f}{F^{1-k}}\right) = I_A^2(f) \circ \ell.$$

Since F is a positive function depending only on x, f/F^{1-k} is also a symmetric tensor of order k, so equation (9.2) allows us to easily compare attenuated transforms for geodesically equivalent metrics for tensors of any order. Thus using the results for negative curvature we can derive analogous results in the flat case and constant positive curvature case (since these are projectively equivalent).

Appendix A. Simple/Anosov manifolds are controlled

This section proves Lemma 7.2, restated here:

Lemma A.1. *If (M, g) is a compact simple/Anosov manifold, there exist $\alpha \in (0, 1]$ and $\kappa > 0$ so that*

$$\|(X + A)Z\|^2 - (RZ, Z) \geq \alpha \|(X + A)Z\|^2 + \kappa \|Z\|^2$$

for any unitary connection A and for any $Z \in \mathcal{Z} \otimes \mathbb{C}^n$ (with $Z|_{\partial(SM)} = 0$ in the boundary case).

The proof of the lemma is based on the following auxiliary proposition. We assume that $U(x, v)$ is a symmetric operator $\{v\}^{\perp} \to \{v\}^{\perp}$, U is continuous, bounded and differentiable along the geodesic flow, and for $(x, v) \in SM$ the map $t \mapsto U(\varphi_t(x, v))$ satisfies the Riccati equation

$$\dot{U} + U^2 + R(\varphi_t(x, v)) = 0.$$

The function U naturally acts on $\mathcal{Z}^n$ as follows. If we let $Z = (Z_1, \ldots, Z_n)$ with $Z_i \in \mathcal{Z}$, then $UZ = (UZ_1, \ldots, UZ_n)$. Recall also that $X + A$ acts on $\mathcal{Z}^n$ as

$$(X + A)Z = (XZ_1, \ldots, XZ_n) + \left(\sum_k a_{1k}Z_k, \ldots, \sum_k a_{nk}Z_k\right)$$

where $A = (a_{jk})$. Clearly $AUZ = UAZ$.

Proposition A.2. *Let (M, g) be a closed manifold without conjugate points, and let U be as above. Then for any $Z \in \mathcal{Z}^n$ with $Z|_{\partial(SM)} = 0$ in the boundary case, one has*

$$(A.1) \qquad \|(X + A)Z - UZ\|^2 = \|(X + A)Z\|^2 - (RZ, Z).$$

Proof. Let us write

$$\|(X + A)Z - UZ\|^2 = \|(X + A)Z\|^2 - 2\operatorname{Re}((X + A)Z, UZ) + \|UZ\|^2.$$

All we need to show is that

$$2\operatorname{Re}((X + A)Z, UZ) = \|UZ\|^2 + (RZ, Z).$$

For this note that $(X + A)^* = -(X + A)$ (A is unitary) and that U and R are symmetric operators, hence using the Riccati equation and $AUZ = UAZ$ we derive

$$((X + A)Z, UZ) = -(Z, (X + A)(UZ)) = -(Z, (XU)Z + U(X + A)Z)$$
$$= (Z, U^2Z) + (Z, RZ) - (Z, U(X + A)Z)$$
$$= \|UZ\|^2 + (RZ, Z) - (UZ, (X + A)Z)$$

and (A.1) is proved. $\qquad\square$

Proof of Lemma A.1 (Anosov case). For the following standard facts see e.g. [PSU15] and references therein. Since the geodesic flow is Anosov we have continuous stable and unstable bundles. It is well known that these are invariant, Lagrangian and contained in the kernel of the contact form. Moreover, the subbundles are transversal to the vertical subbundle. Thus as in [PSU15, Lemma 6.5] we have two continuous symmetric maps U^+ and U^- (Green solutions) satisfying the Riccati equation with the property that the linear map $U^+(x, v) - U^-(x, v)$ is positive definite for every $(x, v) \in SM$.

Let $P := (X + A)Z - U^-Z$ and $Q := (X + A)Z - U^+Z$. Using equation (A.1) we see that $\|P\| = \|Q\|$. Solving for Z we obtain

$$Z = (U^+ - U^-)^{-1}(P - Q).$$

Since $U^\pm$ are continuous, there is a constant $a > 0$ independent of Z and A such that

$$\|Z\| \leq a\|P\|.$$

Since $(X + A)Z = P + U^-Z$ we see that there is another constant $b > 0$ independent of Z and A for which

$$\|(X + A)Z\| \leq b\|P\|.$$

From these inequalities and (A.1) the existence of α and κ easily follows. $\qquad\square$

Proof of Lemma A.1 (simple case). To carry out the proof above from the Anosov case to the simple case we only need to construct two continuous symmetric maps U^+ and U^- satisfying the Riccati equation with the property that the linear map $U^+(x, v) - U^-(x, v)$ is invertible for every $(x, v) \in SM$. We explain this next and note that the construction, which is less subtle than in the Anosov case, will give *smooth* $U^\pm$.

Let $\mathcal{V}$ denote the vertical subbundle associated with the projection $\pi : SM \to M$. Since M is simple we can engulf it by another simple manifold $\hat{M}$. Given $(x, v) \in S\hat{M}$, let $\tau(x, v)$ denote the time it takes the geodesic defined by (x, v) to exit $\hat{M}$ and set $\tau^-(x, v) = \tau(x, -v)$. Given $(x, v) \in S\hat{M}$ define

$$E(x, v) := d\phi_{\tau^-(x,v)}(\mathcal{V}(\phi_{-\tau^-(x,v)}(x, v))).$$

Since τ is smooth on the sphere bundle of the interior of $\hat{M}$ we see that $E|_{SM}$ is smooth. Moreover, since $\hat{M}$ has no conjugate points $E(x, v)$ is transversal to $\mathcal{V}(x, v)$ for $(x, v) \in SM$. The transversality property is equivalent to $d\pi : E(x, v) \to \{v\}^\perp$ being an isomorphism, and hence there exists a linear map $U^+(x, v) : \{v\}^\perp \to \{v\}^\perp$ such that E is the graph of U^+; in other words, given $w \in \{v\}^\perp$, there exists a unique

$w' \in \{v\}^{\perp}$ such that $(w, w') \in E$ and we set $U^+ w := w'$. Since $\mathcal{V}$ is Lagrangian, E is also Lagrangian. This is equivalent to U^+ being symmetric. Finally U^+ solves the Riccati equation along geodesics since by construction $d\phi_t(E(x, v)) = E(\phi_t(x, v))$ (cf. [Pa99, Chapter 2]).

To construct U^- we proceed in a similar fashion with the Lagrangian subbundle

$$F(x, v) := d\phi_{-\tau(x,v)}(\mathcal{V}(\phi_{\tau(x,v)}(x, v))).$$

To check that $U^+(x, v) - U^-(x, v)$ is invertible, suppose otherwise. Hence there exists $0 \neq w \in \{v\}^{\perp}$ such that $U^+ w = U^- w$. In other words, there is a non-zero vector $\xi \in E(x, v) \cap F(x, v)$. By definition of E and F, this implies that there exists $0 \neq \eta \in \mathcal{V}(\phi_{-\tau^-(x,v)}(x, v))$ such that $d\phi_{\tau^-(x,v)+\tau(x,v)}(\eta) \in \mathcal{V}(\phi_{\tau(x,v)}(x, v))$. This means that the Jacobi field $d\pi \circ d\phi_t(\eta)$ vanishes at the entry and exit points to $\hat{M}$ contradicting the fact that $\hat{M}$ has no conjugate points. $\square$

APPENDIX B. SIMPLE SURFACES OF NONPOSITIVE CURVATURE

Let (M, g) be a simple 2D manifold with nonpositive curvature. In this appendix we will show how one can improve the argument in Section 7 by using holomorphic integrating factors [PSU12]. This leads to a new Carleman estimate with linear weights, which can be used to absorb a general connection in addition to a Higgs field, and which would apply to functions u with $\sum e^{2\tau l} \|X_- u_l\|^2 < \infty$ for some fixed $\tau > 0$ (instead of requiring this for all $\tau > 0$). Throughout this section we will use the notation from [PSU15, Appendix B] (see also [PSU12, PSU13]).

If A is a unitary connection, the Pestov identity in Lemma 4.2 will take the form (see also [PSU13, Proposition 3.3])

$$\|V X^A u\|^2 = \|X^A V u\|^2 - (K V u, V u) - (* F_A u, V u) + \|X^A u\|^2$$

for $u \in C^\infty(SM)$ with $u|_{\partial(SM)} = 0$. Since M is two-dimensional we will consider Fourier expansions

$$u = \sum_{k=-\infty}^{\infty} u_k$$

where $u_k \in \Lambda_k$ (i.e. $V u_k = i k u_k$). The point is that if u is holomorphic (i.e. $u_k = 0$ for $k < 0$) and if $i * F_A \geq s > 0$, then

(B.1) $$- (* F_A u, V u) = \sum_{1}^{\infty} (i * F_A u_k, k u_k) \geq s \sum_{1}^{\infty} k \|u_k\|^2.$$

As usual we write $X^A = \mu_+ + \mu_-$. As in Lemma 4.3, writing the Pestov identity in terms of the operators $\mu_{\pm}$ and using the condition $K \leq 0$ together with (B.1) yields the following result.

Lemma B.1. *Let (M, g) be a simple 2D manifold with $K \leq 0$. If A is a unitary connection with $i * F_A \geq s > 0$, then*

$$\|\mu_- u_1\|^2 + \sum_{1}^{\infty} (2k + 2)\|\mu_- u_{k+1}\|^2 + s \sum_{1}^{\infty} k\|u_k\|^2 \leq \sum_{2}^{\infty} (2k - 2)\|\mu_+ u_{k-1}\|^2$$

for any holomorphic $u \in C^\infty(SM)$ with $u|_{\partial(SM)} = 0$.

This estimate is weaker than the one for negative sectional curvature since one has $s \sum k\|u_k\|^2$ instead of $\kappa \sum k^2\|u_k\|^2$ on the left. However, the constant $s > 0$ may be chosen to be very large.

We now consider a weighted version of the above estimate. Let (γ_k) be a sequence of positive numbers (not necessarily polynomially bounded). Applying the previous estimate to $\sum_0^N \gamma_k u_k$ for N finite gives that

$$r_1\|\mu_- u_1\|^2 + \sum_1^{N-1} 2r_{k+1}\|\mu_- u_{k+1}\|^2 + s \sum_1^N r_k\|u_k\|^2 \leq \sum_2^{N+1} 2r_{k-1}\|\mu_+ u_{k-1}\|^2$$

for any holomorphic $u \in C^\infty(SM)$ with $u|_{\partial(SM)} = 0$, where $r_k = k\gamma_k^2$. Writing $\mu_+ u_{k-1} = (X^A u)_k - \mu_- u_{k+1}$ for $k \leq N-1$ and using Cauchy's inequality with ε_k gives

$$(B.2) \quad r_1\|\mu_- u_1\|^2 + \sum_1^{N-1} 2(r_{k+1} - (1+\varepsilon_k)r_{k-1})\|\mu_- u_{k+1}\|^2 + s \sum_1^N r_k\|u_k\|^2$$

$$\leq 2r_{N-1}\|\mu_+ u_{N-1}\|^2 + 2r_N\|\mu_+ u_N\|^2 + \sum_2^{N-1} 2r_{k-1}(1 + \frac{1}{\varepsilon_k})\|(X^A u)_k\|^2$$

We need $0 < \varepsilon_k \leq \frac{r_{k+1}}{r_{k-1}} - 1$ for the coefficients on the left to be ≥ 0. This means that we must have $r_{k+1} > r_{k-1}$ and

$$\varepsilon_k = \delta_{k+1}\left[\frac{r_{k+1}}{r_{k-1}} - 1\right]$$

for some $\delta_{k+1} \in (0,1]$. This leads to

$$(B.3) \quad r_1\|\mu_- u_1\|^2 + \sum_1^{N-1} 2(1 - \delta_{k+1})(r_{k+1} - r_{k-1})\|\mu_- u_{k+1}\|^2 + s \sum_1^N r_k\|u_k\|^2$$

$$\leq 2[r_N + \delta_{N+2}(r_{N+2} - r_N)]\|\mu_- u_{N+2}\|^2 + \sum_2^{N+1} 2\left[1 + \frac{1 - \delta_{k+1}}{\delta_{k+1}}\frac{r_{k-1}}{r_{k+1}}\right]\frac{r_{k+1}r_{k-1}}{r_{k+1} - r_{k-1}}\|(X^A u)_k\|^2$$

We will simplify this estimate by taking $\delta_k = 1/2$ and using the assumption that $r_{k-1} < r_{k+1}$, which implies that

$$r_1\|\mu_- u_1\|^2 + \sum_1^{N-1}(r_{k+1} - r_{k-1})\|\mu_- u_{k+1}\|^2 + s \sum_1^N r_k\|u_k\|^2$$

$$\leq 2r_{N+2}\|\mu_- u_{N+2}\|^2 + 4 \sum_2^{N+1} \frac{r_{k+1}r_{k-1}}{r_{k+1} - r_{k-1}}\|(X^A u)_k\|^2$$

We will now choose $r_k = e^{2\tau k}$ and $\delta_k = 1/2$, where $\tau > 0$, and take the limit $N \to \infty$ to obtain the following estimate.

Lemma B.2. *Let (M, g) be a simple surface with $K \leq 0$, and let A be a unitary connection with $i * F_A \geq s > 0$. Given any $\tau > 0$, one has*

$$e^{2\tau} \|\mu_- u_1\|^2 + \sum_1^\infty e^{2\tau(k+1)}(1 - e^{-2\tau}) \|\mu_- u_{k+1}\|^2 + s \sum_1^\infty e^{2\tau k} \|u_k\|^2$$

$$\leq 2 \frac{1 + e^{-2\tau}}{1 - e^{-2\tau}} \sum_1^\infty e^{2\tau k} \|(X^A u)_{k+1}\|^2 + 2 \lim_{N \to \infty} e^{2\tau(N+2)} \|\mu_- u_{N+2}\|^2$$

for any holomorphic $u \in C^\infty(SM)$ with $u|_{\partial(SM)} = 0$.

The application for this estimate would go as follows: assume that $\tilde{A}$ is a possibly large non-unitary connection, and $\tilde{u} \in C^\infty(SM)$ satisfies

$$(X + \tilde{A})\tilde{u} = f_0 + f_1 + f_{-1}, \qquad \tilde{u}|_{\partial(SM)} = 0.$$

Let $s > 0$ be fixed, and use [PSU13, Proposition 2.4] to find an antiholomorphic integrating factor w with $Xw = i\varphi$ where φ is a 1-form with $d\varphi = \omega_g$ (the area form). Then $A = -is\varphi$ satisfies $i * F_A = s$, and

$$(X + A + \tilde{A})(e^{sw}\tilde{u}) = e^{sw}(f_0 + f_1 + f_{-1}).$$

We let $u = \sum_0^\infty (e^{sw}\tilde{u})_k$. This satisfies

(B.4) $$(X + A + \tilde{A})u = \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_{-1}.$$

Thus in particular

$$\|(X^A u)_k\|^2 \leq R(\|u_{k-1}\|^2 + \|u_{k+1}\|^2), \qquad k \geq 2.$$

Inserting this formally in (B.3) shows that we should choose (r_k) such that

$$2R \left[\frac{r_{k-2} r_k}{r_k - r_{k-2}} + \frac{r_k r_{k+2}}{r_{k+2} - r_k} \right] \leq s r_k.$$

Writing $r_{k+2} = a_k r_k$ where $a_k > 1$ are constants, this can be rewritten as

$$2R \left[\frac{1}{a_{k-2} - 1} + \frac{a_k}{a_k - 1} \right] \leq s.$$

This implies the conditions $a_{k-2} \geq 1 + \frac{2R}{s}$ and $1 - \frac{1}{a_k} \geq \frac{2R}{s}$, that is,

$$1 + \frac{2R}{s} \leq a_k \leq \frac{1}{1 - \frac{2R}{s}}.$$

Recalling that $r_{k+2} = a_k r_k$, and choosing $r_0 = 1$, this shows that the weights (r_k) must satisfy

$$(1 + \varepsilon)^{k/2} \leq r_k \leq \left(\frac{1}{1 - \varepsilon} \right)^{k/2}$$

where $\varepsilon > 0$ is fixed but can be taken arbitrarily small when s is chosen large. Thus exponential growth of the weights (r_k) is exactly the condition required to absorb a large general connection.

We can easily implement the above approach by taking $r_k = e^{2\tau k}$ in (B.3), leading to the following inequality.

Lemma B.3. *Let (M, g) be a simple surface with $K \leq 0$, and let A be a unitary connection with $i * F_A \geq s > 0$. Given any $\tau > 0$, one has*

$$s \sum_1^\infty e^{2\tau k} \|u_k\|^2 \leq \frac{2}{1 - e^{-2\tau}} \sum_2^\infty e^{2\tau(k-1)} \|(X^A u)_k\|^2$$

for any holomorphic $u \in C_F^\infty(SM)$ with $u|_{\partial(SM)} = 0$.

We can also get a similar estimate with μ_- terms on the left, for instance by taking $\varepsilon_k = 1$ and $r_k = e^{2k\tau}$ in (B.2).

Lemma B.4. *Let (M, g) be a simple surface with $K \leq 0$, and let A be a unitary connection with $i * F_A \geq s > 0$. Given any $\tau > 1$, one has*

$$2(e^{4\tau} - 2) \sum_1^\infty e^{2\tau(k-1)} \|\mu_- u_{k+1}\|^2 + s \sum_1^\infty e^{2\tau k} \|u_k\|^2 \leq 4 \sum_2^\infty e^{2\tau(k-1)} \|(X^A u)_k\|^2$$

for any holomorphic $u \in C_F^\infty(SM)$ with $u|_{\partial(SM)} = 0$.

Either of the above estimates could be used to absorb a large non-unitary connection (when s is chosen large enough) when u is a holomorphic solution of (B.4). This would in principle give uniqueness for general connections on simple surfaces with nonpositive curvature. However, the estimates involve exponentially growing weights and thus they should apply to functions for which $\sum e^{2\tau k} \|u_k\|^2 < \infty$, but not in general to functions in $C^\infty(SM)$. These estimates would only give the following type of result:

Lemma B.5. *Let (M, g) be a simple surface with $K \leq 0$, and let $\tilde{A}$ be a general connection. Also, let A be a unitary connection with $i * F_A \geq s$, where $s > 8\|\tilde{A}\|_{L^\infty}$. If $u \in C^\infty(SM)$ is holomorphic and satisfies*

$$(X + A + \tilde{A})u = \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_{-1}, \qquad u|_{\partial(SM)} = 0,$$

and if additionally

(B.5) $$\lim_{k \to \infty} e^{\tau k} \|\mu_- u_k\| = 0$$

for some $\tau > 0$, then $u = 0$.

The challenge is to show (B.5), which would for instance be a consequence of the following kind of claim:

Claim B.6. *Let (M, g) be a simple surface with $K \leq 0$, and let $\tilde{A}$ be a general connection. If $u \in C^\infty(SM)$ satisfies*

$$(X + \tilde{A})u = f, \qquad u|_{\partial(SM)} = 0,$$

and if additionally for some $\tau_0 > 0$ one has

$$\sum_1^\infty e^{2\tau k}\|f_k\|^2 < \infty \quad for\ \tau < \tau_0,$$

then one also has

$$\sum_1^\infty e^{2\tau k}\|\mu_- u_k\|^2 < \infty \quad for\ \tau < \tau_0.$$

There is a partial analogue of Claim B.6 in scattering theory for the Laplacian (see for instance [PSU10, Lemma 2.5]):

Lemma B.7. *Let $\tau_0 > 0$, and assume that f satisfies*

$$\int_{\mathbb{R}^n} e^{2\tau|x|}|f(x)|^2\,dx < \infty \quad for\ \tau < \tau_0.$$

If $\lambda > 0$, if $(-\Delta - \lambda)u = f$ in $\mathbb{R}^n$, and if u is both outgoing and incoming, then u must also satisfy

$$\int_{\mathbb{R}^n} e^{2\tau|x|}|u(x)|^2\,dx < \infty \quad for\ \tau < \tau_0.$$

The condition that u is both outgoing and incoming means that $u = R_+(\lambda)f = R_-(\lambda)f$ (i.e. $f \in \mathrm{Ker}(R_+(\lambda) - R_-(\lambda))$) for the related outgoing and incoming resolvents. This condition "regularises" the solution u to decay as fast as f. The analogue of this condition for X is that u satisfies $u|_{\partial(SM)} = 0$. Following [Gu17] this condition can be written as $u = R_+(0)f = R_-(0)f$ for the flow resolvents (i.e. $f \in \mathrm{Ker}(R_+(0) - R_-(0))$). We have already seen in Proposition 8.1 that the condition $u|_{\partial(SM)} = 0$ "regularises" the solution to be in $C^\infty(SM)$ if f is C^∞.

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Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge CB3 0WB, UK
    E-mail address: g.p.paternain@dpmms.cam.ac.uk

Department of Mathematics and Statistics, University of Jyväskylä
    E-mail address: mikko.j.salo@jyu.fi