DETERMINING AN UNBOUNDED POTENTIAL FROM CAUCHY DATA IN ADMISSIBLE GEOMETRIES

DAVID DOS SANTOS FERREIRA, CARLOS E. KENIG, AND MIKKO SALO

ABSTRACT. In [4] anisotropic inverse problems were considered in certain admissible geometries, that is, on compact Riemannian manifolds with boundary which are conformally embedded in a product of the Euclidean line and a simple manifold. In particular, it was proved that a bounded smooth potential in a Schrödinger equation was uniquely determined by the Dirichlet-to-Neumann map in dimensions $n \geq 3$. In this article we extend this result to the case of unbounded potentials, namely those in $L^{n/2}$. In the process, we derive L^p Carleman estimates with limiting Carleman weights similar to the Euclidean estimates of Jerison-Kenig [10] and Kenig-Ruiz-Sogge [12].

CONTENTS

1. INTRODUCTION

In this paper we consider the problem of proving L^p estimates for limiting Carleman weights on Riemannian manifolds, and the related inverse problem of recovering an $L^{n/2}$ potential from the Dirichlet-to-Neumann map (DN map) related to the Schrödinger equation. The main motivation comes from the inverse conductivity problem posed by

Calderón [1]. This problem asks to determine the conductivity function of a medium from electrical measurements made on its boundary.

In mathematical terms, if $\Omega \Subset \mathbf{R}^n$ is the medium of interest having a positive conductivity coefficient γ, in the Calderón problem one considers the conductivity equation

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega$$

and defines the DN map by

$$\Lambda_\gamma : u|_{\partial\Omega} \mapsto \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

This operator maps the voltage at the boundary to the current given by γ times the normal derivative, which encodes the electrical measurements at the boundary. The inverse problem of Calderón asks to determine γ from the knowledge of Λ_γ. This problem has been extensively studied and we refer to [30] for a recent survey.

The anisotropic Calderón problem considers the case where the conductivity γ is a symmetric positive definite matrix instead of a scalar function. This corresponds to situations where the electrical properties of the medium depend on direction. The problem is open in general in dimensions $n \geq 3$, see [4] for known results and more details. Following [18] the problem may be recast as the determination of the metric g on a compact Riemannian manifold (M, g) with boundary from the corresponding DN map. In [4] progress was made on the anisotropic Calderón problem in the following class of conformal smooth manifolds.

Definition. *A compact Riemannian manifold (M, g), with dimension $n \geq 3$ and with boundary ∂M, is called* admissible *if $M \Subset \mathbf{R} \times M_0$ for some $(n-1)$-dimensional simple manifold (M_0, g_0), and if $g = c(e \oplus g_0)$ where e is the Euclidean metric on $\mathbf{R}$ and c is a smooth positive function on M.*

Here, a compact manifold (M_0, g_0) with boundary is *simple* if for any $p \in M_0$ the exponential map $\exp_p$ with its maximal domain of definition is a diffeomorphism onto M_0, and if ∂M_0 is strictly convex (that is, the second fundamental form of $\partial M_0 \hookrightarrow M_0$ is positive definite).

In [4] it was proved that a Riemannian metric in a conformal class of admissible geometries is uniquely determined by the DN map. This was obtained as a corollary of a result for the Schrödinger equation in a fixed admissible manifold, stating that a bounded smooth potential q is determined by the corresponding DN map. In [4] all coefficients were assumed infinitely differentiable. In this paper we relax this requirement and show that a complex potential $q \in L^{n/2}(M)$ is determined by the DN map.

To state the main result, assume that (M, g) is a compact Riemannian manifold with smooth boundary ∂M, and let Δ_g be the Laplace-Beltrami operator. Given a complex function $q \in L^{n/2}(M)$, where $n \geq 3$ is the dimension of the manifold M, we consider the Dirichlet problem

$$(-\Delta_g + q)u = 0 \text{ in } M, \quad u|_{\partial M} = f.$$

We assume throughout that 0 is not a Dirichlet eigenvalue for this problem, and then standard arguments (see Appendix A) show that there is a well-defined DN map

$$\Lambda_{g,q} : H^{1/2}(\partial M) \to H^{-1/2}(\partial M), \quad f \mapsto \partial_\nu u|_{\partial M}.$$

The following uniqueness theorem is the main result for the inverse problem. (The assumption $q \in L^{n/2}$ may be considered optimal in the context of the standard wellposedness theory for the Dirichlet problem with L^p potentials, and also for the strong unique continuation principle to hold [10].)

Theorem 1.1. *Let (M, g) be admissible and let q_1, q_2 be complex functions in $L^{n/2}(M)$. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.*

In the case where M is a bounded domain in $\mathbf{R}^n$ and g is the Euclidean metric, this result is due to Lavine and Nachman [17] following the earlier result of Jerison and Kenig for $q_j \in L^{n/2+\varepsilon}(M)$ for some $\varepsilon > 0$ (see Chanillo [2] for an account and also for a related result with q_j in a Fefferman-Phong class with small norm). As mentioned above, if q is a smooth function on an admissible manifold M this result was proved in [4] by using L^2 Carleman estimates. In fact, smoothness of q is not essential, and by inspecting the proof of [4] the uniqueness result can be extended to bounded continuous q (with the complex geometrical optics construction in the proof going through for $q \in L^n(M)$). However, the proof for $q \in L^{n/2}$ requires to replace the L^2 Carleman estimates in [4] with corresponding L^p Carleman estimates.

The other main result in this paper is a L^p Carleman estimate for limiting Carleman weights on Riemannian manifolds. The concept of limiting Carleman weights was introduced in [14] as part of a general procedure for producing special complex geometrical optics solutions to elliptic equations, with applications to inverse problems. We refer to [4] for a precise definition and more careful analysis of limiting Carleman weights, also on Riemannian manifolds. For present purposes it is sufficient to mention that the existence of a limiting Carleman weight on (M, g) in dimensions $n \geq 3$ is locally equivalent with the manifold being admissible, and that typical limiting Carleman weights in $\mathbf{R}^n$,

$n \geq 3$, include the linear weight $\varphi(x) = x_1$ and logarithmic weight $\varphi(x) = \log|x|$.

The last two weights are featured in the literature of Carleman estimates and unique continuation, in particular in the scale invariant L^p Carleman estimates of Kenig-Ruiz-Sogge [12] for the linear weight and of Jerison-Kenig [10] for the logarithmic weight. We prove an analogue of these estimates on more general Riemannian manifolds. Note that the existence of a limiting Carleman weight requires at least locally a product structure on the manifold, and therefore the following result is stated for the linear weight on a product manifold. The result, in the case when the manifold (M_0, g_0) below is the standard $n-1$ dimensional torus, is due to Shen [23].

Theorem 1.2. *Let (M_0, g_0) be an $(n-1)$-dimensional compact manifold without boundary, and equip $\mathbf{R} \times M_0$ with the metric $g = e \oplus g_0$ where e is the Euclidean metric. The Euclidean coordinate is denoted by x_1. For any compact interval $I \subseteq \mathbf{R}$ there exists a constant $C_I > 0$ such that if $|\tau| \geq 4$ and*

$$\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$$

then we have

$$\|e^{\tau x_1} u\|_{L^{\frac{2n}{n-2}}(\mathbf{R} \times M_0)} \leq C_I \|e^{\tau x_1} \Delta_g u\|_{L^{\frac{2n}{n+2}}(\mathbf{R} \times M_0)}$$

when $u \in C_0^\infty(I \times M_0)$.

The proof of the L^2 Carleman estimates for limiting Carleman weights in [4] is based on integration by parts and cannot be used in the L^p setting. However, in [13] another proof of the L^2 Carleman estimate is given; this proof is based on Fourier analysis and gives an explicit inverse for the conjugated Laplacian. We will derive the L^p bounds from this explicit inverse operator. This follows the proof of the L^p Carleman estimate of Jerison-Kenig [10] using Jerison's approach [9], [26, Section 5.1] based on the spectral cluster estimates of Sogge [26]. Finally, if one allows strongly pseudoconvex Carleman weights then much stronger estimates are available (see for instance [15, 16]), however for the applications to inverse problems it seems necessary to restrict to limiting Carleman weights.

Remark 1.3. The above theorems are in the setting of (conformal) product manifolds. However, the results also apply to warped products. If $f : \mathbf{R} \to \mathbf{R}$ is a smooth function and (M_0, g_0) is an $(n-1)$-dimensional manifold, the warped product $\mathbf{R} \times_{e^{2f}} M_0$ is the manifold $M = \mathbf{R} \times M_0$

endowed with the metric

$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & e^{2f(x_1)}g_0(x') \end{pmatrix}.$$

We choose coordinates $y_1 = \eta(x_1)$, $y' = x'$ for a suitable smooth strictly increasing function η. In fact, if

$$\eta(t) = \int_0^t e^{-f(s)} \, ds$$

then $\eta'(t)^{-2} = e^{2f(t)}$ and the metric in y coordinates becomes a conformal multiple of a product metric,

$$g(y_1, y') = e^{2f(\eta^{-1}(y_1))} \begin{pmatrix} 1 & 0 \\ 0 & g_0(y') \end{pmatrix}.$$

Warped products have a natural limiting Carleman weight $\varphi(y) = y_1$, and Theorem 1.1 remains true in conformal multiples of warped products whenever (M_0, g_0) is a simple manifold.

The paper is organized as follows. Section 1 is the introduction. In Section 2 we prove the L^p Carleman estimate complemented with the usual L^2 Carleman estimates. Section 3 presents the construction of complex geometrical optics solutions for Schrödinger equations with $L^{n/2}$ potentials in admissible geometries. The proof of Theorem 1.1 is contained in Section 4, modulo a uniqueness result for an analogue of the attenuated geodesic ray transform acting on unbounded functions. This last result has a different character than the rest of the proof, and it is therefore established separately in Section 5. There are two appendices concerning the wellposedness of the Dirichlet problem and the normal operator for the attenuated ray transform.

2. L^p Carleman estimates

The aim of this section is to prove Theorem 1.2, which is an analogue of the L^p Carleman estimates obtained in the Euclidean case by Jerison and Kenig [10] (for logarithmic weights) or by Kenig, Ruiz and Sogge [12] (for linear weights). In fact, we prove a more general result which implies Theorem 1.2 by taking $f = e^{\tau x_1}\Delta_g e^{-\tau x_1}u$ for $u \in C_0^\infty(I \times M_0)$.

The case when (M_0, g_0) is the standard $n-1$ dimensional torus is due to Shen [23].

Proposition 2.1. *Let $I \subseteq \mathbf{R}$ be a compact interval and (M_0, g_0) a compact $(n-1)$-dimensional manifold without boundary. Equip $N = I \times M_0$ with the product metric $g = e \oplus g_0$. For $|\tau| \geq 4$ with $\tau^2 \notin Spec(-\Delta_{g_0})$, there is a linear operator $G_\tau : L^2(N) \to H^2(N)$ such that*

$$\mathrm{e}^{\tau x_1}(-\Delta_g)\mathrm{e}^{-\tau x_1}G_\tau v = v \quad for\ v \in L^2(N),$$
$$G_\tau \mathrm{e}^{\tau x_1}(-\Delta_g)\mathrm{e}^{-\tau x_1}v = v \quad for\ v \in C_0^\infty(N^{int}).$$

This operator satisfies

$$\|G_\tau f\|_{L^2(N)} \leq \frac{C_0}{|\tau|}\|f\|_{L^2(N)},$$

$$\|G_\tau f\|_{H^1(N)} \leq C_0\|f\|_{L^2(N)},$$

$$\|G_\tau f\|_{L^{\frac{2n}{n-2}}(N)} \leq C_0\|f\|_{L^{\frac{2n}{n+2}}(N)},$$

where C_0 is independent of τ (but may depend on I).

Remark 2.2. In the Euclidean case, L^p Carleman estimates with linear weights can be obtained from L^p Carleman estimates with pseudoconvex Carleman weights by scaling. Indeed, suppose that the following Carleman estimate

$$\left\|\mathrm{e}^{\tau(x_1 + x_1^2/2\varepsilon)}u\right\|_{L^{\frac{2n}{n-2}}(\mathbf{R}^n)} \leq C_K\left\|\mathrm{e}^{\tau(x_1 + x_1^2/2\varepsilon)}\Delta u\right\|_{L^{\frac{2n}{n+2}}(\mathbf{R}^n)},$$

holds for all $\varepsilon \leq \varepsilon_0$ and all $u \in C_0^\infty(K)$, then applying this estimate to $u_\mu = u(\mu \cdot)$ with $\mu \geq 1$ and $u \in C_0^\infty(K)$, one gets

$$\left\|\mathrm{e}^{\frac{\tau}{\mu}x_1 + \frac{\tau}{\mu^2}x_1^2/2\varepsilon}u\right\|_{L^{\frac{2n}{n-2}}(\mathbf{R}^n)} \leq C_K\left\|\mathrm{e}^{\frac{\tau}{\mu}x_1 + \frac{\tau}{\mu^2}x_1^2/2\varepsilon}\Delta u\right\|_{L^{\frac{2n}{n+2}}(\mathbf{R}^n)}.$$

Choosing $\mu = \sqrt{\tau}$, and using the fact that $\mathrm{e}^{x_1^2/2\varepsilon} \simeq C_\varepsilon$ on K, one gets the Carleman estimate

$$\left\|\mathrm{e}^{\mu x_1}u\right\|_{L^{\frac{2n}{n-2}}(\mathbf{R}^n)} \leq C_{K,\varepsilon}\left\|\mathrm{e}^{\mu x_1}\Delta u\right\|_{L^{\frac{2n}{n+2}}(\mathbf{R}^n)},$$

for all $u \in C_0^\infty(K)$. However, in the anisotropic case, one has to find another way.

To prepare for the proof of Proposition 2.1 consider the Laplace-Beltrami operator on N,

$$P = \Delta_g = \partial_{x_1}^2 + \Delta_{g_0}$$

and the corresponding conjugated operator (by the limiting Carleman weight x_1)

$$(2.1) \qquad \mathrm{e}^{\tau x_1}P\mathrm{e}^{-\tau x_1} = \partial_{x_1}^2 - 2\tau\partial_{x_1} + \tau^2 + \Delta_{g_0}.$$

We denote by $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \ldots$ the sequence of eigenvalues of $-\Delta_{g_0}$ on M_0 and $(\psi_j)_{j \ge 0}$ the corresponding sequence of eigenfunctions forming an orthonormal basis of $L^2(M_0)$,

$$-\Delta_{g_0}\psi_j = \lambda_j \psi_j.$$

We denote by $\pi_j : L^2(M_0) \to L^2(M_0), u \mapsto (u, \psi_j)\psi_j$ the projection on the linear space spanned by the eigenfunction ψ_j so that

$$\sum_{j=0}^{\infty} \pi_j = \mathrm{Id}, \quad \sum_{j=0}^{\infty} \lambda_j \pi_j = -\Delta_{g_0}$$

and by

$$\widehat{u}(j) = \int_{M_0} u\,\overline{\psi_j}\,\mathrm{d}V_{g_0}$$

the corresponding Fourier coefficients of a function u on M_0. We define the spectral clusters as

$$(2.2) \qquad \chi_k = \sum_{k \le \sqrt{\lambda_j} < k+1} \pi_j, \quad k \in \mathbf{N}.$$

Note that these are projection operators, $\chi_k^2 = \chi_k$, and they constitute a decomposition of the identity

$$(2.3) \qquad \mathrm{Id} = \sum_{k=0}^{\infty} \chi_k.$$

We end this paragraph by recalling the spectral cluster estimates of Sogge [24, 26] that we will need:

$$(2.4) \qquad \|\chi_k u\|_{L^{\frac{2n}{n-2}}(M_0)} \le C(1+k)^{\frac{1}{2}-\frac{1}{n}}\|u\|_{L^2(M_0)},$$

$$\|\chi_k u\|_{L^2(M_0)} \le C(1+k)^{\frac{1}{2}-\frac{1}{n}}\|u\|_{L^{\frac{2n}{n+2}}(M_0)}.$$

The first estimate is given in [26, Corollary 5.1.2] and the second one is a consequence of the first one by duality.

Proof of Proposition 2.1. Recall that our main goal is to prove

$$(2.5) \qquad \|u\|_{L^{\frac{2n}{n-2}}(\mathbf{R}\times M_0)} \le C_I\|f\|_{L^{\frac{2n}{n+2}}(\mathbf{R}\times M_0)}$$

when $u \in C_0^{\infty}(I \times M_0)$ and

$$(2.6) \qquad D_{x_1}^2 u + 2i\tau D_{x_1} u - \Delta_{g_0} u - \tau^2 u = f$$

with $D_{x_1} = -i\partial_{x_1}$. The inverse operator in (2.6) is actually easy to write down, as was done in [13]. The same procedure appears in [9]

and [26, Section 5.1]. Writing $f = \sum_{j=0}^{\infty} \pi_j f$ and similarly for u, the equation formally becomes

$$(D_{x_1}^2 + 2i\tau D_{x_1} - \tau^2 + \lambda_j)\pi_j u = \pi_j f$$

for x_1 on the real line and for $j \geq 0$. The symbol of the operator on the left is $\xi_1^2 + 2i\tau\xi_1 - \tau^2 + \lambda_j$, and this is always nonzero provided that $\tau^2 \neq \lambda_j$ for all j. Thus, if

$$\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0}),$$

an inverse operator may be obtained as

$$\tilde{G}_\tau f(x_1, x') = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} m_\tau\big(x_1 - y_1, \sqrt{\lambda_j}\big)\, \pi_j f(y_1, x')\, dy_1$$

where

$$m_\tau(t, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\eta}}{\eta^2 + 2i\tau\eta - \tau^2 + \mu^2}\, d\eta, \quad \mu > 0.$$

The operator $\tilde{G}_\tau$ is the same as G_τ in [13, Section 4], except that in the present setting $\{\psi_j\}$ is a basis of $L^2(M_0)$ on a compact manifold (M_0, g_0) without boundary instead of being a basis of Dirichlet eigenfunctions on a compact manifold with boundary. Let

$$L_\delta^2(\mathbf{R} \times M_0) = \big\{ f \in L_{\mathrm{loc}}^2(\mathbf{R} \times M_0)\,;\, (1 + x_1^2)^{\delta/2} f \in L^2(\mathbf{R} \times M_0) \big\}$$

and let $H_\delta^s(\mathbf{R} \times M_0)$ by the corresponding Sobolev spaces. The proof of [13, Proposition 4.1] goes through for $\tilde{G}_\tau$ without changes and shows that for any fixed $\delta > 1/2$, if $|\tau| \geq 1$ and $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$ then the equation

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} v = f$$

has a unique solution $v = \tilde{G}_\tau f \in H_{-\delta}^1(\mathbf{R} \times M_0)$ for any $f \in L_\delta^2(\mathbf{R} \times M_0)$. Further, $v \in H_{-\delta}^2(\mathbf{R} \times M_0)$ and

$$\|v\|_{H_{-\delta}^s(\mathbf{R} \times M_0)} \leq C_0 |\tau|^{s-1} \|f\|_{L_\delta^2(\mathbf{R} \times M_0)}, \quad 0 \leq s \leq 2.$$

We define

$$G_\tau f(x_1, x') = \chi(x_1)\tilde{G}_\tau f(x_1, x')$$

with $\chi \in C_0^\infty(\mathbf{R})$ which equals 1 on I. It is then clear that all the statements in the proposition except for the L^p estimate follow from the results for $\tilde{G}_\tau$ explained above.

It is sufficient to prove the L^p estimate in the case where $\tau \geq 4$ and $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$. We first record a lemma.

Lemma 2.3. *If $\tau > 0$, $\mu > 0$, $\tau \neq \mu$ and $t \in \mathbf{R}$ then*

$$|m_\tau(t,\mu)| \leq \frac{1}{\mu}\mathrm{e}^{-|\tau-\mu||t|}.$$

Besides, if $\tau > 0$ then

$$|m_\tau(t,0)| \leq |t|\mathrm{e}^{-\tau|t|}.$$

Proof. This follows by writing

$$\frac{1}{(i\eta - (\tau + \mu))(i\eta - (\tau - \mu))} = \frac{1}{2\mu}\left[\frac{1}{i\eta - (\tau + \mu)} - \frac{1}{i\eta - (\tau - \mu)}\right]$$

and by noting that for $\alpha > 0$

$$\mathscr{F}_\eta^{-1}\left\{\frac{1}{i\eta + \alpha}\right\}(t) = \begin{cases} 0, & t < 0 \\ \mathrm{e}^{-\alpha t}, & t > 0, \end{cases}$$

and similarly for $\alpha < 0$.

Furthermore we have

$$\mathscr{F}_\eta^{-1}\left\{\frac{1}{(\eta + i\tau)^2}\right\}(t) = \begin{cases} t\mathrm{e}^{-\tau|t|}, & t < 0 \\ 0, & t > 0, \end{cases}$$

and this concludes the proof of the lemma. $\qquad\square$

From the decomposition (2.3), the spectral cluster estimate (2.4), and the fact that spectral clusters are projections ($\chi_k^2 = \chi_k$), we get the following string of estimates

$$(2.7) \qquad \|u\|_{L^{\frac{2n}{n-2}}(M_0)} = \left\|\sum_{k=0}^{\infty}\chi_k^2 u\right\|_{L^{\frac{2n}{n-2}}(M_0)}$$

$$\lesssim \sum_{k=0}^{\infty}(1+k)^{\frac{1}{2}-\frac{1}{n}}\|\chi_k u\|_{L^2(M_0)}.$$

Since

$$\|\chi_k u\|_{L^2(M_0)} = \left(\sum_{k \leq \sqrt{\lambda_j} < k+1}|\widehat{u}(j)|^2\right)^{\frac{1}{2}}$$

if we apply the inequality (2.7) to $u = G_\tau f(x_1, \cdot)$, we get for almost every $x_1 \in I$

$$\|G_\tau f(x_1, \cdot)\|_{L^{\frac{2n}{n-2}}(M_0)} \lesssim \sum_{k=0}^{\infty}(1+k)^{\frac{1}{2}-\frac{1}{n}}$$

$$\times \left(\sum_{k \leq \sqrt{\lambda_j} < k+1}\left|\int_{-\infty}^{\infty} m_\tau\left(x_1 - y_1, \sqrt{\lambda_j}\right)\widehat{f}(y_1, j)\,\mathrm{d}y_1\right|^2\right)^{\frac{1}{2}}.$$

By Minkowski's inequality, we have

$$\|G_\tau f(x_1, \cdot)\|_{L^{\frac{2n}{n-2}}(M_0)} \lesssim \sum_{k=0}^{\infty} (1+k)^{\frac{1}{2}-\frac{1}{n}}$$

$$\times \int_{-\infty}^{\infty} \left(\sum_{k \leq \sqrt{\lambda_j} < k+1} \left| m_\tau\big(x_1 - y_1, \sqrt{\lambda_j}\big) \widehat{f}(y_1, j) \right|^2 \right)^{\frac{1}{2}} \mathrm{d}y_1$$

and since

$$\sum_{k \leq \sqrt{\lambda_j} < k+1} \left| m_\tau\big(x_1 - y_1, \sqrt{\lambda_j}\big) \widehat{f}(y_1, j) \right|^2$$

$$\leq \sup_{k \leq \sqrt{\lambda_j} < k+1} \left| m_\tau(x_1 - y_1, \sqrt{\lambda_j}) \right|^2 \sum_{k \leq \sqrt{\lambda_j} < k+1} \left| \widehat{f}(y_1, j) \right|^2$$

$$\leq \sup_{k \leq \sqrt{\lambda_j} < k+1} \left| m_\tau(x_1 - y_1, \sqrt{\lambda_j}) \right|^2 \times \|\chi_k f(y_1, \cdot)\|_{L^2(M_0)}^2$$

using once again the spectral cluster estimate (2.4), we finally get

$$(2.8) \quad \|G_\tau f(x_1, \cdot)\|_{L^{\frac{2n}{n-2}}(M_0)} \lesssim \sum_{k=0}^{\infty} (1+k)^{1-\frac{2}{n}}$$

$$\times \int_{-\infty}^{\infty} \sup_{k \leq \sqrt{\lambda_j} < k+1} \left| m_\tau(x_1 - y_1, \sqrt{\lambda_j}) \right| \times \|f(y_1, \cdot)\|_{L^{\frac{2n}{n+2}}(M_0)} \mathrm{d}y_1.$$

Using Lemma 2.3, we estimate

$$\sup_{k \leq \sqrt{\lambda_j} < k+1} \left| m_\tau(t, \sqrt{\lambda_j}) \right| \leq \frac{1}{k} \begin{cases} \mathrm{e}^{-(k-\tau)|t|} & \text{when } \tau < k \\ 1 & \text{when } k \leq \tau < k+1 \\ \mathrm{e}^{-(\tau-k-1)|t|} & \text{when } \tau \geq k+1 \end{cases}$$

with $k > 0$. (Note that when $k = 0$, a majorant is $\mathrm{e}^{-(\tau/2)|t|}$ for $\tau \geq 4$). This allows us to estimate the series

$$\sum_{k=0}^{\infty} (1+k)^{1-\frac{2}{n}} \sup_{k \leq \sqrt{\lambda_j} < k+1} \left| m_\tau(t, \sqrt{\lambda_j}) \right|$$

$$\lesssim \sum_{1 \leq k \leq \tau-2} k^{-\frac{2}{n}} \mathrm{e}^{-(\tau-k-1)|t|} + \tau^{-\frac{2}{n}} + \sum_{k > \tau+1} k^{-\frac{2}{n}} \mathrm{e}^{-(k-\tau)|t|} + \mathrm{e}^{-(\tau/2)|t|}$$

$$\lesssim \int_0^{\tau-2} r^{-\frac{2}{n}} \mathrm{e}^{-(\tau-r-2)|t|} \mathrm{d}r + 1 + \int_\tau^{\infty} r^{-\frac{2}{n}} \mathrm{e}^{-(r-\tau)|t|} \mathrm{d}r.$$

By an obvious change of variables we have

$$\int_0^{\tau-2} r^{-\frac{2}{n}} e^{-(\tau-r-2)|t|} \, dr + \int_\tau^\infty r^{-\frac{2}{n}} e^{-(r-\tau)|t|} \, dr$$

$$\leq 2|t|^{-1+\frac{2}{n}} \left(\int_0^1 r^{-\frac{2}{n}} \, dr + \int_1^\infty e^{-s} \, ds \right) \lesssim |t|^{-1+\frac{2}{n}}$$

whence

$$(2.9) \qquad \sum_{k=0}^\infty (1+k)^{1-\frac{2}{n}} \sup_{k \leq \sqrt{\lambda_j} < k+1} \left| m_\tau(t, \sqrt{\lambda_j}) \right| \lesssim 1 + |t|^{-1+\frac{2}{n}}.$$

From the estimates (2.8) and (2.9), we obtain

$$\|G_\tau f(x_1, \cdot)\|_{L^{\frac{2n}{n-2}}(M_0)}$$

$$\lesssim \int_{-\infty}^\infty \left(1 + |x_1 - y_1|^{-1+\frac{2}{n}} \right) \|f(y_1, \cdot)\|_{L^{\frac{2n}{n+2}}(M_0)} \, dy_1$$

$$\lesssim \int_{-\infty}^\infty |x_1 - y_1|^{-1+\frac{2}{n}} \|f(y_1, \cdot)\|_{L^{\frac{2n}{n+2}}(M_0)} \, dy_1 + |I|^{\frac{1}{2}-\frac{1}{n}} \|f\|_{L^{\frac{2n}{n+2}}(I \times M_0)}$$

and we conclude using the Hardy-Littlewood-Sobolev inequality

$$\|G_\tau f\|_{L^{\frac{2n}{n-2}}(I \times M_0)} \lesssim \|f\|_{L^{\frac{2n}{n+2}}(I \times M_0)}.$$

This completes the proof of Proposition 2.1. $\qquad\qquad\square$

3. Complex geometrical optics

In this section we will construct the complex geometrical optics solutions that will be used to recover an $L^{n/2}$ potential. Throughout the section, let (M, g) be a compact manifold with smooth boundary, and let $(M, g) \Subset (T, g) \Subset (\tilde{T}, g)$ where $T = \mathbf{R} \times M_0$, $\tilde{T} = \mathbf{R} \times \tilde{M}_0$, and $g = e \oplus g_0$, and $(M_0, g_0) \Subset (\tilde{M}_0, g_0)$ are two $(n-1)$-dimensional simple manifolds. We also assume that $n \geq 3$.

First we state a consequence of Proposition 2.1 for the manifold M (this follows easily by embedding $(\tilde{M}_0, g_0)$ in some compact manifold without boundary and using suitable restrictions and extensions by zero).

Proposition 3.1. *For $|\tau| \geq 4$ outside a countable set, there is a linear operator $G_\tau : L^2(M) \to H^2(M)$ such that*

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} G_\tau v = v \quad for \; v \in L^2(M),$$
$$G_\tau e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} v = v \quad for \; v \in C_0^\infty(M^{int}).$$

This operator satisfies

$$\|G_\tau f\|_{L^2(M)} \leq \frac{C_0}{|\tau|}\|f\|_{L^2(M)},$$

$$\|G_\tau f\|_{H^1(M)} \leq C_0\|f\|_{L^2(M)},$$

$$\|G_\tau f\|_{L^{\frac{2n}{n-2}}(M)} \leq C_0\|f\|_{L^{\frac{2n}{n+2}}(M)},$$

where C_0 is independent of τ.

Let us first construct the required complex geometrical optics solutions for the case where no potential is present. This is analogous to [4, Proposition 5.1] for $q = 0$.

Proposition 3.2. *Let $\omega \in \tilde{M}_0 \setminus M_0$ be a fixed point, let $\lambda \in \mathbf{R}$ be fixed, and let $b \in C^\infty(S^{n-2})$ be a function. Write $x = (x_1, r, \theta)$ where (r, θ) are polar normal coordinates with center ω in $(\tilde{M}_0, g_0)$. For $|\tau|$ sufficiently large outside a countable set, there exists $u_0 \in H^1(M)$ satisfying*

$$-\Delta_g u_0 = 0 \quad in\ M,$$

$$u_0 = e^{-\tau x_1}(e^{-i\tau r}|g|^{-1/4}e^{i\lambda(x_1+ir)}b(\theta) + r_0)$$

where r_0 satisfies

$$|\tau|\|r_0\|_{L^2(M)} + \|r_0\|_{H^1(M)} + \|r_0\|_{L^{\frac{2n}{n-2}}(M)} \lesssim 1.$$

Proof. The claim follows if one can find r_0 satisfying

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}r_0 = f$$

with the required norm estimates, where

$$f = e^{\tau x_1}\Delta_g e^{-\tau x_1}(e^{-i\tau r}e^{i\lambda(x_1+ir)}b(\theta)).$$

It is enough to take $r_0 = G_\tau f$ and to note that

$$\|f\|_{L^2(M)} = \|\Delta_g(e^{i\lambda(x_1+ir)}b(\theta))\|_{L^2(M)} \lesssim 1.$$

The L^2 and H^1 estimates follow from Proposition 3.1. The $L^{\frac{2n}{n-2}}$ estimate follows from the H^1 estimate and Sobolev embedding, or alternatively from the $L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}$ estimate for G_τ. $\square$

We next consider the case with a potential $q \in L^{n/2}(M)$, and try to find a solution to $(-\Delta_g + q)u = 0$ in M of the form

$$u = u_0 + e^{-\tau x_1}r_1.$$

Since $-\Delta_g u_0 = 0$, the function r_1 should satisfy

(3.1) $$e^{\tau x_1}(-\Delta_g + q)e^{-\tau x_1}r_1 = -qe^{\tau x_1}u_0.$$

Since q is only in $L^{n/2}(M)$, here we need to use the $L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}$ estimates for G_τ. We follow the argument of Lavine and Nachman [17]. It will be convenient to symmetrize the situation slightly. Later on, the L^n functions m_j in the next lemma are chosen to be essentially $|q|^{1/2}$.

Lemma 3.3. *Let $m_1, m_2 \in L^n(M)$ be two fixed functions. Then for $|\tau| \geq \tau_0$ outside a countable set,*

$$(3.2) \qquad \|m_1 G_\tau m_2 f\|_{L^2} \leq C_0 \|m_1\|_{L^n} \|m_2\|_{L^n} \|f\|_{L^2}.$$

Further,

$$(3.3) \qquad \|m_1 G_\tau m_2 f\|_{L^2(M) \to L^2(M)} \to 0$$

as $|\tau| \to \infty$.

Proof. The Hölder inequality and Proposition 3.1 imply that

$$\|m_1 G_\tau m_2 f\|_{L^2} \leq \|m_1\|_{L^n} \|G_\tau m_2 f\|_{L^{\frac{2n}{n-2}}} \leq C_0 \|m_1\|_{L^n} \|m_2 f\|_{L^{\frac{2n}{n+2}}}$$
$$\leq C_0 \|m_1\|_{L^n} \|m_2\|_{L^n} \|f\|_{L^2}.$$

Let $\varepsilon > 0$ and decompose $m_j = m_j^\sharp + m_j^\flat$ where $m_j^\sharp \in L^\infty(M)$,

$$\|m_j^\sharp\|_{L^n} \leq \|m_j\|_{L^n}$$
$$\text{and} \quad \|m_j^\flat\|_{L^n} \leq \frac{\varepsilon}{3C_0 \max(\|m_1\|_{L^n}, \|m_2\|_{L^n})}.$$

(One can take for instance $m_j^\sharp = m_j \chi_{\{|m_j| \leq \mu\}}$ for large enough μ.) It follows from the L^2 estimates for G_τ and (3.2) that

$$\|m_1 G_\tau m_2 f\|_{L^2} \leq \|m_1^\sharp G_\tau m_2^\sharp f\|_{L^2} + \|m_1^\sharp G_\tau m_2^\flat f\|_{L^2} + \|m_1^\flat G_\tau m_2 f\|_{L^2}$$
$$\leq \left(\frac{C_0 \|m_1^\sharp\|_{L^\infty} \|m_2^\sharp\|_{L^\infty}}{|\tau|} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \right) \|f\|_{L^2}.$$

The last expression is bounded by $\varepsilon \|f\|_{L^2}$ if $|\tau|$ is sufficiently large. This proves (3.3). $\square$

We now finish the construction of complex geometrical optics solutions.

Proposition 3.4. *Assume that $q \in L^{n/2}(M)$. Let $\omega \in \tilde{M}_0 \setminus M_0$ be a fixed point, let $\lambda \in \mathbf{R}$ be fixed, and let $b \in C^\infty(S^{n-2})$ be a function. Write $x = (x_1, r, \theta)$ where (r, θ) are polar normal coordinates with center ω in $(\tilde{M}_0, g_0)$. For $|\tau|$ sufficiently large outside a countable set, there exists a solution $u \in H^1(M)$ of $(-\Delta_g + q)u = 0$ in M of the form*

$$u = e^{-\tau x_1}(e^{-i\tau r}|g|^{-1/4} e^{i\lambda(x_1+ir)} b(\theta) + \tilde{r})$$

where $\tilde{r}$ satisfies

$$\|\tilde{r}\|_{L^{\frac{2n}{n-2}}(M)} \lesssim 1, \qquad \|\tilde{r}\|_{L^2(M)} \to 0 \ as \ |\tau| \to \infty.$$

Proof. As explained above, we let u_0 be the harmonic function given in Proposition 3.2, and look for a solution of the form $u = u_0 + e^{-\tau x_1}r_1$. We write $q(x) = |q(x)|e^{i\alpha(x)} = |q(x)|^{1/2}m(x)$ where $m(x) = |q(x)|^{1/2}e^{i\alpha(x)}$. Then $|q|^{1/2}, m \in L^n(M)$ with L^n norms equal to $\|q\|_{L^{n/2}}^{1/2}$.

We obtain a solution u provided that (3.1) holds. Trying r_1 in the form $r_1 = G_\tau|q|^{1/2}v$, we see that v should satisfy

$$(\mathrm{Id} + mG_\tau|q|^{1/2})v = -me^{\tau x_1}u_0.$$

By Lemma 3.3, for $|\tau|$ sufficiently large one has $\|mG_\tau|q|^{1/2}\|_{L^2 \to L^2} \leq 1/2$. One then obtains a solution

$$v = -(\mathrm{Id} + mG_\tau|q|^{1/2})^{-1}(me^{\tau x_1}u_0).$$

Since $\|me^{\tau x_1}u_0\|_{L^2} \leq \|m\|_{L^n}\|e^{\tau x_1}u_0\|_{L^{\frac{2n}{n-2}}} \lesssim 1$, it follows that $\|v\|_{L^2} \lesssim 1$. Consequently

$$\|r_1\|_{L^{\frac{2n}{n-2}}} \leq C_0\||q|^{1/2}v\|_{L^{\frac{2n}{n+2}}} \lesssim 1.$$

Now u is of the form given in the statement of the proposition, provided that

$$\tilde{r} = r_0 + r_1.$$

This remainder term satisfies $\|\tilde{r}\|_{L^{\frac{2n}{n-2}}} \lesssim 1$.

To study $\|\tilde{r}\|_{L^2}$ we fix $\varepsilon > 0$ and make a decomposition $|q|^{1/2} = s^\sharp + s^\flat$ where $s^\sharp \in L^\infty(M)$, $\|s^\sharp\|_{L^n} \leq \|q\|_{L^{n/2}}^{1/2}$, and $\|s^\flat\|_{L^n} \leq \varepsilon$. Then

$$\|r_1\|_{L^2} \leq \|G_\tau s^\sharp v\|_{L^2} + C_1\|G_\tau s^\flat v\|_{L^{\frac{2n}{n-2}}}$$
$$\leq \left(\frac{C_0\|s^\sharp\|_{L^\infty}}{|\tau|} + C_0C_1\|s^\flat\|_{L^n}\right)\|v\|_{L^2}.$$

Choosing $|\tau|$ sufficiently large, we see that $\|r_1\|_{L^2} \lesssim \varepsilon$ for $|\tau|$ large. Since also $\|r_0\|_{L^2(M)} \lesssim |\tau|^{-1}$, it follows that $\|\tilde{r}\|_{L^2} \to 0$ as $|\tau| \to \infty$.

Finally, to prove that $u \in H^1(M)$, it is enough to consider a compact manifold $(\hat{M}, g)$ which is slightly larger than (M, g) and extend q by zero outside M, and to perform the above construction of solutions in $\hat{M}$. One obtains a solution $u \in L^{\frac{2n}{n-2}}(\hat{M}) \subseteq L^2(\hat{M})$, and $\Delta_g u = qu \in L^{\frac{2n}{n+2}}(\hat{M}) \subseteq H^{-1}(\hat{M})$ by Sobolev embedding. Elliptic regularity implies that $u \in H^1_{\mathrm{loc}}(\hat{M}^{\mathrm{int}})$, thus also $u \in H^1(M)$. $\qquad\square$

4. RECOVERING THE POTENTIAL

We are now ready to give the proof of the main uniqueness result.

Proof of Theorem 1.1. Assume, as before, that $(M, g) \Subset (T, g) \Subset (\tilde{T}, g)$ where $T = \mathbf{R} \times M_0$, $\tilde{T} = \mathbf{R} \times \tilde{M}_0$, and $(M_0, g_0) \Subset (\tilde{M}_0, g_0)$ are two $(n-1)$-dimensional simple manifolds where $n \geq 3$. Also assume that $g = e \oplus g_0$.

From the assumption $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, writing $q = q_1 - q_2$, we know from Lemma A.1 that

$$(4.1) \qquad \int_M q u_1 u_2 \, \mathrm{d}V_g = 0$$

where $u_1, u_2 \in H^1(M)$ are solutions of $(-\Delta_g + q_1)u_1 = 0$ in M and $(-\Delta_g + q_2)u_2 = 0$ in M. By Proposition 3.4, for τ sufficiently large outside a countable set there exist solutions u_j of the form

$$u_1 = \mathrm{e}^{-\tau(x_1+ir)}(|g|^{-1/4}\mathrm{e}^{i\lambda(x_1+ir)}b(\theta) + r_1),$$
$$u_2 = \mathrm{e}^{\tau(x_1+ir)}(|g|^{-1/4} + r_2).$$

Here λ is a fixed real number, $b \in C^\infty(S^{n-2})$ is a fixed function, and $x = (x_1, r, \theta)$ are coordinates in $\tilde{T}$ where (r, θ) are polar normal coordinates in $(\tilde{M}_0, g_0)$ with center at a fixed point $\omega \in \tilde{M}_0 \setminus M_0$. Also, the remainder terms satisfy

$$\|r_j\|_{L^{\frac{2n}{n-2}}(M)} = O(1), \quad \|r_j\|_{L^2(M)} = o(1)$$

as $\tau \to \infty$.

Inserting the solutions in (4.1) and noting that $\mathrm{d}V_g = |g|^{1/2}\mathrm{d}x_1 \, \mathrm{d}r \, \mathrm{d}\theta$, we obtain that

$$(4.2) \qquad \int_M q\mathrm{e}^{i\lambda(x_1+ir)}b(\theta) \, \mathrm{d}x_1 \, \mathrm{d}r \, \mathrm{d}\theta = \int_M q(a_1 r_2 + a_2 r_1 + r_1 r_2) \, \mathrm{d}V$$

where a_1, a_2 are smooth functions in M independent of τ. We show that the right hand side converges to 0 as $\tau \to \infty$. To do this, fix $\varepsilon > 0$ and write $q = q^\sharp + q^\flat$ where $q^\sharp \in L^\infty(M)$, $\|q^\sharp\|_{L^{n/2}} \leq \|q\|_{L^{n/2}}$, and $\|q^\flat\|_{L^{n/2}} \leq \varepsilon$. The right hand side of (4.2) is bounded by

$$\left| \int_M q(a_1 r_2 + a_2 r_1 + r_1 r_2) \, \mathrm{d}V \right|$$
$$\lesssim \|q^\sharp\|_{L^\infty}(\|r_1\|_{L^2} + \|r_2\|_{L^2} + \|r_1\|_{L^2}\|r_2\|_{L^2})$$
$$+ \|q^\flat\|_{L^{n/2}}(\|r_1\|_{L^{\frac{2n}{n-2}}} + \|r_2\|_{L^{\frac{2n}{n-2}}} + \|r_1\|_{L^{\frac{2n}{n-2}}}\|r_2\|_{L^{\frac{2n}{n-2}}}).$$

Using the bounds for r_j, if τ is sufficiently large then the last quantity is $\lesssim \varepsilon$. This shows that the right hand side of (4.2) goes to 0 as $\tau \to \infty$.

Extend q by zero into T and interpret the left hand side of (4.2) as an integral over T. Taking the limit as $\tau \to \infty$, we obtain that

$$\int_{-\infty}^{\infty} \int_0^{\infty} \int_{S^{n-2}} q(x_1, r, \theta) e^{i\lambda(x_1 + ir)} b(\theta) \, dx_1 \, dr \, d\theta = 0.$$

This statement is true for all choices of $\omega \in \tilde{M}_0 \backslash M_0$, for all real numbers λ, and for all functions $b \in C^{\infty}(S^{n-2})$. Since $q \in L^1(M)$, Fubini's theorem shows that $q(\cdot, r, \theta)$ is in $L^1(\mathbf{R})$ for a.e. (r, θ). Consequently

$$(4.3) \qquad \int_{S^{n-2}} \int_0^{\infty} f_{\lambda}(r, \theta) e^{-\lambda r} b(\theta) \, dr \, d\theta = 0$$

where $f_{\lambda} \in L^1(M_0)$ is the function given by

$$f_{\lambda}(r, \theta) = \int_{-\infty}^{\infty} e^{i\lambda x_1} q(x_1, r, \theta) \, dx_1.$$

If $|\lambda|$ is sufficiently small, it follows from Lemma 5.1 below that the vanishing of the integrals (4.3) for all choices ω and b implies that $f_{\lambda} = 0$. Since $q(\cdot, r, \theta)$ is a compactly supported function in $L^1(\mathbf{R})$ for a.e. (r, θ), the Paley-Wiener theorem shows that $q(\cdot, r, \theta) = 0$ for such (r, θ). Consequently $q_1 = q_2$. $\qquad \square$

5. Attenuated ray transform

It remains to show the following lemma which was used in the proof of Theorem 1.1.

Lemma 5.1. *Let (M_0, g_0) be an $(n-1)$-dimensional simple manifold, and let $f \in L^1(M_0)$. Consider the integrals*

$$\int_{S^{n-2}} \int_0^{\tau(\omega, \theta)} f(r, \theta) e^{-\lambda r} b(\theta) \, dr \, d\theta$$

where (r, θ) are polar normal coordinates in (M_0, g_0) centered at some $\omega \in \partial M_0$, and $\tau(\omega, \theta)$ is the time when the geodesic $r \mapsto (r, \theta)$ exits M_0. If $|\lambda|$ is sufficiently small, and if these integrals vanish for all $\omega \in \partial M_0$ and all $b \in C^{\infty}(S^{n-2})$, then $f = 0$.

The last result is related to the vanishing of the attenuated geodesic ray transform of the function f on M_0. For the following facts see [3], [20], [22]. To define the ray transform, we consider the unit sphere bundle

$$SM_0 = \bigcup_{x \in M_0} S_x, \qquad S_x = \left\{ (x, \xi) \in T_x M_0 \, ; \, |\xi| = 1 \right\}.$$

This manifold has boundary $\partial(SM_0) = \{(x,\xi) \in SM_0\,;\, x \in \partial M_0\}$ which is the union of the sets of inward and outward pointing vectors,

$$\partial_\pm(SM_0) = \{(x,\xi) \in SM_0\,;\, \pm\langle \xi, \nu \rangle \leq 0\}.$$

Here ν is the outer unit normal vector to ∂M_0. Note that $\partial_+(SM_0)$ is a manifold whose boundary consists of all the tangential directions $\{(x,\xi) \in \partial(SM_0)\,;\, \langle \xi, \nu \rangle = 0\}$. Thus the space $C_0^\infty((\partial_+(SM_0))^{\mathrm{int}})$ contains all smooth functions on $\partial_+(SM_0)$ vanishing near tangential directions.

Denote by $t \mapsto \gamma(t,x,\xi)$ the unit speed geodesic starting at x in direction ξ, and let $\tau(x,\xi)$ be the time when this geodesic exits M_0. Since (M_0, g_0) is simple, $\tau(x,\xi)$ is finite for each $(x,\xi) \in SM_0$. We also write $\varphi_t(x,\xi) = (\gamma(t,x,\xi), \dot\gamma(t,x,\xi))$ for the geodesic flow.

The geodesic ray transform, with constant attenuation $-\lambda$, acts on functions on M_0 by

$$T_\lambda f(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi)) \mathrm{e}^{-\lambda t}\, \mathrm{d}t, \qquad (x,\xi) \in \partial_+(SM_0).$$

In Lemma 5.1, if f were a continuous function, one could choose $b(\theta)$ to approximate a delta function at fixed angles θ and obtain that

$$\int_0^{\tau(\omega,\theta)} f(r,\theta)\mathrm{e}^{-\lambda r}\, \mathrm{d}r = 0$$

for any $\omega \in \partial M_0$ and any $\theta \in S^{n-2}$. Since (r,θ) are polar normal coordinates the curves $r \mapsto (r,\theta)$ are geodesics in (M_0, g_0), and this would imply that

$$T_\lambda f(x,\xi) = 0 \qquad \text{for all } (x,\xi) \in \partial_+(SM_0).$$

One has the following injectivity result from [4, Theorem 7.1]. (If M_0 is two-dimensional the smallness assumption on the attenuation coefficient was recently removed in [21].)

Proposition 5.2. *Let (M_0, g_0) be a simple manifold. There exists $\varepsilon > 0$ such that if λ is a real number with $|\lambda| < \varepsilon$ and if $f \in C^\infty(M)$, then the condition $T_\lambda f(x,\xi) = 0$ for all $(x,\xi) \in \partial_+(SM_0)$ implies that $f = 0$.*

The previous argument together with Proposition 5.2 proves Lemma 5.1 for smooth f. However, this requires well defined restrictions of f to all geodesics and it is not obvious how to do this when $f \in L^1$. We circumvent this problem by using duality and the ellipticity of the normal operator $T_\lambda^* T_\lambda$.

We will need a few facts. Below we write

$$h_\psi(x,\xi) = h(\varphi_{-\tau(x,-\xi)}(x,\xi)), \quad (x,\xi) \in SM_0$$

for $h \in C^\infty(\partial_+(SM_0))$, and

$$(h,\tilde{h})_{L^2_\mu(\partial_+(SM_0))} = \int_{\partial_+(SM_0)} h\tilde{h}\mu\, \mathrm{d}(\partial(SM_0))$$

where $\mu(x,\xi) = -\langle \xi, \nu(x)\rangle$ and $\mathrm{d}N$ is the natural Riemannian volume form on a manifold N.

Lemma 5.3. *(Santaló formula) If $F : SM_0 \to \mathbf{R}$ is continuous then*

$$\int_{SM_0} F\, \mathrm{d}(SM_0)$$
$$= \int_{\partial_+(SM_0)} \int_0^{\tau(x,\xi)} F(\varphi_t(x,\xi))\mu(x,\xi)\, \mathrm{d}t\, \mathrm{d}(\partial(SM_0))(x,\xi).$$

Proof. See [3, Lemma A.8]. $\qquad\square$

Lemma 5.4. *If $f \in C^\infty(M_0)$ and $h \in C_0^\infty((\partial_+(SM_0))^{int})$ then*

$$(T_\lambda f, h)_{L^2_\mu(\partial_+(SM_0))} = (f, T_\lambda^* h)_{L^2(M_0)}$$

where

$$T_\lambda^* h(x) = \int_{S_x} \mathrm{e}^{-\lambda\tau(x,-\xi)} h_\psi(x,\xi)\, \mathrm{d}S_x(\xi), \qquad x \in M_0.$$

Proof. By the Santaló formula

$$(T_\lambda f, h)_{L^2_\mu(\partial_+(SM_0))}$$
$$= \int_{\partial_+(SM_0)} \int_0^{\tau(x,\xi)} \mathrm{e}^{-\lambda t} f(\gamma(t,x,\xi))h\mu\, \mathrm{d}t\, \mathrm{d}(\partial(SM_0))$$
$$= \int_{\partial_+(SM_0)} \int_0^{\tau(x,\xi)} \mathrm{e}^{-\lambda t} f(\varphi_t(x,\xi))h_\psi(\varphi_t(x,\xi))\mu\, \mathrm{d}t\, \mathrm{d}(\partial(SM_0))$$
$$= \int_{SM_0} \mathrm{e}^{-\lambda\tau(x,-\xi)} f(x)h_\psi(x,\xi)\, \mathrm{d}(SM_0)$$
$$= \int_{M_0} f(x)\left(\int_{S_x} \mathrm{e}^{-\lambda\tau(x,-\xi)} h_\psi(x,\xi)\, \mathrm{d}S_x(\xi)\right) \mathrm{d}V(x).$$

This proves Lemma 5.4. $\qquad\square$

Lemma 5.5. *$T_\lambda^* T_\lambda$ is a self-adjoint elliptic pseudodifferential operator of order -1 in M_0^{int}.*

Proof. This is contained in [6, Proposition 2], but for completeness we also include a proof in Appendix B. $\qquad\square$

Proof of Lemma 5.1. The first step is to extend (M_0, g_0) to a slightly larger simple manifold and to extend f by zero. In this way we can assume that f is compactly supported in M_0^{int}.

We let b also depend on ω and change notations to write the assumption in the lemma in the form

$$\int_{S_x} \int_0^{\tau(x,\xi)} e^{-\lambda t} f(\gamma(t,x,\xi)) b(x,\xi) \, dt \, dS_x(\xi) = 0$$

for all $x \in \partial M_0$ and $b \in C_0^\infty((\partial_+(SM_0))^{\text{int}})$. Next we make the choice $b(x,\xi) = h(x,\xi)\mu(x,\xi)$ for $h \in C_0^\infty((\partial_+(SM_0))^{\text{int}})$ and integrate the last identity over ∂M_0 to obtain

$$\int_{\partial_+(SM_0)} \int_0^{\tau(x,\xi)} e^{-\lambda t} f(\gamma(t,x,\xi)) h(x,\xi) \mu \, dt \, d(\partial(SM_0)) = 0.$$

We are now in the same situation as in the proof of Lemma 5.4, and invoking the Santaló formula implies

$$\int_{M_0} f(x) T_\lambda^* h(x) \, dV(x) = 0$$

for all $h \in C_0^\infty((\partial_+(SM_0))^{\text{int}})$. Note that the last integral is absolutely convergent because $f \in L^1(M_0)$, and also the previous steps are justified by Fubini's theorem.

It remains to choose $h = T_\lambda \varphi$ for $\varphi \in C_0^\infty(M_0^{\text{int}})$ to obtain that

$$\int_{M_0} f(x) T_\lambda^* T_\lambda \varphi(x) \, dV(x) = 0.$$

Since $T_\lambda^* T_\lambda$ is self-adjoint, we have

$$\int_{M_0} (T_\lambda^* T_\lambda f(x)) \varphi(x) \, dV(x) = 0.$$

This is valid for all test functions φ, so $T_\lambda^* T_\lambda f = 0$. By ellipticity, since f was compactly supported in M_0^{int}, it follows that $f \in C_0^\infty(M_0^{\text{int}})$. One can now use the argument for smooth f given above, together with the injectivity result (Proposition 5.2), to conclude the proof that $f = 0$. $\qquad\square$

Appendix A. Wellposedness

Here we recall the standard arguments that show wellposedness of the Dirichlet problem for $-\Delta_g + q$ on a compact oriented manifold (M, g) with smooth boundary and with $q \in L^{n/2}(M)$, $n \geq 3$. Consider first the inhomogeneous problem for the Schrödinger equation,

$$(A.1) \qquad\qquad (-\Delta_g + q)u = F \text{ in } M, \quad u|_{\partial M} = 0.$$

The bilinear form related to this problem is

$$B(u, v) = \int_M \left(\langle \mathrm{d}u, \mathrm{d}\bar{v} \rangle + qu\bar{v} \right) \mathrm{d}V, \quad u, v \in H_0^1(M),$$

where $\langle \cdot , \cdot \rangle$ is the complex-linear inner product of 1-forms and $\mathrm{d}V$ is the volume form on (M, g). By the Sobolev embedding $H^1(M) \subseteq L^{\frac{2n}{n-2}}(M)$ and by Hölder's inequality, B is a bounded bilinear form on $H_0^1(M)$. Writing $q = q^\sharp + q^\flat$ where $q^\sharp \in L^\infty(M)$ and $\|q^\flat\|_{L^{n/2}(M)}$ is small, we obtain from Poincaré's inequality that

$$B(u, u) \geq c\|u\|_{H^1(M)}^2 - C\|u\|_{L^2(M)}^2, \quad u \in H_0^1(M).$$

This shows that $B + C$ is coercive, and by the Lax-Milgram lemma, compact Sobolev embedding and the Fredholm theorem, the equation (A.1) has a unique solution $u \in H_0^1(M)$ for any $F \in H^{-1}(M)$ if one is outside a countable set of eigenvalues.

We can now consider the Dirichlet problem

(A.2) $$(-\Delta_g + q)u = 0 \text{ in } M, \quad u|_{\partial M} = f.$$

We assume that 0 is not a Dirichlet eigenvalue, and it follows from the above discussion that for any $f \in H^{1/2}(\partial M)$ there is a unique solution $u \in H^1(M)$. The DN map is formally defined as the map

$$\Lambda_{g,q} : H^{1/2}(\partial M) \to H^{-1/2}(\partial M)$$

$$f \mapsto \partial_\nu u|_{\partial M}.$$

More precisely, if $f \in H^{1/2}(\partial M)$ we define $\Lambda_{g,q}f$ weakly as the function in $H^{-1/2}(\partial M)$ which satisfies

$$\int_{\partial M} (\Lambda_{g,q}f)\bar{h} \, \mathrm{d}S = \int_M \left(\langle \mathrm{d}u, \mathrm{d}\bar{v} \rangle + qu\bar{v} \right) \mathrm{d}V$$

where u is the unique solution of (A.2), and v is any extension in $H^1(M)$ of h ($v|_{\partial M} = h$). Then $\Lambda_{g,q}$ is a bounded map $H^{1/2}(\partial M) \to H^{-1/2}(\partial M)$ again by Hölder and Sobolev embedding.

The DN map satisfies in the weak sense

$$\int_{\partial M} (\Lambda_{g,q}f)\bar{h} \, \mathrm{d}S = \int_{\partial M} f \overline{\Lambda_{g,\bar{q}}h} \, \mathrm{d}S.$$

To see this, let $u, v \in H^1(M)$ solve $(-\Delta_g + q)u = 0$, $u|_{\partial M} = f$ and $(-\Delta_g + \bar{q})v = 0$, $v|_{\partial M} = h$. Then

$$\int_{\partial M} (\Lambda_{g,q}f)\bar{h} \, \mathrm{d}S = \int_M \left(\langle \mathrm{d}u, \mathrm{d}\bar{v} \rangle + qu\bar{v} \right) \mathrm{d}V$$

$$= \overline{\int_M \left(\langle \mathrm{d}v, \mathrm{d}\bar{u} \rangle + \bar{q}v\bar{u} \right) \mathrm{d}V} = \int_{\partial M} \overline{(\Lambda_{g,\bar{q}}h)} f \, \mathrm{d}S.$$

As a consequence, we have the basic integral identity used in the uniqueness proof.

Lemma A.1. *If $q_1, q_2 \in L^{n/2}(M)$ and $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then*

$$\int_M (q_1 - q_2) u_1 u_2 \, \mathrm{d}V = 0$$

for any $u_j \in H^1(M)$ with $(-\Delta_g + q_1)u_1 = 0$ in M, $(-\Delta_g + q_2)u_2 = 0$ in M.

Proof. Follows from the computation

$$
\begin{aligned}
0 &= \int_{\partial M} (\Lambda_{g,q_1} - \Lambda_{g,q_2})(u_1|_{\partial M}) u_2 \, \mathrm{d}S \\
&= \int_{\partial M} \left(\Lambda_{g,q_1}(u_1|_{\partial M}) u_2 - u_1 \overline{\Lambda_{g,\bar{q}_2}(\bar{u}_2|_{\partial M})} \right) \mathrm{d}S
\end{aligned}
$$

and the definition of the DN maps. $\qquad\qquad\square$

Appendix B. Normal operator

The setting is the compact simple Riemannian manifold (M_0, g_0) of dimension $n - 1$. Let T_λ be the attenuated ray transform as in Section 5. We will prove Lemma 5.5. Write

$$\psi(x, \xi) = \varphi_{-\tau(x,-\xi)}(x, \xi).$$

We compute the normal operator $T_\lambda^* T_\lambda f$ for $f \in C_0^\infty(M_0^{\mathrm{int}})$

$$
\begin{aligned}
&T_\lambda^* T_\lambda f(x) \\
&= \int_{S_x} e^{-\lambda \tau(x,-\xi)} (T_\lambda f)_\psi(x, \xi) \, \mathrm{d}S_x(\xi) \\
&= \int_{S_x} e^{-\lambda \tau(x,-\xi)} \int_0^{\tau(\psi(x,\xi))} e^{-\lambda t} f(\gamma(t, \psi(x, \xi))) \, \mathrm{d}t \, \mathrm{d}S_x(\xi) \\
&= \int_{S_x} e^{-\lambda \tau(x,-\xi)} \int_0^{\tau(x,-\xi)+\tau(x,\xi)} e^{-\lambda t} f(\gamma(t, \psi(x, \xi))) \, \mathrm{d}t \, \mathrm{d}S_x(\xi)
\end{aligned}
$$

and using changes of variables we get for the last integral expression

$$T_\lambda^* T_\lambda f(x)$$

$$= \int_{S_x} e^{-\lambda \tau(x, -\xi)} \int_{-\tau(x, -\xi)}^{\tau(x, \xi)} e^{-\lambda(s + \tau(x, -\xi))} f(\gamma(s, x, \xi)) \, ds \, dS_x(\xi)$$

$$= \int_{S_x} \left[\int_{-\tau(x, -\xi)}^{0} + \int_{0}^{\tau(x, \xi)} \right] e^{-\lambda(s + 2\tau(x, -\xi))} f(\gamma(s, x, \xi)) \, ds \, dS_x(\xi)$$

$$= \int_{S_x} \int_{0}^{\tau(x, \xi)} \left[e^{-2\lambda \tau(x, -\xi)} e^{-\lambda s} + e^{-2\lambda \tau(x, \xi)} e^{\lambda s} \right] f(\gamma(s, x, \xi)) \, ds \, dS_x(\xi).$$

Changing variables $y = \exp_x(s\xi)$ shows that

$$T_\lambda^* T_\lambda f(x) = \int_{M_0} K_\lambda(x, y) f(y) \, dV(y)$$

where

$$K_\lambda(x, y) = \frac{(e^{-\lambda \varphi_+(x, y)} + e^{-\lambda \varphi_-(x, y)})}{d_{g_0}^{n-2}(x, y)} \left(\frac{\det g_0(x)}{\det g_0(y)} \right)^{\frac{1}{2}} |\det(\exp_x^{-1})'(x, y)|.$$

with

$$\varphi_\pm = 2\tau(x, \mp \mathrm{grad}_y \, d_{g_0}(x, y)) \pm d_{g_0}(x, y).$$

The functions $\varphi_\pm$ are smooth away from the diagonal $x = y$, and their k-th order derivatives behave as $d_{g_0}(x, y)^{-k}$. Note that $\det(\exp^{-1})'$ stands for the Jacobian determinant of

$$\exp^{-1} : M_0 \times M_0 \to \mathbf{R}^{n-1}$$

$$(x, y) \mapsto \exp_x^{-1}(y).$$

The kernel of the normal operator is symmetric

$$K_\lambda(y, x) = K_\lambda(x, y)$$

and the singular support of this kernel is the diagonal in $M_0 \times M_0$.

We will now prove that the operator P_λ with kernel K_λ is actually a pseudodifferential operator. The first observation in that direction is that in coordinates

(B.1) $$d_{g_0}^2(x, y) = a_{jk}(x, y)(x^j - y^j)(x^k - y^k)$$

with $a^{jk}(x, x) = g_0^{jk}(x)$. Indeed the square of the distance vanishes at second order and its Hessian at $x = y$ is twice the metric. This can be seen from the well known formula

$$\nabla^2 \varphi(y)(\theta, \theta) = \frac{\partial^2}{\partial t^2} \varphi(\exp_y t\theta) \Big|_{t=0}$$

and the fact that if $|\theta|_{g_0} = 1$ then $d_{g_0}^2(\exp_y t\theta, y) = t^2$. To prove that P_λ is a pseudodifferential operator in $\Psi^{-1}(M^{\text{int}})$ we need to show that for any couple of cutoff functions (ψ, χ) supported in charts of M^{int}, the operator with kernel

$$\tilde{K}_\lambda(x, y) = \psi(x) K_\lambda(x, y) \sqrt{\det g_0(y)} \chi(y)$$

expressed in coordinates[1], is a pseudodifferential operator on $\mathbf{R}^{n-1}$ with symbol in S^{-1}. Because of its form and of (B.1), the kernel satisfies

(B.2) $$|\partial_x^\alpha \partial_y^\beta \tilde{K}_\lambda(x, x - y)| \leq C_\alpha |y|^{-n+2-|\beta|}$$

and has compact support in $\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$.

Such operators are pseudodifferential operators and this can easily be seen in the following way: the symbol associated with such an operator is

$$\tilde{p}_\lambda(x, \xi) = \int \tilde{K}_\lambda(x, x - y) e^{-iy \cdot \xi} \, dy$$

For cutoff functions ψ and χ whose supports don't intersect, the previous symbol is a Schwartz function because the kernel is a smooth compactly supported function. So we are only interested in those symbols corresponding to kernels $\tilde{K}_\lambda(x, x - y)$ which are supported close to $\mathbf{R}^{n-1} \times \{0\}$. In that case, we use a dyadic partition of unity

$$1 = \sum_{\mu=-\infty}^{\infty} \chi(2^{-\mu} z),$$

with χ a function supported in an annulus, to decompose the symbol as a sum of terms of the form

$$2^{\mu(n-1)} \int e^{i2^\mu y \cdot \xi} \chi(y) \tilde{K}_\lambda(x, x - 2^\mu y) \, dy.$$

Note that because of the compact support of the kernel, these terms vanish when μ is large, so we are mainly concerned with the terms where μ is less than some positive integer, say N. Because of the behaviour (B.2), the rescaled kernel $\tilde{K}_\lambda(x, x - 2^\mu y)$ is uniformly bounded by $2^{-\mu(n-2)}$ as well as all its derivatives. Applying the non-stationary phase when $|\xi| \geq 1$ and $2^\mu \xi$ is large we get

$$|\tilde{p}_\lambda(x, \xi)| \lesssim \sum_{\mu \leq N, \, 2^\mu |\xi| \geq 1} 2^\mu (2^\mu |\xi|)^{-N} + \sum_{\mu \leq N, \, 2^\mu |\xi| \leq 1} 2^\mu$$

$$\lesssim (1 + |\xi|)^{-1}$$

---

[1] By a slight abuse of notations, to lighten the exposition, we don't write the pullback by the coordinates and think of x and y as variables in $\mathbf{R}^{n-1}$.

Repeating this argument for the derivatives of this function, we get that $\tilde{p}_\lambda$ is a classical symbol of order -1.

Let us concentrate on p_0: we have

$$\tilde{K}_0(x,y) = \psi(x)\frac{\det(\exp_x^{-1})'(x,y)}{d_{g_0}^{n-2}(x,y)}\sqrt{\det g_0(y)}\chi(y)$$

from the previous computation, we see that taking $x = y$ in the non-singular factors, yields error terms whose kernel are less singular by an order of $|x-y|$, i.e. errors with symbols of order $(1+|\xi|)^{-2}$. Therefore in terms of the principal symbol, it suffices to compute

$$\psi(x)\chi(x) \times \sqrt{\det g_0(x)}\int \frac{\mathrm{e}^{-iy\cdot\xi}}{|g_0(x)y\cdot y|^{\frac{n-2}{2}}}\,\mathrm{d}y$$
$$= c_n|g_0^{-1}(x)\xi\cdot\xi|^{-\frac{1}{2}}\psi(x)\chi(x).$$

Finally, these observations show that P_0 has as principal symbol a multiple of

$$|\xi|_{g_0}^{-1} = \frac{1}{\sqrt{g_0^{jk}(x)\xi_j\xi_k}}$$

and since the principal symbol of P_λ depends smoothly on λ, it doesn't vanish for λ small enough. This means that for λ small enough, P_λ is an elliptic self-adjoint pseudodifferential operator of order -1.

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Université Paris 13, Cnrs, Umr 7539 Laga, 99, avenue Jean-Baptiste Clément, F-93430 Villetaneuse, France
E-mail address: ddsf@math.univ-paris13.fr

Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637-1514, USA
E-mail address: cek@math.uchicago.edu

Department of Mathematics and Statistics, University of Helsinki, PO Box 68, 00014 Helsinki, Finland
E-mail address: mikko.salo@helsinki.fi