Convexity Recognition Using Multi-Scale Autoconvolution

Esa Rahtu[1], Mikko Salo[2] and Janne Heikkilä[1]

[1]Machine Vision Group
Department of Electrical Engineering
PO Box 4500, 90014 Univ. of Oulu, Finland
E-mail: erahtu@ee.oulu.fi, jth@ee.oulu.fi

[2]Rolf Nevanlinna Institute
Department of Mathematics
PO Box 4, 00014 Univ. of Helsinki, Finland
E-mail: msa@rni.helsinki.fi

Abstract

This paper introduces a novel measure for object convexity using the recently introduced Multi-Scale Autoconvolution transform. The proposed measure is computationally efficient and recognizes even small errors in a convex domain. We also consider its implementation and give a complete Matlab algorithm for computing this measure for digital images. Finally, we give examples to verify its applicability and accuracy. The examples also consider convexity as a measure for complexity.

1. Introduction

Object shape analysis is of great importance in many areas of computer vision and image analysis. One of the main concepts in this field is object convexity, and for instance in integral geometry and mathematical morphology it is one of the very essential properties. When operating with real digital images, convexity in a strict mathematical sense is rarely encountered. However, in many applications a certain amount of nonconvexity is allowed and it is desirable to find measures for the convexity of an object. Some methods, like approximate convexity measures [2] and fuzzy convexity measures [3], have been proposed for this kind of characterization.

In this paper, we propose a novel measure for object convexity, based on the Multi-Scale Autoconvolution (MSA) transform introduced in [1]. This affine invariant transform has many applications in object classification as discussed in [4], and it can also be used in analyzing properties such as convexity. It evolves, and we will give a mathematical proof of this fact, that the value of the MSA transform at a certain point distinguishes between convex and nonconvex sets. This value will be used as the convexity measure.

The rest of the paper considers the implementation of the convexity measure and contains examples which illustrate its properties. The MSA convexity measure is simple and efficient to compute, and we will give a complete Matlab program for doing this. In the examples, we will find this measure for several objects verifying that the measure behaves correctly. We will also consider the behaviour of the measure in the presence of noise or holes in a convex region, and consider its applicability as a measure of object complexity.

2. Multi-Scale Autoconvolution

Let f be an image intensity function in $L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$ with $f \geq 0$, and let X_0,X_1 and X_2 be independent random variables with values in $\mathbf{R}^2$ so that $P(X_j = x_j) = \frac{1}{\|f\|_{L^1}} f(x_j)$. For $\alpha, \beta \in \mathbf{R}$, define the random variable

$$U_{\alpha,\beta} = X_0 + \alpha(X_1 - X_0) + \beta(X_2 - X_0).$$

Then one has $U_{\alpha,\beta} = \alpha X_1 + \beta X_2 + \gamma X_0$, where $\gamma = 1 - \alpha - \beta$. Now it can be easily shown that $U_{\alpha,\beta}$ has a probability density function

$$P(U_{\alpha,\beta} = u) = \frac{1}{\|f\|_{L^1}^3} (f_\alpha * f_\beta * f_\gamma)(u) \tag{1}$$

where $f_a(x) = a^{-2} f(x/a)$ for $a \neq 0$ and $f_a = \|f\|_{L^1} \delta_0$ for $a = 0$ (the Dirac delta). For $\alpha, \beta \in \mathbf{R}$, define the MSA transform of f by

$$F(\alpha, \beta) = E[f(U_{\alpha,\beta})]$$

Writing this out in terms of the probability density function gives

$$\begin{aligned} F(\alpha, \beta) &= \int f(u) P(U_{\alpha,\beta} = u) du \\ &= \frac{1}{\|f\|_{L^1}^3} \int f(u)(f_\alpha * f_\beta * f_\gamma)(u) du \\ &= \frac{1}{\|f\|_{L^1}^3} \frac{1}{(\alpha\beta\gamma)^2} \iiint f(u) f(\frac{u-x-y}{\gamma}) \\ & \quad f(\frac{x}{\alpha}) f(\frac{y}{\beta}) dx\, dy\, du \end{aligned} \tag{2}$$

if $\alpha, \beta, \gamma \neq 0$, and straightforward modifications if one of these numbers is zero. Taking the Fourier transform and using the convolution and correlation theorems, one has

$$F(\alpha,\beta) = \frac{1}{(2\pi)^2}\frac{1}{\hat{f}(0)^3}\int \hat{f}(\xi)^*\hat{f}(\alpha\xi)\hat{f}(\beta\xi)\hat{f}(\gamma\xi)d\xi$$

which holds for all α, β.

3. MSA for Convex Sets

We first consider the MSA transform for binary images. Such images are given by sets $K \subseteq \mathbf{R}^2$, and the image intensity function is the characteristic function χ_K of K,

$$\chi_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

It is natural to assume that K is a bounded Lebesgue measurable set, for then $\chi_K \in L^1 \cap L^\infty$, and the MSA transform is well defined. We will further assume that K is closed. For nice sets K this results in no loss of generality since if $|\overline{K} \smallsetminus K| = 0$ ($|\cdot|$ is a Lebesgue measure and the $\overline{K}$ closure of K) then $\chi_{\overline{K}} = \chi_K$ as L^1 functions, which means that the MSA transform sees the sets $\overline{K}$ and K as identical.

Thus, let K be a compact subset of $\mathbf{R}^2$ and let $f = \chi_K$. Then the MSA transform of f may be written as

$$F(\alpha,\beta) = \int f(u)P(U_{\alpha,\beta} = u)\,du = P(U_{\alpha,\beta} \in K).$$

This immediately implies that $F(\alpha, \beta) \leq 1$ for any α and β. We also note that for $f = \chi_K$ the affine invariant moments $E[f(U_{\alpha,\beta})^k]$ for $k \in \mathbf{Z}_+$ are the same for all k, so in the binary case the MSA transform (the first moment) carries all information from all the moments.

In this article, we show that the MSA transform behaves particularly well for convex sets. A set $A \subseteq \mathbf{R}^2$ is by definition convex if for any $x, y \in A$ the line segment $[x, y] = \{(1-\alpha)x + \alpha y\,;\, 0 \leq \alpha \leq 1\}$ lies in A. If A is closed, it is enough to assume that $\frac{1}{2}x + \frac{1}{2}y \in A$ whenever $x, y \in A$, since then by iteration $(1-\alpha)x + \alpha y \in A$ for any dyadic rational number α between 0 and 1, and the whole segment $[x, y]$ must lie in A, since A is closed. For more information on convex sets see [5].

The first result states that for convex sets one knows $F(\alpha, \beta)$ exactly for certain α and β.

Property 1. Let K be a compact subset of $\mathbf{R}^2$ and let $f = \chi_K$. If K is convex then $F(\alpha, \beta) = 1$ whenever $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1 - \alpha$.

Proof. If $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1 - \alpha$, then one has $0 \leq \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta + \gamma = 1$. In this case, $U_{\alpha,\beta} = \alpha X_1 + \beta X_2 + \gamma X_0$ is a convex combination of the X_j. Since $X_j \in K$ almost surely, and K is convex, we have $U_{\alpha,\beta} \in K$ almost surely, so $F(\alpha, \beta) = P(U_{\alpha,\beta} \in K) = 1$. □

In particular, if K is convex then $F(1/2, 1/2) = 1$. For any set K one has

$$F(\frac{1}{2}, \frac{1}{2}) = P(\frac{1}{2}X_1 + \frac{1}{2}X_2 \in K)$$

which is the probability that the midpoint of two random points from K lies in K. This value may be thought of as a measure of convexity of the set K. Indeed, the following result shows that $F(1/2, 1/2) = 1$ characterizes convex sets K assuming the mild condition $K = \overline{\mathrm{int}(K)}$, where $\mathrm{int}(K)$ is the interior of K.

Property 2. Let K be a compact subset of $\mathbf{R}^2$ which satisfies $K = \overline{\mathrm{int}(K)}$. Then, if $f = \chi_K$ and $F(\frac{1}{2}, \frac{1}{2}) = 1$, K is convex.

Proof. From the definition

$$F(\frac{1}{2}, \frac{1}{2}) = \frac{16}{|K|^2}\int\int f(u)f(2x)f(2(u-x))\,dx\,du.$$

We make the change of variables $x = \frac{1}{2}s$, $u = \frac{1}{2}s + \frac{1}{2}t$. The Jacobian is equal to $1/16$, and we obtain

$$F(\frac{1}{2}, \frac{1}{2}) = \frac{1}{|K|^2}\int_K\int_K \chi_K(\frac{1}{2}s + \frac{1}{2}t)\,ds\,dt.$$

Since $\chi_K \leq 1$ the last expression is ≤ 1. But we had $F(\frac{1}{2}, \frac{1}{2}) = 1$, which implies that there is a set $E \subseteq K \times K$ so that $|E| = 0$ and $\frac{1}{2}s + \frac{1}{2}t \in K$ for any (s, t) in $K \times K \smallsetminus E$.

Let now $(s, t) \in \mathrm{int}(K) \times \mathrm{int}(K)$. Then there is a sequence (s_j, t_j) in $K \times K \smallsetminus E$ with $(s_j, t_j) \to (s, t)$, for an otherwise some small ball in $\mathrm{int}(K) \times \mathrm{int}(K)$ with its center (s, t) would only contain points of E, contradicting the fact that $|E| = 0$. Now, given such a sequence (s_j, t_j), we know that $\frac{1}{2}s_j + \frac{1}{2}t_j \in K$ for any j. Since K is closed, we get that $\frac{1}{2}s + \frac{1}{2}t = \lim_{j\to\infty}(\frac{1}{2}s_j + \frac{1}{2}t_j) \in K$.

Finally, since $K = \overline{\mathrm{int}(K)}$ the points in K are limits of sequences of points in $\mathrm{int}(K)$, and the midpoint of any two points s, t in K must also lie in K since this holds for $\mathrm{int}(K)$ and K is closed. Consequently K is convex. □

4. Implementation

Using the results in Section 3, one can now build a classifier that distinguishes convex sets from nonconvex ones. However this is somewhat difficult to perform with digital images, as illustrated later in this section and it is better to use a convexity measure. The reliability of this measure is dependent on the MSA implementation, so we present a description and the Matlab code for the one which was used in the experiments in Section 5.

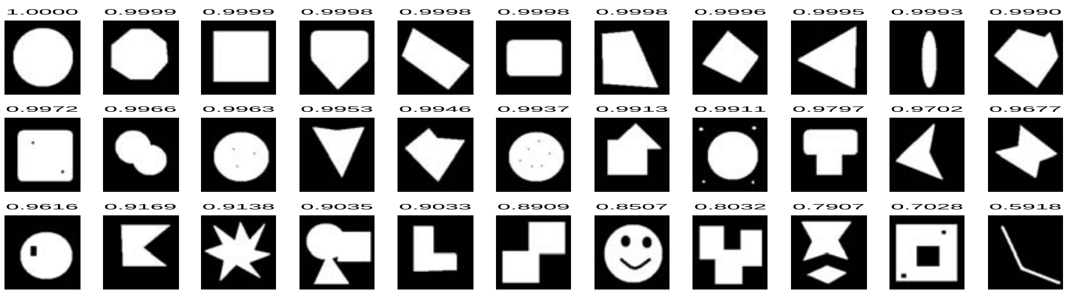

Figure 2. 33 objects arranged according to MSA based convexity measure.

The algorithm is a straightforward Matlab implementation of the Fourier transformed form of MSA. The program contains only seven lines and will be given later. Before applying the MSA to the digital image f, the continuous presentation given in Section 2 must be converted to discrete form. Taking into account that in our case $\alpha = \beta = 1/2$ and $\gamma = 0$ we have:

$$F(\frac{1}{2}, \frac{1}{2}) = \frac{1}{N_1 N_2} \frac{1}{\mathcal{F}(0)^2} \sum_{i=0}^{N_1 N_2 - 1} \mathcal{F}^*(w_i) \mathcal{F}(\frac{1}{2} w_i)^2, \quad (3)$$

where $\mathcal{F}$ is the discrete Fourier transform of f. If we assume that f is an $M_1 \times M_2$ matrix, the transformation length N_i must be taken to be $N_i \geq (|\alpha| + |\beta| + |\gamma|) M_i - 2$ to avoid the wrap-around error. In our case, where $\alpha = \beta = 1/2$ and $\gamma = 0$, N_i was chosen to equal M_i. Calculating the values $\mathcal{F}(1/2 \cdot w_i)$ for $i = 1, \ldots, N_j$ requires a decimation of the image function f to half of its original size. In our algorithm, this is done so that the four closest pixels in f are summed up to form one pixel in the decimated image f', for example $f'(1,1) = f(1,1) + f(1,2) + f(2,1) + f(2,2)$.

As an example, here is the Matlab implementation which was used for computing $F(1/2, 1/2)$ in Section 5:

```
function F=convexity(f)
[m,n]=size(f);
f0=sum(f(:));
G0=conj(fft2(f,m,n));
f1=f(1:2:m,1:2:n)+f(2:2:m,1:2:n) ...
+f(1:2:m,2:2:n)+f(2:2:m,2:2:n);
G1=fft2(f1,m,n);
F=real(sum(sum(G0.*(G1.^2)))) ...
/(n*m*f0^2);
```

Now one can use the value $F(1/2, 1/2)$ as a convexity measure and classify all sets with this value above a given threshold as convex. However digital images, like the ones in Figure 1, are usually not convex in the strict sense, so instead of a strict classification one can use the convexity measure and interpret this information depending on the application. This convexity value can also be used as a certain kind of measure of object complexity. This is comparable to the usual perimeter squared per area measure, but due to the affine invariance property of the MSA transform, see [1], our measure is also invariant under any spatial affine transformations.

5. Experiments

In this section, we present some experiments to illustrate the properties introduced in Section 3, using the MSA algorithm specified in Section 4.

As a first illustration, we organized 33 128×128 binary images based on convexity and complexity. The convexity was measured using the $F(1/2, 1/2)$ value and the complexity using the perimeter squared per area measure. Figure 2 shows the images organized based on $F(1/2, 1/2)$ values so that the image having the highest value is on the top left. One can also see the $F(1/2, 1/2)$ values above each image. Looking at the figure, one can see that objects which look convex to the human eye are placed at the top of the list and have MSA values higher than 0.999.

Figure 3 contains the same images as Figure 2 organized now according to the perimeter squared per area values. Again, the values are given above each image. One can now compare how the two methods measure object complexity. By first looking at Figure 2 one can observe that, at least

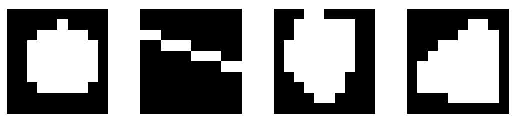

Figure 1. Samples of digital convex objects which are nonconvex in the strict sense.

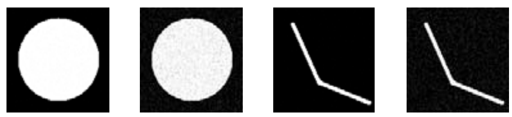

Figure 4. Samples of objects with Gaussian noise with variances 0.0001 and 0.01.

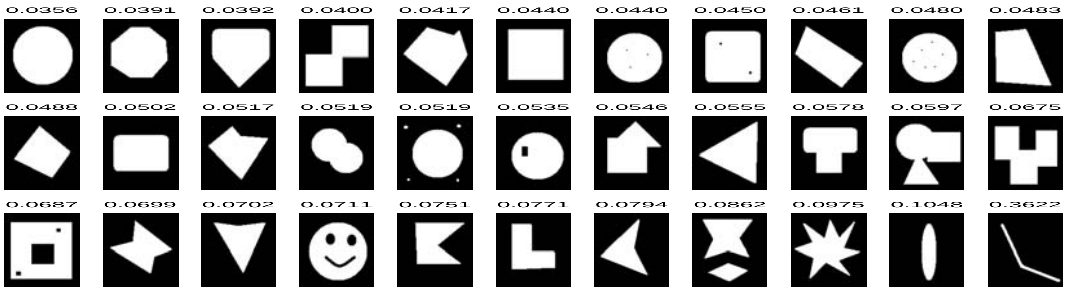

Figure 3. 33 objects arranged according to perimeter squared per area measure.

subjectively, convexity seems to work quite well as a complexity measure. The same thing can been seen when comparing the orders in Figures 2 and 3. Moreover the convexity measure is insensitive to spatial affine transforms, which can be a desirable property in computer vision applications.

The second experiment measures how the $F(1/2, 1/2)$ value reacts if some interior pixels of a convex region are changed to zero. We started with an image containing only ones, and randomly chose certain pixels that were then changed to zero. The original image will have $F(1/2, 1/2) = 1.0000$ and Table 1 contains the $F(1/2, 1/2)$ values with different relative amounts of error pixels. In every case, we ran 1000 tests and computed the mean of the obtained $F(1/2, 1/2)$ values. As one can observe, the MSA value is effective in finding holes in a convex region. Our test image was 100×100, so the method already recognized one error pixel. It seems that the values follow the equation $F(1/2, 1/2) = 1 - P_e$, where P_e is the relative amount of error pixels.

Relative amount of error pixels	$F(1/2, 1/2)$ value
20 %	0.7999
10 %	0.9000
5 %	0.9499
1 %	0.9900
0.1 %	0.9990
0.0 %	1.0000

Table 1. The $F(1/2, 1/2)$ values of convex regions with error pixels.

Noise variance	The first image	The last image
0	1.0000	0.5918
0.0001	0.9949	0.5129
0.0005	0.9893	0.4416
0.001	0.9846	0.3996
0.005	0.9652	0.2877
0.01	0.9499	0.2437

Table 2. The $F(1/2, 1/2)$ values for the first and last object from Figure 2 with Gaussian noise

As a final illustration, we tested how the $F(1/2, 1/2)$ values change if the binary image is disturbed with Gaussian noise. As a test set, we took the first and last images in Figure 2 and added zero mean independently distributed Gaussian noise. Samples of these are shown in Figure 4. In Table 2, one can see the results and observe that the MSA value reacts quite quickly. As expected, the value decreases as noise power increases, so in the sense that the disturbed image is less convex than the original binary image, the measure works correctly.

6. Conclusions

In this paper, we have presented a novel affine invariant method and algorithm for measuring object convexity. As shown by the experiments, this measure is easy to compute and can recognize even small errors in a convex domain. When using such a measure with digital images, one has to interpret the results depending on the application. However, our experiments indicate that MSA is applicable to convex shape analysis for these images.

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