A SHARP STABILITY ESTIMATE FOR TENSOR TOMOGRAPHY IN NON-POSITIVE CURVATURE

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Abstract. We consider the geodesic X-ray transform acting on solenoidal tensor fields on a compact simply connected manifold with strictly convex boundary and non-positive curvature. We establish a stability estimate of the form $L^2 \mapsto H_T^{1/2}$, where the $H_T^{1/2}$-space is defined using the natural parametrization of geodesics as initial boundary points and incoming directions (fan-beam geometry); only tangential derivatives at the boundary are used. The proof is based on the Pestov identity with boundary term localized in frequency.

1. Introduction

To motivate our results, let us begin with the simplest case of the Radon transform in $\mathbb{R}^2$ in parallel beam geometry (see [Na01] for more details).

Example. If $f \in C_c^\infty(\mathbb{R}^2)$, the Radon transform of f is

$$Rf(s,v) = \int_{-\infty}^{\infty} f(sv + tv^\perp)\,dt, \qquad s \in \mathbb{R},\ v \in S^1,$$

where $v^\perp$ is the rotation of v by 90° counterclockwise. The Fourier transform of Rf in the s variable, denoted by $(Rf)\tilde{\ }(\,\cdot\,,v)$, satisfies the *Fourier slice theorem*

$$(Rf)\tilde{\ }(\sigma,v) = (2\pi)^{1/2}\hat{f}(\sigma v), \qquad \sigma \in \mathbb{R},\ v \in S^1.$$

Using the Plancherel theorem and polar coordinates, we obtain that

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^2)} = \|\hat{f}\|^2_{L^2(\mathbb{R}^2)} &= \int_0^\infty \int_{S^1} |\hat{f}(\sigma v)|^2 \sigma\,dv\,d\sigma \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{S^1} |\hat{f}(\sigma v)|^2 |\sigma|\,dv\,d\sigma \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{S^1} |(Rf)\tilde{\ }(\sigma,v)|^2|\sigma|\,dv\,d\sigma.\end{aligned}$$

In particular, this implies the stability estimate

$$\|f\|_{L^2(\mathbb{R}^2)} \leq \frac{1}{(4\pi)^{1/2}}\|Rf\|_{H_T^{1/2}(\mathbb{R}\times S^1)} \tag{1.1}$$

with the mixed Sobolev norm $\|h\|_{H_T^{1/2}(\mathbb{R}\times S^1)} = \|(1+\sigma^2)^{1/4}\tilde{h}(\sigma,v)\|_{L^2(\mathbb{R}\times S^1)}$.

The main question we address in the present paper is the existence of a stability estimate analogous to (1.1) but in a geometric setting, namely, when $\mathbb{R}^2$ and the lines in the plane are replaced by a Riemannian manifold and its geodesics. There are two features we wish to preserve from (1.1): one is its $L^2 \to H^{1/2}$ nature and the other is that the $H^{1/2}_T$ only incorporates "half of the derivatives" of the target space (space of geodesics).

Let us first be more precise about the geometric setting. The geodesic X-ray transform acts on functions defined on the unit sphere bundle of a compact oriented d-dimensional Riemannian manifold (M, g) with smooth boundary ∂M ($d \geq 2$). Let SM denote the unit sphere bundle on M, i.e.

$$SM := \{(x, v) \in TM : |v|_g = 1\}.$$

We define the volume form on SM by $d\Sigma^{2d-1}(x, v) = dV^d(x) \wedge dS_x(v)$, where dV^d is the volume form on M and dS_x is the volume form on the fibre S_xM. The boundary of SM is $\partial SM := \{(x, v) \in SM : x \in \partial M\}$. On ∂SM the natural volume form is $d\Sigma^{2d-2}(x, v) = dV^{d-1}(x) \wedge dS_x(v)$, where dV^{d-1} is the volume form on ∂M. We distinguish two subsets of ∂SM (incoming and outgoing directions)

$$\partial_\pm SM := \{(x, v) \in \partial SM : \pm\langle v, \nu(x)\rangle_g \leq 0\},$$

where $\nu(x)$ is the outward unit normal vector on ∂M at x. It is easy to see that

$$\partial_+ SM \cap \partial_- SM = S(\partial M).$$

Given $(x, v) \in SM$, we denote by $\gamma_{x,v}$ the unique geodesic with $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$ and let $\tau(x, v)$ be the first time when the geodesic $\gamma_{x,v}$ exits M.

We say that (M, g) is *non-trapping* if $\tau(x, v) < \infty$ for all $(x, v) \in SM$. In this case the space of geodesics is naturally parametrized by $\partial_+ SM$ (fan-beam geometry).

Definition 1.1. The *geodesic X-ray transform* of a function $F \in C^\infty(SM)$ is the function

$$IF(x, v) := \int_0^{\tau(x,v)} F(\gamma_{x,v}(t), \dot\gamma_{x,v}(t))\, dt, \quad (x, v) \in \partial_+ SM.$$

If the manifold (M, g) is non-trapping and has strictly convex boundary, then $I : C^\infty(SM) \to C^\infty(\partial_+ SM)$, and extends as a bounded operator $I : H^k(SM) \to H^k(\partial_+ SM)$ for all $k \geq 0$ [Sh94, Theorem 4.2.1], where the Sobolev spaces are defined using the L^2-inner products arising from the volume forms introduced above.

We shall consider I acting on special functions $F \in C^\infty(SM)$ induced by symmetric tensor fields. We denote by $C^\infty(S^m(T^*M))$ the space of smooth covariant symmetric tensor fields of rank m on M with L^2 inner product:

$$(u, w) := \int_M u_{i_1\cdots i_m} w^{i_1\cdots i_m}\, dV^d,$$

where $w^{i_1\cdots i_m} = g^{i_1 j_1} \cdots g^{i_m j_m} w_{j_1\cdots j_m}$. There is a natural map

$$\ell_m : C^\infty(S^m(T^*M)) \to C^\infty(SM)$$

given by $\ell_m(f)(x,v) := f_x(v,\dots,v)$. We can now define the geodesic ray transform acting on symmetric m-tensors simply by setting $I_m := I \circ \ell_m$. Let $d^s = \sigma\nabla$ be the symmetric inner differentiation, where ∇ is the Levi-Civita connection associated with g, and σ denotes symmetrization. It is easy to check that if $f = d^s p$ for some $p \in C^\infty(S^{m-1}(T^*M))$ with $p|_{\partial M} = 0$, then $I_m f = 0$. The tensor tomography problem asks the following question: are such tensors the only obstructions for I_m to be injective? If this is the case, then we say I_m is *solenoidal injective or s-injective* for short. This terminology is explained by the following well known decomposition (cf. [Sh94]). Given $f \in H^k(S^m(T^*M))$, $k \geq 0$, there exist uniquely determined $f_s \in H^k(S^m(T^*M))$ and $p \in H^{k+1}(S^{m-1}(T^*M))$, such that

$$f = f_s + d^s p, \quad \delta^s f_s = 0, \quad p|_{\partial M} = 0,$$

where δ^s is the divergence. We call f_s and $d^s p$ the *solenoidal* part and *potential* part of f respectively.

There is one important instance in which the tensor tomography problem is solved for tensors of any order m and in any dimension d. This is when we assume in addition that the sectional curvature of M is non-positive. Moreover, in this case a stability estimate is available as follows:

Theorem 1.2. ([PS88] *and* [Sh94, Theorem 4.3.3]) *Let (M,g) be a simply connected compact manifold with strictly convex boundary and non-positive sectional curvature. Given $m \geq 0$ there is a constant $C > 0$ such that for any $f \in H^1(S^m(T^*M))$*

$$\|f_s\|^2_{L^2} \leq C(\|I_m f\|^2_{H^1(\partial_+ SM)} + m\|f\|_{H^1}\|I_m f\|_{L^2}).$$

(We note that a manifold as in the theorem is necessarily non-trapping.) There are two notorious differences between the stability estimate above and that in (1.1). Firstly, the stability estimate in Theorem 1.2 has in the right hand side the term $\|f\|_{H^1}\|I_m f\|_{L^2}$ when $m \neq 0$. Secondly, it is not sharp in the sense that it is $L^2 \to H^1$. In [BS18] Boman and Sharafutdinov resolved these issues for strictly convex domains in Euclidean space and asked whether the same was true for the more general setting of non-positively curved Riemannian manifolds. This paper provides a positive answer to these questions. Moreover, the 1/2-Sobolev space on the target space of I_m is naturally suggested by the geometry and the most relevant L^2-energy identity for the problem: *the Pestov identity.* The Pestov identity with boundary term in the way that we shall use it here was derived for instance in [IP18, Lemma 8]. It contains a boundary term given by

$$(Tu, \overset{\mathtt{v}}{\nabla} u)_{L^2(\partial SM)} \tag{1.2}$$

where $u \in C^\infty(\partial SM)$, $\overset{\mathtt{v}}{\nabla}$ is the vertical gradient, and T is a *tangential operator* defined by

$$T = \langle \nu(x), v\rangle \overset{\mathtt{h}}{\nabla} - \nu X,$$

where X is the geodesic vector field and $\overset{\mathtt{h}}{\nabla}$ the full horizontal gradient (we refer to Sections 2 and 3 for the precise definitions). The operator T acts on ∂SM and it only involves horizontal derivatives. This suggests that only horizontal derivatives of $I_m f$ on ∂SM should appear in the stability estimate.

We can define the *tangential* (or *horizontal*) $H^1(\partial SM)$-norm by setting

$$\|u\|^2_{H^1_T(\partial SM)} := \|u\|^2_{L^2(\partial SM)} + \|\overset{\mathtt{h}}{\nabla}{}^{\|} u\|^2_{L^2(\partial SM)}$$

where $\overset{\mathtt{h}}{\nabla}{}^{\|} u$ contains the tangential derivatives in $\overset{\mathtt{h}}{\nabla} u$ along ∂M. For example, if M is a ball in $\mathbb{R}^n$ with Euclidean metric, then $\partial SM = \partial M \times S^{d-1}$ and

$$\|u\|^2_{H^1_T(\partial SM)} = \int_{\partial M} \int_{S^{d-1}} (|u(x,v)|^2 + |\nabla_x u(x,v)|^2) \, dS(v) \, dS(x)$$

where ∇_x is the gradient on ∂M. The space $H^1_T(\partial_+ SM)$ is defined by restriction, and $H^{1/2}_T(\partial_+ SM)$ is defined by complex interpolation between $L^2(\partial_+ SM)$ and $H^1_T(\partial_+ SM)$.

With this definition we may now state our main result:

Theorem 1.3. *Let (M, g) be a simply connected compact manifold with strictly convex boundary and non-positive sectional curvature. Given $m \geq 0$ there is a constant $C > 0$ such that for any $f \in H^1(S^m(T^*M))$*

$$\|f_s\|_{L^2} \leq C \|I_m f\|_{H^{1/2}_T(\partial_+ SM)}.$$

The constant C can be estimated in terms of m and (M, g). In fact, for the related stability result for the transport equation in Theorem 5.1, one can take $C = 1$.

Most of work in the proof of Theorem 1.3 lies in the upgrade from the $H^1(\partial_+ SM)$-norm in Theorem 1.2 to the $H^{1/2}_T(\partial_+ SM)$-norm. The upgrade is possible thanks to the localization in frequency of the Pestov identity first noted in full generality in [PS18] (in two dimensions this was proved in [PSU15]). However, in [PS18] we did not consider the boundary term. It turns out, quite remarkably, that the boundary term (1.2) also localizes in frequency. This allows us to change the norm for $I_m f$ from H^1 to $H^{1/2}_T$, thus producing the upgrade. We also mention that for $\dim(M) = 2$ the proof would simplify substantially because spherical harmonics decompositions and the T operator are simpler; the two-dimensional proof will be given in [PSU19].

1.1. **Related results and alternative approaches.** There are many earlier results on stability estimates for I_m, using different techniques. One approach is to consider the normal operator $I_m^* I_m$ where the adjoint I_m^* is computed using a natural L^2_μ-inner product on $\partial_+ SM$ suggested by the Santaló formula. When M is free of conjugate points, it turns out that $I_m^* I_m$ is an (elliptic) ΨDO of order -1 on a slightly larger open manifold engulfing M. This approach has produced stability estimates for the normal operator, cf. [SU04], and has proved to be of fundamental importance in the solution of several geometric inverse problems. One drawback is that one needs to work on the slightly extended manifold, unless one is willing to incorporate modified transmission conditions to account for boundary effects [MNP19]. Another drawback is that the approach does not give estimates for the constants due to a compactness argument. Still, quite recently, a sharp stability estimate has been obtained in [AS19], by defining a suitable $H^{1/2}$-norm based on this extension or equivalently on a different parametrization of the space of geodesics. Our approach in Theorem 1.3 deals directly with the boundary and with the space of geodesics in "fan-beam" geometry as given by $\partial_+ SM$. In this sense our theorem addresses the open problem stated at the end of the introduction in [AS19]. Also our tangential derivatives are naturally suggested by the geometry of the problem.

The microlocal approach can actually be pushed further, using scattering calculus and a combination of a local theorem with a global strict convexity assumption as in [UV16, SUV18]. This is also very powerful, and allows even to consider situations with conjugate points as long as $d \geq 3$. However, the stability estimates thus produced are $L^2 \to H^1$.

One drawback of Theorem 1.3 is the curvature assumption. In [AS19] the estimates hold for compact simple manifolds for $m = 0, 1$ and for $m = 2$ when I_m is known to be injective, e.g. when $d = 2$ [Sh07, PSU13]. Another possible improvement would be to replace the assumption $f \in H^1$ by $f \in L^2$ and to prove the two-sided inequality

$$c\|f_s\|_{L^2} \leq \|I_m f\|_{H_T^{1/2}(\partial_+ SM)} \leq C\|f_s\|_{L^2}.$$

For this, one would like to prove that I_m is bounded from L^2 to $H_T^{1/2}$. This is true if f vanishes near ∂M since I_m is a Fourier integral operator, but it is not clear how to prove this with uniform bounds when the support of f extends up to ∂M.

Finally, we mention that quite recently, Monard [M19] has studied very detailed mapping properties of I_0 for 2D discs of constant curvature at all Sobolev scales; for these cases, he also obtains a stability estimate with a suitable $H^{1/2}$-norm. Further references to stability estimates for I_m may be found in [AS19].

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2. Geometric preliminaries

In this section we collect some geometric preliminaries for subsequent use.

Unit sphere bundle. We start by recalling some standard notions related to the geometry of the unit sphere bundle. We follow the setup and notation of [PSU15]; for other approaches and background information see [GK80, Sh94, Pa99, Kn02, DS11].

Let (M, g) be a d-dimensional compact Riemannian manifold with or without boundary, having unit sphere bundle $\pi : SM \to M$, and let X be the geodesic vector field. We equip SM with the Sasaki metric. If $\mathcal{V}$ denotes the vertical subbundle given by $\mathcal{V} = \operatorname{Ker} d\pi$, then there is an orthogonal splitting with respect to the

Sasaki metric:

$$TSM = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}. \tag{2.1}$$

The subbundle $\mathcal{H}$ is called the horizontal subbundle. Elements in $\mathcal{H}(x,v)$ and $\mathcal{V}(x,v)$ are canonically identified with elements in the codimension one subspace $\{v\}^\perp \subset T_xM$ by the isomorphisms

$$d\pi_{x,v} : \mathcal{V}(x,v) \to \{v\}^\perp, \quad \mathcal{K}_{x,v} : \mathcal{H}(x,v) \to \{v\}^\perp,$$

here $\mathcal{K}_{x,v}$ is the connection map coming from Levi-Civita connection. We will use these identifications freely below.

We shall denote by $\mathcal{Z}$ the set of smooth functions $Z : SM \to TM$ such that $Z(x,v) \in T_xM$ and $\langle Z(x,v), v\rangle = 0$ for all $(x,v) \in SM$. Alternatively we may describe the elements of $\mathcal{Z}$ is a follows. Consider the pull-back bundle π^*TM over SM and let N denote the subbundle of π^*TM whose fiber over (x,v) is given by $N_{(x,v)} = \{v\}^\perp$. Then $\mathcal{Z}$ coincides with the smooth sections of the bundle N. Note that N carries a natural scalar product and thus an L^2-inner product (using the Liouville measure on SM for integration).

Given a smooth function $u \in C^\infty(SM)$ we can consider its gradient ∇u with respect to the Sasaki metric. Using the splitting above we may write uniquely in the decomposition (2.1)

$$\nabla u = ((Xu)X, \overset{\mathtt{h}}{\nabla} u, \overset{\mathtt{v}}{\nabla} u).$$

The derivatives $\overset{\mathtt{h}}{\nabla} u \in \mathcal{Z}$ and $\overset{\mathtt{v}}{\nabla} u \in \mathcal{Z}$ are called horizontal and vertical derivatives respectively. (This differs from the definitions in [Kn02, Sh94] since here all objects are defined on SM as opposed to TM.)

The geodesic vector X acts on $\mathcal{Z}$ as follows:

$$XZ(x,v) := \frac{DZ(\varphi_t(x,v))}{dt}\Big|_{t=0} \tag{2.2}$$

where D/dt is the covariant derivative with respect to Levi-Civita connection and φ_t is the geodesic flow. With respect to the L^2-product on N, the formal adjoints of $\overset{\mathtt{v}}{\nabla} : C^\infty(SM) \to \mathcal{Z}$ and $\overset{\mathtt{h}}{\nabla} : C^\infty(SM) \to \mathcal{Z}$ are denoted by $-\overset{\mathtt{v}}{\mathrm{div}}$ and $-\overset{\mathtt{h}}{\mathrm{div}}$ respectively. Note that since X leaves invariant the volume form of the Sasaki metric we have $X^* = -X$ for both actions of X on $C^\infty(SM)$ and $\mathcal{Z}$.

Let $R(x,v) : \{v\}^\perp \to \{v\}^\perp$ be the operator determined by the Riemann curvature tensor by $R(x,v)w = R(w,v)v$, and let $d = \dim M$.

Spherical harmonics decomposition. There is a natural spherical harmonics decomposition with respect to the vertical Laplacian $\Delta = -\overset{\mathtt{v}}{\mathrm{div}}\overset{\mathtt{v}}{\nabla}$ (cf. [PSU15, Section 3] and [GK80]):

$$L^2(SM) = \bigoplus_{m=0}^{\infty} H_m(SM),$$

so that any $f \in L^2(SM)$ has the orthogonal decomposition

$$f = \sum_{m=0}^{\infty} f_m.$$

We write $\Omega_m = H_m(SM) \cap C^\infty(SM)$. Then $\Delta u = m(m+d-2)u$ for $u \in \Omega_m$ and we let $\lambda_m := m(m+d-2)$.

Decomposition of X. The geodesic vector field has a special behaviour with respect to the decomposition into fibrewise spherical harmonics: it maps Ω_m into $\Omega_{m-1} \oplus \Omega_{m+1}$ [GK80, Proposition 3.2]. Hence on Ω_m we can write

$$X = X_- + X_+$$

where $X_- : \Omega_m \to \Omega_{m-1}$ and $X_+ : \Omega_m \to \Omega_{m+1}$. By [GK80, Proposition 3.7] the operator X_+ is overdetermined elliptic (i.e. it has injective principal symbol). We can explain the decomposition $X = X_- + X_+$ as follows. Fix $x \in M$ and consider local coordinates which are geodesic at x (so $\partial_{x_j} g_{kl}(x) = 0$ for all j, k, l). Then $Xu(x,v) = v^i \frac{\partial u}{\partial x^i}$. We now use the following basic fact about spherical harmonics: the product of a spherical harmonic of degree m with a spherical harmonic of degree one decomposes as the sum of spherical harmonics of degree $m-1$ and $m+1$.

3. Pestov identity with boundary term

We recall the following commutator formulas from [PSU15]:

$$\begin{aligned} &[X, \overset{\mathtt{v}}{\nabla}] = -\overset{\mathtt{h}}{\nabla}, \\ &[X, \overset{\mathtt{h}}{\nabla}] = R\overset{\mathtt{v}}{\nabla}, \\ &\overset{\mathtt{h}}{\operatorname{div}}\overset{\mathtt{v}}{\nabla} - \overset{\mathtt{v}}{\operatorname{div}}\overset{\mathtt{h}}{\nabla} = (d-1)X. \end{aligned} \tag{3.1}$$

Taking adjoints gives the following commutator formulas on $\mathcal{Z}$:

$$\begin{aligned} &[X, \overset{\mathtt{v}}{\operatorname{div}}] = -\overset{\mathtt{h}}{\operatorname{div}}, \\ &[X, \overset{\mathtt{h}}{\operatorname{div}}] = -\overset{\mathtt{v}}{\operatorname{div}}R. \end{aligned} \tag{3.2}$$

Using these relations one can establish a Pestov identity with boundary term. Let $\mu(x,v) := \langle v, \nu(x) \rangle$. We let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the L^2-norm and L^2-inner product respectively determined by the volume form $d\Sigma^{2d-1}$ on SM; we let $(\cdot,\cdot)_{\partial SM}$ stand for the L^2-inner product on ∂SM determined by $d\Sigma^{2d-2}$.

Proposition 3.1 (Pestov identity with boundary term, cf. Lemma 8 in [IP18])**.** *Let (M,g) be a compact manifold with smooth boundary. If $u \in C^\infty(SM)$, then*

$$\|\overset{\mathtt{v}}{\nabla} Xu\|^2 = \|X\overset{\mathtt{v}}{\nabla} u\|^2 - (R\overset{\mathtt{v}}{\nabla} u, \overset{\mathtt{v}}{\nabla} u) + (d-1)\|Xu\|^2 + P(u,u),$$

where P is the quadratic form defined by

$$P(u,w) = (Tu, \overset{\mathtt{v}}{\nabla} w)_{\partial SM},$$

and $Tu := \mu \overset{\mathtt{h}}{\nabla} u - Xu \overset{\mathtt{v}}{\nabla}\mu$.

We can express T using the full horizontal derivative $\overset{\mathtt{h}}{\overline{\nabla}} u = \overset{\mathtt{h}}{\nabla} u + (Xu)v$ as $T = \mu \overset{\mathtt{h}}{\overline{\nabla}} - \nu X$ since $\overset{\mathtt{v}}{\nabla}\mu = \nu - \mu v$. It turns out that T can also be rewritten in such a way that it acts on functions $u \in C^\infty(\partial SM)$. To see this, consider the operators

$$\nabla^{\|} u := \nabla u - \langle \nabla u, (\nu, 0)\rangle (\nu, 0) \tag{3.3}$$

and

$$X^{\|} := X - \langle X, (\nu, 0)\rangle (\nu, 0) = (v - \langle v, \nu\rangle \nu, 0) = (v^{\|}, 0) \tag{3.4}$$

at the boundary. We also define the horizontal part of $\nabla^{\|}$ as

$$\overset{\mathtt{h}}{\overline{\nabla}}{}^{\|} u := d\pi(\nabla^{\|} u) = \overset{\mathtt{h}}{\overline{\nabla}} u - \langle \overset{\mathtt{h}}{\overline{\nabla}} u, \nu\rangle \nu.$$

The following simple lemma is proved in [IP18, Lemma 14]:

Lemma 3.2. *We have*

$$T = \mu \overset{\mathtt{h}}{\overline{\nabla}}{}^{\|} - \nu X^{\|}. \tag{3.5}$$

From this form we can clearly see that $T : C^\infty(\partial SM) \to \mathcal{Z}|_{\partial SM}$.

Remark 3.3. In 2D, $Tu = (\mathbb{T}u)iv$, where $\mathbb{T}$ is the tangential horizontal vector field $(i\nu, 0)$ and i is the complex structure of the surface. The vector field $\mathbb{T}$ and the vertical vector field V form a commuting frame for ∂SM.

We next rewrite the Pestov identity in terms of X_+ and X_- as in [PS18]. To do this, we need some notation: for a polynomially bounded sequence $\alpha = (\alpha_l)_{l=0}^\infty$ of real numbers, we define a corresponding "inner product"

$$(u, w)_\alpha = \sum_{l=0}^\infty \alpha_l (u_l, w_l)_{L^2(SM)}, \qquad u, w \in C^\infty(SM).$$

We also write $\|u\|_\alpha^2 = \sum_{l=0}^\infty \alpha_l \|u_l\|^2$. (If each α_l is positive one gets an actual inner product and norm, but it is notationally convenient to allow zero or negative α_l.)

The Pestov identity can then be written in the following form. Define

$$\alpha_l = \lambda_l \left[\left(1 + \frac{1}{l+d-2}\right)^2 - 1 \right] + (d-1), \tag{3.6}$$

$$\beta_l = \lambda_l \left[1 - \left(1 - \frac{1}{l}\right)^2 \right] - (d-1). \tag{3.7}$$

The next result extends [PS18, Proposition 4.4] to the case with boundary terms.

Proposition 3.4 (Pestov identity in terms of $X_\pm$ with boundary term)**.** *Let* (M, g) *be a compact manifold with smooth boundary. If* $u \in C^\infty(SM)$*, then*

$$\|X_- u\|_\alpha^2 - (R \overset{\mathtt{v}}{\nabla} u, \overset{\mathtt{v}}{\nabla} u) + \|Z(u)\|^2 + P(u, u) = \|X_+ u\|_\beta^2,$$

where $Z(u)$ *is* $\overset{\mathtt{v}}{\mathrm{div}}$*-free.*

Proof. Recall from [PSU15, Lemma 4.4] that

$$\overset{\mathtt{h}}{\nabla} u = \overset{\mathtt{v}}{\nabla}\left[\sum_{l=1}^{\infty}\left(\frac{1}{l}X_+u_{l-1} - \frac{1}{l+d-2}X_-u_{l+1}\right)\right] + Z(u) \tag{3.8}$$

where $Z(u) \in \mathcal{Z}$ satisfies $\overset{\mathtt{v}}{\mathrm{div}}\, Z(u) = 0$. Thus by (3.1)

$$X\overset{\mathtt{v}}{\nabla} u = \overset{\mathtt{v}}{\nabla}\sum_{l=1}^{\infty}\left[\left(1-\frac{1}{l}\right)X_+u_{l-1} + \left(1+\frac{1}{l+d-2}\right)X_-u_{l+1}\right] - Z(u). \tag{3.9}$$

This gives

$$\begin{aligned}
&\|X\overset{\mathtt{v}}{\nabla} u\|^2 \\
&= \sum_{l=1}^{\infty}\lambda_l\left(\left(1-\frac{1}{l}\right)X_+u_{l-1} + \left(1+\frac{1}{l+d-2}\right)X_-u_{l+1}, \left(1-\frac{1}{l}\right)X_+u_{l-1} + \left(1+\frac{1}{l+d-2}\right)X_-u_{l+1}\right) \\
&\quad + \|Z(u)\|^2 \\
&= \sum_{l=1}^{\infty}\lambda_l\left[\left(1-\frac{1}{l}\right)^2\|X_+u_{l-1}\|^2 + \left(1+\frac{1}{l+d-2}\right)^2\|X_-u_{l+1}\|^2\right] \\
&\quad + \sum_{l=1}^{\infty}\lambda_l\left(1-\frac{1}{l}\right)\left(1+\frac{1}{l+d-2}\right)[(X_+u_{l-1}, X_-u_{l+1}) + (X_-u_{l+1}, X_+u_{l-1})] + \|Z(u)\|^2.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
&\|\overset{\mathtt{v}}{\nabla} Xu\|^2 - (d-1)\|Xu\|^2 \\
&= -(d-1)\|X_-u_1\|^2 + \sum_{l=1}^{\infty}(\lambda_l - (d-1))(X_+u_{l-1} + X_-u_{l+1}, X_+u_{l-1} + X_-u_{l+1}) \\
&= -(d-1)\|X_-u_1\|^2 + \sum_{l=1}^{\infty}(\lambda_l - (d-1))\left[\|X_+u_{l-1}\|^2 + \|X_-u_{l+1}\|^2\right] \\
&\qquad + \sum_{l=1}^{\infty}(\lambda_l - (d-1))\left[(X_+u_{l-1}, X_-u_{l+1}) + (X_-u_{l+1}, X_+u_{l-1})\right].
\end{aligned}$$

Somewhat miraculously, we observe that

$$\lambda_l\left(1-\frac{1}{l}\right)\left(1+\frac{1}{l+d-2}\right) = \lambda_l - (d-1).$$

This means that the two sums above involving $[(X_+u_{l-1}, X_-u_{l+1}) + (X_-u_{l+1}, X_+u_{l-1})]$ terms are equal. The Pestov identity from Proposition 3.1 now yields

$$\sum_{l=0}^{\infty}\alpha_l\|X_-u_{l+1}\|^2 - (R\overset{\mathtt{v}}{\nabla} u, \overset{\mathtt{v}}{\nabla} u) + \|Z(u)\|^2 + P(u,u) = \sum_{l=1}^{\infty}\beta_l\|X_+u_{l-1}\|^2$$

where α_l, β_l are as in (3.6)–(3.7). The result follows. ☐

Later on we shall need the following useful property.

Lemma 3.5 (Adjoint of T)**.** *The formal adjoint of* $T : C^\infty(\partial SM) \to \mathcal{Z}|_{\partial SM}$ *satisfies*

$$\overset{\mathrm{v}}{\mathrm{div}} T = -T^* \overset{\mathrm{v}}{\nabla}$$

and the operator $\overset{\mathrm{v}}{\mathrm{div}} T$ *is self-adjoint in* $L^2(\partial SM, d\Sigma^{2d-2})$.

Proof. We use the Pestov identity with boundary term, to claim first that the operator $\overset{\mathrm{v}}{\mathrm{div}} T$ is self adjoint. Proposition 3.4 and the polarization identity imply that

$$P(u,w) = (X_+ u, X_+ w)_\beta - (X_- u, X_- w)_\alpha + (R\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} w) - (Z(u), Z(w))$$

and since R is symmetric, it follows that $P(u,w) = P(w,u)$. But $P(u,w) = -(\overset{\mathrm{v}}{\mathrm{div}} T u, w)$ and thus $\overset{\mathrm{v}}{\mathrm{div}} T$ is self-adjoint. Hence

$$\overset{\mathrm{v}}{\mathrm{div}} T = (\overset{\mathrm{v}}{\mathrm{div}} T)^* = -T^* \overset{\mathrm{v}}{\nabla}$$

as desired. □

4. Frequency localization

Recall from Section 2 that any $u \in C^\infty(SM)$ admits an L^2-orthogonal decomposition

$$u = \sum_{l=0}^{\infty} u_l, \qquad u_l \in \Omega_l,$$

where Ω_l corresponds to the set of vertical spherical harmonics of degree l. Since $X_\pm$ maps Ω_l to $\Omega_{l\pm 1}$, it is immediate that the Pestov identity with boundary term (Proposition 3.4) reduces to the following identity when applied to functions in Ω_l (i.e. frequency localized Pestov identity).

Proposition 4.1 (Pestov identity on Ω_l with boundary term)**.** *Let* (M,g) *be a compact manifold with smooth boundary, and let* $l \geq 0$. *One has*

$$\alpha_{l-1}\|X_- u\|^2 - (R\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u) + \|Z(u)\|^2 + P(u,u) = \beta_{l+1}\|X_+ u\|^2, \qquad u \in \Omega_l.$$

(We define $\alpha_{-1} = 0$*.)*

It was proved in [PS18] (and in [PSU15, Appendix B] when $\dim(M) = 2$) that the frequency localized Pestov identity for all l is equivalent with the standard Pestov identity. The same is true in the boundary case:

Lemma 4.2. *The Pestov identity with boundary term on* Ω_l *is equivalent with the Pestov identity with boundary term in the following sense: for any* $u \in C^\infty(SM)$*, one has*

$$\sum_{l=0}^{\infty}\left[\alpha_{l-1}\|X_- u_l\|^2 - (R\overset{\mathrm{v}}{\nabla} u_l, \overset{\mathrm{v}}{\nabla} u_l) + \|Z(u_l)\|^2 + P(u_l,u_l) - \beta_{l+1}\|X_+ u_l\|^2\right]$$
$$= \|X_- u\|_\alpha^2 - (R\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u) + \|Z(u)\|^2 + P(u,u) - \|X_+ u\|_\beta^2.$$

The result will follow if we can show that the curvature, Z and P terms localise. Thus Lemma 4.2 is a corollary of the next result.

Lemma 4.3. *If (M, g) is a compact Riemannian manifold, then*

$$(R\overset{\mathtt{v}}{\nabla}u, \overset{\mathtt{v}}{\nabla}w) = 0, \qquad (Z(u), Z(w)) = 0, \qquad P(u, w) = 0$$

whenever $u \in \Omega_m$, $w \in \Omega_l$ and $m \neq l$. In particular

$$\overset{\mathtt{v}}{\operatorname{div}}T : \Omega_m \to \Omega_m.$$

Proof. The localization of the curvature term was proved [PS18, Lemma 5.4]. We shall prove here that the Z-term localizes. That is enough to obtain also the conclusion for P since Proposition 3.4 and the polarization identity imply that

$$P(u, w) = (X_+u, X_+w)_\beta - (X_-u, X_-w)_\alpha + (R\overset{\mathtt{v}}{\nabla}u, \overset{\mathtt{v}}{\nabla}w) - (Z(u), Z(w)).$$

Hence the statements for the curvature and Z-term imply that $P(u, w) = 0$ when $m \neq l$. The last claim follows since $P(u, w) = -(\overset{\mathtt{v}}{\operatorname{div}}Tu, w)$.

The claim for $Z(u)$ for $d = 2$ follows from [PS18, Remark 6.5] using the explicit representation for $Z(u)$. To prove the claim when $d \geq 3$, recall that $Z(u)$ is the $\overset{\mathtt{v}}{\operatorname{div}}$-free part of $\overset{\mathtt{h}}{\nabla}u$ (the $\overset{\mathtt{v}}{\operatorname{div}}$-free part is uniquely defined since there are no nontrivial harmonic 1-forms on S_xM when $d \geq 3$). Using the bracket relation $\overset{\mathtt{h}}{\nabla} = \overset{\mathtt{v}}{\nabla}X - X\overset{\mathtt{v}}{\nabla}$ we can relate $X\overset{\mathtt{v}}{\nabla}$ and $Z(u)$. Indeed this is done explicitly in equation (3.9), which shows that $Z(u)$ is the $\overset{\mathtt{v}}{\operatorname{div}}$-free part of $-X\overset{\mathtt{v}}{\nabla}u$. If we consider a coordinate system around a point x such that $\partial_{x_j}g_{kl}(x) = 0$ for all j, k, l and write $\overset{\mathtt{v}}{\nabla}u = (\partial^k u)\partial_{x_k}$ as in [PSU15, Appendix A], then at x

$$X\overset{\mathtt{v}}{\nabla}u = v^j\partial_{x_j}(\partial^k u)\partial_{x_k} = v^j\partial^k(\partial_{x_j}u)\partial_{x_k} = v^j\overset{\mathtt{v}}{\nabla}(\partial_{x_j}u).$$

Hence if we think of each v^j as 1-form it is enough to analyze the vertical Fourier decomposition of $A\overset{\mathtt{v}}{\nabla}w$, where A is a scalar 1-form and $w = \partial_{x_j}u \in \Omega_m$. This is precisely the content of Lemma A.1, and combining the statement of that lemma with (3.9) we see that $Z(u) = -B(u)$ where B is the operator in Lemma A.1 for $X\overset{\mathtt{v}}{\nabla}$. Since B localizes in frequency, the lemma is proved. □

5. Stability for the transport equation

In this section we will prove the main stability estimate for solutions of the transport equation $Xu = f$ in SM when f has finite degree. In the next section we will give the more standard form where the solenoidal part of f is estimated in terms of $I_m f$.

Theorem 5.1. *Let (M, g) be a compact Riemannian manifold with smooth boundary and sectional curvature ≤ 0, let $u \in C^\infty(SM)$, and write $f := Xu$. Suppose that f has finite degree m. If $m = 0$, then*

$$\|f\|_{L^2(SM)} \leq \|u\|_{H^{1/2}_T(\partial SM)}$$

whereas if $m \geq 1$, then

$$\|f - X(u_0 + \ldots + u_{m-1})\|_{L^2(SM)} \leq \|u\|_{H_T^{1/2}(\partial SM)}.$$

5.1. **Shifted Pestov identity with boundary terms.** To prove Theorem 5.1 we first assume that $m \geq 1$, and discuss the case $m = 0$ later. We will try to estimate f in terms of $u|_{\partial SM}$ in suitable norms. The starting point is the identity from Proposition 4.1 with $l \geq 1$:

$$\alpha_{l-1}\|X_- u_l\|^2 - (R\overset{\mathrm{v}}{\nabla} u_l, \overset{\mathrm{v}}{\nabla} u_l) + \|Z(u_l)\|^2 + P(u_l, u_l) = \beta_{l+1}\|X_+ u_l\|^2.$$

Since we are assuming non-positive sectional curvature, we have

$$-(R\overset{\mathrm{v}}{\nabla} u_l, \overset{\mathrm{v}}{\nabla} u_l) + \|Z(u_l)\|^2 \geq 0$$

and thus

$$\alpha_{l-1}\|X_- u_l\|^2 + P(u_l, u_l) \leq \beta_{l+1}\|X_+ u_l\|^2.$$

We divide this estimate by α_{l-1} (always different from zero since $l \geq 1$), which corresponds to shifting the estimate down by one half vertical derivatives since $\alpha_{l-1} \sim l$. It follows that

$$\|X_- u_l\|^2 + \frac{1}{\alpha_{l-1}} P(u_l, u_l) \leq \frac{\beta_{l+1}}{\alpha_{l-1}}\|X_+ u_l\|^2.$$

The constant $\frac{\beta_{l+1}}{\alpha_{l-1}}$ is exactly $D_d(l)^2$ where $D_d(l)$ is as in [PSU15, Lemma 5.1]. Note that $D_d(l) \leq 1$ for $d \geq 4$ and in the remaining cases it is sufficiently close to one for all practical purposes (when reading the proof it may be helpful to think that $D_d(l) \equiv 1$).

Thus we have the following inequality:

$$\|X_- u_l\|^2 + \frac{1}{\alpha_{l-1}} P(u_l, u_l) \leq D_d(l)^2\|X_+ u_l\|^2. \tag{5.1}$$

For $l \geq m$ we have $X_- u_{l+2} + X_+ u_l = 0$ and using (5.1) we may write

$$\|X_- u_l\|^2 + \frac{1}{\alpha_{l-1}} P(u_l, u_l) \leq D_d(l)^2\|X_- u_{l+2}\|^2.$$

Starting at $l = m$ and iterating this inequality N times leads to

$$\|X_- u_m\|^2 \leq \left[\prod_{j=0}^{N-1} D_d(m+2j)^2\right]\|X_- u_{m+2N}\|^2 - \sum_{j=0}^{N-1} \frac{\prod_{k=0}^{j-1} D_d(m+2k)^2}{\alpha_{m-1+2j}} P(u_{m+2j}, u_{m+2j})$$

Write $\gamma_{d,m,j} = \prod_{k=0}^{j-1} D_d(m+2k)^2$ and $\gamma_{d,m,0} = 1$. In the notation of [PSU15, Theorem 1.1] one has $\gamma_{d,m,j} = \prod_{k=0}^{j-1} C_d(m-1+2k)^2$, and thus $\gamma_{d,m,j} \leq c_d$ where

$$c_d = \begin{cases} 2, & d = 2, \\ 1.28, & d = 3, \\ 1, & d \geq 4. \end{cases} \tag{5.2}$$

Since $\|X_- u_l\|^2 \to 0$ as $l \to \infty$, we may take the limit as $N \to \infty$ to obtain

$$\|X_- u_m\|^2 \leq -\sum_{j=0}^{\infty} \frac{\gamma_{d,m,j}}{\alpha_{m-1+2j}} P(u_{m+2j}, u_{m+2j}). \tag{5.3}$$

The argument above gives a completely analogous inequality for $\|X_- u_{m+1}\|^2$, and adding these two inequalities leads to

$$\|X_- u_m\|^2 + \|X_- u_{m+1}\|^2 \leq -\sum_{k=0}^{\infty} b_{m,k} P(u_{m+k}, u_{m+k}) \tag{5.4}$$

where

$$b_{m,k} = \begin{cases} \frac{\gamma_{d,m,j}}{\alpha_{m-1+2j}}, & k = 2j, \\ \frac{\gamma_{d,m+1,j}}{\alpha_{m+2j}}, & k = 2j+1. \end{cases}$$

Define $q := u_0 + u_1 + \cdots + u_{m-1}$. Then the transport equation $Xu = f$ also gives

$$Xq + X_- u_m + X_- u_{m+1} = f \tag{5.5}$$

and thus $\|f - Xq\|^2 = \|X_- u_m\|^2 + \|X_- u_{m+1}\|^2$. This yields

$$\|f - Xq\|^2 \leq -\sum_{k=0}^{\infty} b_{m,k} P(u_{m+k}, u_{m+k}). \tag{5.6}$$

If we assume $m = 0$, then the equation $Xu = f$ implies $X_- u_1 = f$, and (5.3) gives

$$\|f\|^2 \leq -\sum_{j=0}^{\infty} b_{1,2j} P(u_{1+2j}, u_{1+2j}) \leq -\sum_{k=0}^{\infty} b_{1,k} P(u_{1+k}, u_{1+k}). \tag{5.7}$$

Thus to prove Theorem 5.1 for $m \geq 0$, it remains to bound the right hand side of (5.6) for all $m \geq 1$.

5.2. **The right hand side of** (5.6)**: the space** $H_T^{1/2}(\partial SM)$**.** Let $m \geq 1$. Motivated by (5.6) and the fact that P is defined in terms of the (horizontal) tangential operator T, we define a natural $H^{1/2}$-space as follows. Define the $H_T^1(\partial SM)$-norm by setting

$$\|u\|_{H_T^1}^2 := \|u\|_{L^2}^2 + \|\overset{\mathtt{h}}{\nabla}{}^{\|} u\|_{L^2}^2.$$

The space $H_T^{1/2}(\partial SM)$ is defined by complex interpolation between L^2 and H_T^1. The norm $H_T^{-1/2}$ is defined by duality, and then $H_T^{-1/2}$ is also the interpolation space between L^2 and H_T^{-1} (see [BL76, Corollary 4.5.2]).

Remark 5.2. Note that from (3.5) we have

$$\|Tu\|_{L^2} \leq \|\overset{\mathtt{h}}{\nabla}{}^{\|} u\|_{L^2} \tag{5.8}$$

since $|Tu|^2 = \mu^2 |\overset{\mathtt{h}}{\nabla}{}^{\|} u|^2 + |X^{\|} u|^2 \leq (\mu^2 + |v^{\|}|^2) |\overset{\mathtt{h}}{\nabla}{}^{\|} u|^2 = |v|^2 |\overset{\mathtt{h}}{\nabla}{}^{\|} u|^2 = |\overset{\mathtt{h}}{\nabla}{}^{\|} u|^2$.

Now we use the key property of localization given by Lemma 4.3 to observe that

$$\sum_{k=0}^{\infty} b_{m,k} P(u_{m+k}, u_{m+k}) = P\left(\sum_{k=0}^{\infty} u_{m+k}, \sum_{l=0}^{\infty} b_{m,l} u_{m+l}\right).$$

We define an operator $B = B_m : C^\infty(\partial SM) \to C^\infty(\partial SM)$ by setting

$$Bu := \sum_{l=0}^{\infty} b_{m,l} u_{m+l}.$$

Since $m \geq 1$, the constant $b_{m,l}$ is well defined also when $l = 0$. Now (5.6) becomes

$$\|f - Xq\|^2 \leq -P(u, Bu) = -(Tu, \overset{\mathrm{v}}{\nabla} Bu). \tag{5.9}$$

Here is the main claim:

Lemma 5.3. *Given $u \in C^\infty(\partial SM)$ we have*

$$(Tu, \overset{\mathrm{v}}{\nabla} Bu) \leq \|u\|^2_{H_T^{1/2}}.$$

Proof. We may write

$$(Tu, \overset{\mathrm{v}}{\nabla} Bu) = -(B \overset{\mathrm{v}}{\mathrm{div}} Tu, u).$$

By the definitions, it suffices to show that

$$\|B \overset{\mathrm{v}}{\mathrm{div}} Tu\|_{H_T^{-1/2}} \leq \|u\|_{H_T^{1/2}}.$$

By interpolation, this follows from the next two inequalities

$$\|B \overset{\mathrm{v}}{\mathrm{div}} Tu\|_{L^2} \leq \|u\|_{H_T^1}, \tag{5.10}$$

$$\|B \overset{\mathrm{v}}{\mathrm{div}} Tu\|_{H_T^{-1}} \leq \|u\|_{L^2}. \tag{5.11}$$

To prove these estimates we first establish the property

$$\|\overset{\mathrm{v}}{\nabla} Bu\|_{L^2} \leq \|u\|_{L^2}. \tag{5.12}$$

Indeed using the definition of B,

$$\|\overset{\mathrm{v}}{\nabla} Bu\|^2_{L^2} = (Bu, \Delta Bu) = \sum_{l=0}^{\infty} \lambda_{m+l} b_{m,l}^2 \|u_{m+l}\|^2_{L^2}.$$

To prove (5.12) we will show that $\lambda_{m+l} b_{m,l}^2 \leq 1$ for $m \geq 1$ and $l \geq 0$. If $m = 1$ and $l = 0$, then $\lambda_1 b_{1,0}^2 = \frac{\lambda_1}{\alpha_0^2} = \frac{1}{d-1} \leq 1$, so we may assume $m, l \geq 1$. One has

$\gamma_{d,m,j} \leq c_d$, which gives $\lambda_{m+l} b_{m,l}^2 \leq c_d^2 \frac{\lambda_{m+l}}{\alpha_{m-1+l}^2}$. Observe that simplifying (3.6) gives $\alpha_l = \frac{(2l+d-2)(l+d-1)}{l+d-2}$ for all $l \geq 1$. We thus have, writing $k = m + l \geq 2$,

$$\begin{aligned} \frac{\lambda_k}{\alpha_{k-1}^2} &= \frac{k(k+d-2)(k+d-3)^2}{(2k+d-4)^2(k+d-2)^2} \\ &= \frac{1}{4} \frac{k^3 + (2d-6)k^2 + (d-3)^2 k}{(k^2 + (d-4)k + (\frac{d-4}{2})^2)(k+d-2)} \\ &= \frac{1}{4} \frac{k^3 + (2d-6)k^2 + (d^2 - 6d + 9)k}{k^3 + (2d-6)k^2 + (d^2 - 6d + 8 + (\frac{d-4}{2})^2)k + (d-2)(\frac{d-4}{2})^2}. \end{aligned}$$

Thus if $d = 2$ or $d \geq 6$, one has $(\frac{d-4}{2})^2 \geq 1$ and hence $\frac{\lambda_k}{\alpha_{k-1}^2} \leq \frac{1}{4}$. If $d = 3, 4, 5$ we estimate

$$\frac{\lambda_k}{\alpha_{k-1}^2} \leq \frac{1}{4} \frac{k^3 + (2d-6)k^2 + (d^2 - 6d + 9)k}{k^3 + (2d-6)k^2 + (d^2 - 6d + 8)k} \leq \frac{1}{4}\left[1 + \frac{1}{(k+d-3)^2 - 1}\right] \leq \frac{1}{3}$$

using $k \geq 2$. Combining these estimates with (5.2), we have

$$\lambda_{m+l} b_{m,l}^2 \leq c_d^2 \frac{\lambda_{m+l}}{\alpha_{m-1+l}^2} \leq 1.$$

The estimate (5.12) follows. Since $-\overset{\vee}{\nabla} B$ is the adjoint of $B\overset{\vee}{\mathrm{div}}$, using (5.12) and (5.8) yields

$$\|B\overset{\vee}{\mathrm{div}} Tu\|_{L^2} \leq \|Tu\|_{L^2} \leq \|u\|_{H_T^1}$$

thus proving (5.10).

Finally to prove (5.11), we note that $B\overset{\vee}{\mathrm{div}} T = \overset{\vee}{\mathrm{div}} TB$ by Lemma 4.3. Using Lemma 3.5 we may write

$$\begin{aligned} \|B\overset{\vee}{\mathrm{div}} Tu\|_{H_T^{-1}} &= \sup_{\|h\|_{H_T^1}=1} (\overset{\vee}{\mathrm{div}} TBu, h) \\ &= \sup_{\|h\|_{H_T^1}=1} -(T^* \overset{\vee}{\nabla} Bu, h) \\ &= \sup_{\|h\|_{H_T^1}=1} -(\overset{\vee}{\nabla} Bu, Th) \\ &\leq \sup_{\|h\|_{H_T^1}=1} \|u\|_{L^2} \|Th\|_{L^2} \\ &\leq \|u\|_{L^2}, \end{aligned}$$

where in the penultimate line we used (5.12) and (5.8). □

Theorem 5.1 for $m \geq 1$ now follows from (5.9) and Lemma 5.3. When $m = 0$, it follows from (5.7) and Lemma 5.3.

6. Stability for the solenoidal part

We now rewrite Theorem 5.1 in terms of the solenoidal part of f and extend the result to H^1 regularity. Recall that the map

$$\ell_m : C^\infty(S^m(T^*M)) \to \bigoplus_{k=0}^{[m/2]} \Omega_{m-2k},$$

is an isomorphism and it gives a natural identification between functions in Ω_m and *trace-free* symmetric m-tensors (for details on this see [GK80, DS11, PSU15]). The identification actually holds pointwise for every $x \in M$. Moreover, the L^2-norms on trace free symmetric m-tensors and functions in Ω_m are the same up to a constant depending only on d and m (cf. [DS11, Lemma 2.4]).

If we let $\tilde{f} := \ell_m^{-1} f$ and $\tilde{q} := \ell_{m-1}^{-1} q$, the well-known relation $X\ell_{m-1} = \ell_m d^s$ implies that $\ell_m^{-1}(f - Xq) = \tilde{f} - d^s\tilde{q}$. To simplify the notation we shall the drop the tildes, identify f with $\tilde{f}$, q with $\tilde{q}$ and use that the L^2-norms are equivalent.

We first collect regularity properties of solutions of transport equations involving H^1 tensor fields.

Lemma 6.1. *Let (M, g) be a compact simple manifold. Given $f \in H^1(S^m(T^*M))$, there is $u^f \in H^1(SM)$ satisfying*

$$Xu^f = -f \text{ in } SM, \qquad u^f|_{\partial_- SM} = 0, \qquad u^f|_{\partial_+ SM} = If. \tag{6.1}$$

Moreover, one has $u^f|_{\partial SM} \in H^1(\partial SM)$ and $If \in H^1_0(\partial_+ SM)$.

Proof. If $f \in C^\infty(S^m(T^*M))$, define u^f on SM by

$$u^f(x, v) = \int_0^{\tau(x,v)} f(\varphi_t(x, v))\, dt$$

where φ_t is the geodesic flow on SM. One has $u^f \in C^\infty(SM \setminus S(\partial M)) \cap C(SM)$ since the same is true for τ, and (6.1) holds for u^f. By [Sh95, Corollary 1], the map $f \mapsto u^f$ extends as a bounded map $H^1(S^m(T^*M)) \to H^1(SM)$. (This boils down to the fact that $\overset{\mathtt{v}}{\nabla}\tau$ and $\overset{\mathtt{h}}{\nabla}{}^{\|}\tau$, where the operator $\overset{\mathtt{h}}{\nabla}{}^{\|}$ is extended smoothly to SM, are uniformly bounded on $SM \setminus S(\partial M)$, see [Sh94, Lemma 4.1.3] and [DPSU07, Lemma 5.1].) Moreover, by [Sh94, Theorem 4.2.1] the map $f \mapsto If$ extends as a bounded map $H^1(S^m(T^*M)) \to H^1(\partial_+ SM)$. Then the properties (6.1) remain valid for $f \in H^1$ (the boundary value of u^f is in $H^{1/2}(\partial SM)$ by the trace theorem). Since If vanishes on the boundary of $\partial_+ SM$ when $f \in C^\infty$, one has $If \in H^1_0(\partial_+ SM)$ first for $f \in C^\infty$ and then for $f \in H^1$ by density. Since $u^f|_{\partial SM} = E_0(If)$ where E_0 denotes extension by zero from $\partial_+ SM$ to ∂SM, we have $u^f|_{\partial SM} \in H^1(\partial SM)$ when $f \in H^1$. □

Next we give a version of Theorem 5.1 for H^1 tensor fields.

Theorem 6.2. *Let (M, g) be a simply connected compact manifold with strictly convex boundary and sectional curvature ≤ 0, and let $f \in H^1(S^m(T^*M))$. If $m = 0$, then*

$$\|f\|_{L^2(M)} \leq C\|u^f\|_{H^{1/2}_T(\partial SM)} \tag{6.2}$$

whereas if $m \geq 1$, then

$$\|f - d^s q\|_{L^2(M)} \leq C\|u^f\|_{H_T^{1/2}(\partial SM)}. \tag{6.3}$$

Here C only depends on d and m.

Proof. Let $m \geq 1$ (the case $m = 0$ is analogous). Going back to (5.3) and using Lemma 5.3, one has the inequality

$$\|X_- u_m\|^2 + \|X_- u_{m+1}\|^2 \leq \|u\|^2_{H_T^{1/2}(\partial SM)}, \qquad u \in C^\infty(SM).$$

Since functions in $H^1(SM)$ have traces in $H^{1/2}(\partial SM)$, and hence also in $H_T^{1/2}(\partial SM)$, the above inequality holds for $u \in H^1(SM)$ by density. Then it is enough to take $u = u^f$, where $u^f \in H^1(SM)$ by Lemma 6.1, and to note that by (5.5) and by equivalence of the L^2 norms

$$\|X_- u_m\|^2 + \|X_- u_{m+1}\|^2 = \|f - Xq\|^2_{L^2(SM)} \geq c(d,m)\|f - d^s q\|^2_{L^2(M)}. \qquad \square$$

The estimate (6.2) for $m = 0$ is already in the form that we want, so we will focus on the case $m \geq 1$. Using the potential and solenoidal decomposition, we may write $f = f_s + d^s p$ where $\delta^s f_s = 0$ and p is an $(m-1)$-tensor such that $p|_{\partial M} = 0$. Let $w = p - q$. Then integrating by parts

$$\begin{aligned} \|f - d^s q\|^2 &= \|f_s + d^s w\|^2 \\ &= \|f_s\|^2 + 2(f_s, d^s w) + \|d^s w\|^2 \\ &= \|f_s\|^2 + 2(\iota_\nu f_s, w)_{\partial M} + \|d^s w\|^2 \\ &\geq \|f_s\|^2 - 2|(\iota_\nu f_s, q)_{\partial M}|. \end{aligned} \tag{6.4}$$

Next we observe that for any $\varepsilon > 0$

$$2|(\iota_\nu f_s, q)_{\partial M}| \leq \frac{1}{\varepsilon}\|q\|^2_{H^{1/2}(\partial M)} + \varepsilon\|\iota_\nu f_s\|^2_{H^{-1/2}(\partial M)}. \tag{6.5}$$

We now claim:

Lemma 6.3. *We have*

$$\|\iota_\nu f_s\|_{H^{-1/2}(\partial M)} \lesssim \|f_s\|_{L^2(M)}.$$

Proof. This is a duality argument, but it is important that $\delta^s f_s = 0$. Consider a bounded extension map for symmetric $(m-1)$-tensors, $e : H^{1/2}(\partial M) \to H^1(M)$ (such a map can be constructed from a corresponding extension map for functions by working in local coordinates and using a partition of unity). Now write

$$\begin{aligned} \|\iota_\nu f_s\|_{H^{-1/2}(\partial M)} &= \sup_{\|h\|_{H^{1/2}(\partial M)}=1} \int_{\partial M} \langle \iota_\nu f_s, h\rangle \, dS \\ &= \sup_{\|h\|_{H^{1/2}(\partial M)}=1} \left(-(\delta^s f_s, e(h)) + (f_s, d^s e(h))\right) \\ &= \sup_{\|h\|_{H^{1/2}(\partial M)}=1} (f_s, d^s e(h)) \\ &\lesssim \|f_s\|_{L^2(M)}. \end{aligned} \qquad \square$$

Combining (6.3)–(6.5) with Lemma 6.3 and choosing ε small enough, it follows that

$$\|f_s\|^2_{L^2(M)} \lesssim \|u^f\|^2_{H^{1/2}_T(\partial SM)} + \|q\|^2_{H^{1/2}(\partial M)}. \tag{6.6}$$

The next two lemmas will be useful when estimating the last term on the right.

Lemma 6.4. *Given $m \geq 0$, there is a constant $C > 0$ such that for any tensor q of order m*

$$\|q\|_{H^{1/2}(\partial M)} \leq C\|\ell_m q\|_{H^{1/2}_T(\partial SM)}.$$

Proof. Recall that we identify symmetric m-tensors with functions in $\bigoplus_{k=0}^{[m/2]} \Omega_{m-2k}$ via ℓ_m as explained at the beginning of this section. By interpolation, it is enough to show that $\|\ell_m^{-1} h\|_{L^2} \lesssim \|h\|_{L^2}$ and $\|\ell_m^{-1} h\|_{H^1} \lesssim \|h\|_{H^1_T}$ when $h \in \bigoplus_{k=0}^{[m/2]} \Omega_{m-2k}$. The first inequality follows from the equivalence of the L^2-norms. For the second inequality, observe that locally a symmetric m-tensor field can be written as $q = q_{i_1 \dots i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m}$. The H^1-norm of q in ∂M consists of the L^2-norm of q plus the L^2 norm of the components $q_{i_1 \dots i_m}(x)$ *tangentially* to M. Locally $\ell_m q$ has the form $q_{i_1 \dots i_m} v^{i_1} \dots v^{i_m}$. When we apply $\overset{\mathtt{h}}{\nabla}{}^{\parallel}$ to $\ell_m q$ all the tangential derivatives in the direction of ∂M will appear. There will also be some vertical derivatives (involving the Christoffel symbols), but since $\ell_m q$ is a polynomial of degree m in v, these terms can all be controlled by the L^2-norm of $\ell_m q$. Thus $\|q\|_{H^1} \lesssim \|\ell_m q\|_{H^1_T}$ follows, and this may be rewritten as $\|\ell_m^{-1} h\|_{H^1} \lesssim \|h\|_{H^1_T}$. □

Lemma 6.5 (The $H^1_T(\partial SM)$ norm localizes in frequency)**.** *One has*

$$\|u\|^2_{H^1_T(\partial SM)} = \sum_{m=0}^{\infty} \|u_m\|^2_{H^1_T(\partial SM)}$$

for all $u \in H^1_T(\partial SM)$. In particular, $\|\sum_{l=0}^m u_l\|_{H^1_T(\partial SM)} \leq \|u\|_{H^1_T(\partial SM)}$ when $m \geq 0$.

Proof. The proof is somewhat indirect and is based on the following observations.

(1) Let W be a vector field on M and let $\mathbb{W} = (W, 0)$ be its horizontal lift to SM. Then $\Delta \mathbb{W} = \mathbb{W} \Delta$ where Δ is the vertical Laplacian. This can be seen by taking a geodesic coordinate neighbourhood around a point x, so that $\partial_j g_{kl}(x) = 0$ for all $j, k.l$. In that case if we write $W = w^i(x)\partial_{x_i}$, then $(\mathbb{W}u)(x, v) = w^i(x)\frac{\partial u}{\partial x_i}$ and thus $\mathbb{W} : \Omega_m \to \Omega_m$. (Another way to prove this is to check that $[\delta_j, \Delta] = 0$, using the notation and commutator formulas in [PSU15, Appendix A].)

(2) There is a neighbourhood U_ε of ∂M in M diffeomorphic to $\partial M \times [0, \varepsilon)$ via $\partial M \times [0, \varepsilon) \ni (x, t) \mapsto \exp_x(-t\nu(x)) \in U_\varepsilon$. This allows us to naturally extend to U_ε the exterior unit normal ν to a vector field, still denoted by ν.

(3) A smooth function $u \in C^\infty(\partial SM)$ can be extended to a smooth function $u^{\mathbb{v}} \in C^\infty(SU_\varepsilon)$ simply by making it constant on the orbits of the flow of $\mathbb{v}$, the horizontal lift of ν. By item (1) we have

$$(u_m)^{\mathbb{v}} = (u^{\mathbb{v}})_m \tag{6.7}$$

(4) Let f_t be the flow of ν in SM, and let V_ε be the neighbourhood of ∂SM in SM diffeomorphic to $\partial SM \times [0,\varepsilon)$ via $(x,v,t) \mapsto f_{-t}(x,v)$. Since $f_{-t}(x,v) = (x(t), v(t))$ where $x(t) = \exp_x(-t\nu)$ is the normal geodesic and $v(t)$ is the parallel transport of v along $x(t)$, one has $V_\varepsilon = SU_\varepsilon$ (the map $v \mapsto v(t)$ is bijective from S_xM onto $S_{x(t)}M$).

Let $u \in C^\infty(\partial SM)$. The fact that $\nu(u^\nu) = 0$ implies that $\overset{\mathtt{h}}{\nabla}{}^{\|} u = (\overset{\mathtt{h}}{\nabla} u^\nu)|_{\partial SM}$ and thus for $(x,v) \in \partial SM$ we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon |\overset{\mathtt{h}}{\nabla} u^\nu(x,v,t)|^2 \, dt = |\overset{\mathtt{h}}{\nabla}{}^{\|} u(x,v)|^2.$$

Integrating over ∂SM and using that $V_\varepsilon = SU_\varepsilon$ by (4), this gives

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \|\overset{\mathtt{h}}{\nabla} u^\nu\|^2_{L^2(SU_\varepsilon)} = \|\overset{\mathtt{h}}{\nabla}{}^{\|} u\|^2_{L^2(\partial SM)}. \tag{6.8}$$

We now recall that it is possible to write for any $w \in C^\infty(SU_\varepsilon)$ (all norms in $L^2(SU_\varepsilon)$), cf. [LRS18, proof of Lemma 5.1]:

$$\|\overset{\mathtt{h}}{\nabla} w\|^2 = \|Z(w)\|^2 + \|X_- w_1\|^2 + \sum_{l=1}^\infty A(d,l)\|X_+ w_{l-1}\|^2 + B(d,l)\|X_- w_{l+1}\|^2, \tag{6.9}$$

where $A(d,l) = 2 + \frac{d-2}{2}$ and $B(d,l) = 2 + \frac{1}{d+l-2}$. By Lemma 4.3 one may write $\|Z(w)\|^2 = \sum \|Z(w_m)\|^2$, and thus using (6.9) for $w = u^\nu$ and $w = (u^\nu)_m$ for each m we deduce

$$\|\overset{\mathtt{h}}{\nabla} u^\nu\|^2_{L^2(SU_\varepsilon)} = \sum_{m=0}^\infty \|\overset{\mathtt{h}}{\nabla} (u^\nu)_m\|^2_{L^2(SU_\varepsilon)}.$$

Dividing this by ε and taking the limit as $\varepsilon \to 0$, the identities (6.7) and (6.8) yield

$$\|\overset{\mathtt{h}}{\nabla}{}^{\|} u\|^2_{L^2(\partial SM)} = \sum_{m=0}^\infty \|\overset{\mathtt{h}}{\nabla}{}^{\|} u_m\|^2_{L^2(\partial SM)}.$$

This implies the desired claim for $u \in C^\infty(\partial SM)$, and the result follows since $C^\infty(\partial SM)$ is dense in $H^1_T(\partial SM)$. (The density claim can be proved by using a partition of unity, convolution approximation in coordinate neighborhoods, and the Friedrichs lemma [Hö85, Lemma 17.1.5].) ☐

We now put the arguments above together to derive:

Theorem 6.6. *Let (M,g) be a simply connected compact manifold with strictly convex boundary and non-positive sectional curvature. Given $m \geq 0$ there is a constant $C > 0$ such that for any $f \in H^1(S^m(T^*M))$, one has*

$$\|f_s\|_{L^2(M)} \leq C \|u^f\|_{H^{1/2}_T(\partial SM)}.$$

Proof. For $m = 0$ this is just (6.2), so we assume $m \geq 1$. Combining (6.6) and Lemma 6.4 we derive

$$\|f_s\| \lesssim \|u^f\|_{H^{1/2}_T} + \|\ell_{m-1} q\|^2_{H^{1/2}_T}.$$

Recall that $\ell_{m-1}q = \sum_{l\leq m-1} u_l^f$. Interpolating the bound $\|\sum_{l\leq m-1} u_l\|_{L^2(\partial SM)} \leq \|u\|_{L^2(\partial SM)}$ for $u \in L^2(\partial SM)$ with the bound in Lemma 6.5 gives

$$\|\sum_{l\leq m-1} u_l\|_{H^{1/2}_T} \leq \|u\|_{H^{1/2}_T}, \qquad u \in H^{1/2}_T(\partial SM),$$

and the result follows by taking $u = u^f$. □

We can refine this further and prove Theorem 1.3 in the introduction. Define the space $H^{1/2}_T(\partial_+ SM)$ as the interpolation space between $H^1_T(\partial_+ SM)$ and $L^2(\partial_+ SM)$.

Proof of Theorem 1.3. Theorem 6.6 gives

$$\|f_s\|_{L^2(M)} \leq C\|E_0(If)\|_{H^{1/2}_T(\partial SM)} \tag{6.10}$$

where E_0 is extension by zero from $\partial_+ SM$ to ∂SM. We define $H^1_{T,0}(\partial_+ SM)$ as the closure of $C^\infty_c((\partial_+ SM)^{\mathrm{int}})$ with respect to the H^1_T-norm, and $H^{1/2}_{T,0}(\partial_+ SM)$ as the interpolation space between L^2 and $H^1_{T,0}(\partial_+ SM)$. Since E_0 is bounded $H^1_{T,0}(\partial_+ SM) \to H^1_T(\partial SM)$ and $L^2(\partial_+ SM) \to L^2(\partial SM)$, it is also bounded

$$E_0 : H^{1/2}_{T,0}(\partial_+ SM) \to H^{1/2}_T(\partial SM). \tag{6.11}$$

Now, if $f \in H^1(S^m(T^*M))$, then If is in $H^1_0(\partial_+ SM)$ by Lemma 6.1 and hence also in the larger space $H^{1/2}_{T,0}(\partial_+ SM)$. Combining (6.10) and (6.11) proves the result. □

Remark 6.7. Theorem 1.3 remains true for $f \in H^{1/2}(S^m(T^*M))$ by density, since I is bounded $H^{1/2}(S^m(T^*M)) \to H^{1/2}(\partial_+ SM)$ by [Sh94, Theorem 4.2.1] and interpolation. It would be interesting if one could prove Theorem 1.3 for $f \in L^2(S^m(T^*M))$. However, in general we do not know if I is bounded $L^2(S^m(T^*M)) \to H^{1/2}_T(\partial_+ SM)$. Also, our approach with the Pestov identity as it stands is unable to produce stability estimates for higher order Sobolev norms.

Appendix A. Vector spherical harmonics

In Lemma 4.3, we want to prove the frequency localization statement

$$(Z(u_m), Z(w_l)) = 0, \qquad m \neq l,$$

where $u_m \in \Omega_m$ and $w_l \in \Omega_l$. As discussed in the proof of Lemma 4.3, this follows from the next localization statement. Recall that if $A \in \Omega_1$, i.e. $A(x,v) = A_j(x)v^j$, then $Au = A_+u + A_-u$ for $u \in C^\infty(SM)$ where $A_\pm : \Omega_m \to \Omega_{m\pm 1}$.

Lemma A.1. *Let $d = dim(M) \geq 3$ and let $A \in \Omega_1$. For any $u_m \in \Omega_m$ one has*

$$A\overset{\mathrm{v}}{\nabla} u_m = \overset{\mathrm{v}}{\nabla}\left[\left(1 - \frac{1}{m+1}\right) A_+ u_m + \left(1 + \frac{1}{m+d-3}\right) A_- u_m\right] + B(u_m)$$

where $B(u_m)$ is the $\overset{\mathrm{v}}{\mathrm{div}}$-free part of $A\overset{\mathrm{v}}{\nabla} u_m$. The map $B : C^\infty(SM) \to \mathcal{Z}$ satisfies

$$(B(u_m), B(w_l)) = 0, \qquad m \neq l,$$

for any $u_m \in \Omega_m$ and $w_l \in \Omega_l$.

The proof is based on understanding vector spherical harmonics expansions on SM. This will be done next, and the proof of Lemma A.1 will be given in the end of this appendix.

If $u_m \in \Omega_m$, then $\overset{\mathrm{v}}{\nabla} u_m$ is an element of the space

$$\mathcal{Z} = \{Z \in C^\infty(SM, TM)\,;\, Z(x,v) \in T_xM \text{ and } \langle Z(x,v), v\rangle = 0\}.$$

Thus each $Z(x,\,\cdot\,)$ is a vector field on S_xM, and using the Sasaki metric it can be identified with a 1-form on S_xM. The Hodge Laplacian $d\delta + \delta d$ acting on 1-forms on each S_xM induces a vertical Laplacian on $\mathcal{Z}$,

$$\Delta_1 : \mathcal{Z} \to \mathcal{Z}.$$

Spherical harmonics expansions on $\mathcal{Z}$ are eigenfunction expansions of Δ_1. However, since each S_xM is isometric to the standard round sphere S^{d-1}, it is really enough to understand spherical harmonics expansions of 1-forms on S^{d-1} as studied e.g. in [Bo09]. The case $d = \dim(M) = 2$ is easy and reduces to Fourier expansions on the circle. The cases $d = 3$ and $d \geq 4$ will have different features.

The case $d = 2$. Using the formulas in [PSU15, Appendix B], when $d = \dim(M) = 2$ the space $\mathcal{Z}$ can be identified with $C^\infty(SM)$ via $\mathcal{Z} = \{z(x,v)iv\,;\, z \in C^\infty(SM)\}$ and one has

$$\Delta_1(z(x,v)iv) = -\overset{\mathrm{v}}{\nabla}\overset{\mathrm{v}}{\mathrm{div}}(z(x,v)iv) = -(V^2 z)iv.$$

Thus the eigenfunction expansions for Δ_1 in $\mathcal{Z}$ are the same as eigenfunction expansions for the vertical Laplacian $-V^2$ on $L^2(SM)$. The eigenvalues of Δ_1 are $\{\lambda_m\}_{m=0}^\infty$ where $\lambda_m = m(m + d - 2) = m^2$, and one has the orthogonal decomposition

$$L^2(SM, N) = \bigoplus_{m=0}^\infty H_m^1$$

where $H_m^1 = \{Z \in L^2(SM, N)\,;\, \Delta_1 Z = \lambda_m Z\} = \{z(x,v)iv\,;\, -V^2 z = \lambda_m z\}$. In particular, any $Z \in \mathcal{Z}$ has the L^2-orthogonal expansion

$$Z = \sum_{m=0}^\infty z_m(x,v)\, iv$$

where $z_m \in \Omega_m$.

The case $d = 3$. It turns out that when $d = 3$, the study of vector spherical harmonics still reduces to Fourier expansions on functions but there is extra structure.

Thus let $d = \dim(M) = 3$, and fix $x \in M$. Recall that $Z(x,\,\cdot\,)$ is a vector field on S_xM. Since S_xM is two-dimensional (and isometric to S^2), there is a well-defined rotation by $90°$ on each $T_v(S_xM)$ which induces a map

$$* : \mathcal{Z} \to \mathcal{Z}, \quad *Z(x,v) = (*_{S_xM}(Z(x,v)^\flat))^\sharp$$

where $*_{S_xM}$ is the Hodge star operator on S_xM and the musical isomorphisms act with respect to the metric g_x on S_xM.

Spherical harmonics expansions on $\mathcal{Z}$ are described as follows:

Lemma A.2. *Let (M,g) be a Riemannian manifold with $d = dim(M) = 3$. The eigenvalues of Δ_1 are $\{\lambda_m\}_{m=1}^{\infty}$ where $\lambda_m = m(m+d-2)$, and one has the orthogonal decomposition*

$$L^2(SM,N) = \bigoplus_{m=1}^{\infty} H_m^1$$

where $H_m^1 = \{Z \in L^2(SM,N)\,;\, \Delta_1 Z = \lambda_m Z\}$. Moreover, writing $\Omega_m^1 = H_m^1 \cap C^\infty(SM,N)$, one has the L^2-orthogonal decomposition

$$\Omega_m^1 = \overset{\mathrm{v}}{\nabla}\Omega_m \oplus *\overset{\mathrm{v}}{\nabla}\Omega_m.$$

In particular, any $Z \in \mathcal{Z}$ has the L^2-orthogonal expansion

$$Z = \sum_{m=1}^{\infty} (\overset{\mathrm{v}}{\nabla} u_m + *\overset{\mathrm{v}}{\nabla} w_m)$$

where $u_m, w_m \in \Omega_m$.

Proof. It is enough to work on S_xM for fixed x, and since S_xM is isometric to S^2 it is enough to determine the eigenvalues and eigenspaces of the Hodge Laplacian $\Delta = d\delta + \delta d$ in S^2 when acting on 1-forms. Recall that $\delta = - * d *$ and $** = (-1)^k$ on k-forms.

If α is a 1-form on S^2, one has the Hodge decomposition $\alpha = du + \delta\beta$ where u is a 0-form and β is a 2-form (there are no nontrivial harmonic 1-forms on S^2). Moreover, $\beta = - * w$ for some 0-form w and one has

$$\alpha = du + *dw.$$

Since u and w are smooth functions on S^2, they have spherical harmonics expansions

$$u = \sum_{m=0}^{\infty} u_m, \qquad w = \sum_{m=0}^{\infty} w_m, \qquad u_m, w_m \in \Omega_m(S^2).$$

The Hodge Laplacian satisfies $\Delta d = d\Delta$ and $\Delta * d = *d\Delta$. It follows that

$$\Delta\alpha = \sum_{m=1}^{\infty} \lambda_m (du_m + *dw_m). \tag{A.1}$$

In the L^2 inner product on S^2 one has $(du_m, dw_l) = (\Delta u_m, w_l) = \lambda_m(u_m, w_l)$ if $u_m \in \Omega_m, w_l \in \Omega_l$. Thus it follows that

$$(du_m, dw_l) = (*du_m, *dw_l) = 0 \text{ if } m \neq l,$$

and $(du_m, *dw_l) = 0$ for all m, l. From (A.1) we see by orthogonality that the eigenvalues of Δ on 1-forms are $\{\lambda_m\}_{m=1}^{\infty}$ and the eigenspace corresponding to λ_m consists of elements of the form $du_m + *dw_m$ with $u_m, w_m \in \Omega_m$. □

Next we will see how multiplication by $A(x,v) \in \Omega_1$ affects Fourier decompositions on $\mathcal{Z}$. Recall that on $C^\infty(SM)$, multiplication by A splits as $A = A_+ + A_-$ where $A_\pm : \Omega_m \to \Omega_{m\pm1}$. In the case of $\mathcal{Z}$, there is a new operator A_0 which allows us to describe how multiplication by A affects Fourier expansions.

Lemma A.3. *Let $d = \dim(M) = 3$ and let $A \in \Omega_1$. The operator*

$$A_0 : C^\infty(SM) \to C^\infty(SM), \quad A_0 u = \langle \overset{\mathrm{v}}{\nabla} A, *\overset{\mathrm{v}}{\nabla} u \rangle$$

satisfies $A_0 : \Omega_m \to \Omega_m$ for any m. For any $u \in C^\infty(SM)$ one has

$$A\overset{\mathrm{v}}{\nabla} u = \sum_{l=1}^{\infty} \overset{\mathrm{v}}{\nabla}\Big[\left(1 - \frac{1}{l}\right) A_+ u_{l-1} + \left(1 + \frac{1}{l+d-2}\right) A_- u_{l+1}\Big] + \sum_{l=1}^{\infty} *\overset{\mathrm{v}}{\nabla}\Big[\frac{1}{\lambda_l} A_0 u_l\Big]$$

and

$$A * \overset{\mathrm{v}}{\nabla} u = -\sum_{l=1}^{\infty} \overset{\mathrm{v}}{\nabla}\Big[\frac{1}{\lambda_l} A_0 u_l\Big] + \sum_{l=1}^{\infty} *\overset{\mathrm{v}}{\nabla}\Big[\left(1 - \frac{1}{l}\right) A_+ u_{l-1} + \left(1 + \frac{1}{l+d-2}\right) A_- u_{l+1}\Big].$$

Proof. Again it is enough to fix x and to work with differential forms on S^2. So we are reduced to the following: if $A(v) = a_j v^j$ is a scalar function in S^2 with $a_j \in \mathbb{C}$, and if d, δ and $\Delta = d\delta + \delta d$ are the corresponding operators on S^2, we need to show that the operator

$$A_0(u) = \langle dA, *du \rangle$$

maps $\Omega_m(S^2)$ to itself, and for any $u \in C^\infty(S^2)$ we have

$$Adu = \sum_{l=1}^{\infty} d\Big[\left(1 - \frac{1}{l}\right) A_+ u_{l-1} + \left(1 + \frac{1}{l+d-2}\right) A_- u_{l+1}\Big] + \sum_{l=1}^{\infty} *d\Big[\frac{1}{\lambda_l} A_0 u_l\Big]. \tag{A.2}$$

The last statement will then follow since A commutes with $*$.

Define vector fields on S^2, $\partial_j u = \partial_{y_j}(u(y/|y|))|_{S^2}$ (see [PSU15, Appendix A] for more details). Then $du = \partial_j u \, dv^j$, and $*du$ is the cross product of v and du,

$$*du = (v_2\partial_3 u - v_3 \partial_2 u)\, dv^1 + (v_3 \partial_1 u - v_1 \partial_3 u)\, dv^2 + (v_1 \partial_2 u - v_2 \partial_1 u)\, dv^3$$

which can be written as $*du = (V_j u)\, dv^j$ with $V_1 = v_2 \partial_3 - v_3 \partial_2$ etc. In the following we will raise and lower indices with respect to the Euclidean metric in $\mathbb{R}^3$ (i.e. write v^j for v_j etc). Since $A(v) = a_k v^k$ we have $\partial_j A = a_k(\delta_j^k - v_j v^k) = a_j - A v_j$. Since $\langle dv^j, dv^k \rangle = \delta^{jk} - v^j v^k$ and since $v^j \partial_j = v^j V_j = 0$, it follows that

$$A_0 u = (\partial_j A) V^j u = a_j V^j u.$$

But now, on SM in any dimension $d = \dim(M)$, the vertical Laplacian $\Delta_0 = -\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{v}}{\nabla}$ on scalar functions satisfies

$$[\Delta_0, v_a] = -2\partial_a + (d-1) v_a, \qquad [\Delta_0, \partial_b] = -(d-3)\partial_b + 2 v_b \Delta_0$$

and consequently

$$[\Delta_0, v_a \partial_b - v_b \partial_a] = 0.$$

It follows that each V^j maps $\Omega_m(S^2)$ to itself, and the same is then true for A_0.

It remains to show (A.2) for $u \in C^\infty(S^2)$. We begin with the Hodge decomposition

$$Adu = d\alpha + *d\beta$$

where $\alpha, \beta \in C^\infty(S^2)$ are determined from

$$\Delta\alpha = \delta(Adu),$$
$$\Delta\beta = - * d(Adu).$$

To solve for the Fourier components of α and β, we expand the right hand sides of the above equations in Fourier series. One has

$$\begin{aligned}\delta(Adu) &= A(\delta du) - \langle dA, du\rangle = A\Delta u + \frac{1}{2}(\Delta(Au) - (\Delta A)u - A(\Delta u)) \\ &= \frac{1}{2}(\Delta(Au) + A\Delta u - (\Delta A)u).\end{aligned}$$

Expanding u in spherical harmonics and using that $\Delta A = (d-1)A$ we get

$$\delta(Adu) = \frac{1}{2}\sum_{l=1}^{\infty}\left[(\lambda_l - (d-1))(A_+u_{l-1} + A_-u_{l+1}) + \lambda_{l-1}A_+u_{l-1} + \lambda_{l+1}A_-u_{l+1}\right].$$

(Note that $(\delta(Adu))_0 = \frac{1}{2}(A_-\Delta u_1 - (d-1)A_-u_1) = 0$.) It follows that for $l \geq 1$,

$$\alpha_l = \frac{1}{2\lambda_l}\left[(\lambda_l + \lambda_{l-1} - (d-1))A_+u_{l-1} + (\lambda_l + \lambda_{l+1} - (d-1))A_-u_{l+1}\right].$$

For the second equation, we have

$$- * d(Adu) = - * (dA \wedge du) = *(dA \wedge *(*du)) = \langle dA, *du\rangle = A_0 u$$

and therefore for $l \geq 1$ one has

$$\beta_l = \frac{1}{\lambda_l}A_0 u_l.$$

Substituting α and β in the Hodge decomposition $Adu = d\alpha + *d\beta$ proves (A.2). $\square$

The case $d \geq 4$. If $d = \dim(M) \geq 4$, the Laplacian Δ_1 acting on $\mathcal{Z}$ will have two sets of eigenvalues corresponding to the Hodge demposition of 1-forms. If x is fixed, then S_xM is isometric to S^{d-1} and any 1-form α on S_xM has a Hodge decomposition

$$\alpha = d_{S_xM}u + \delta_{S_xM}\beta$$

where u is a 0-form and β is an exact 2-form on S_xM. Correspondingly, any $Z \in \mathcal{Z}$ has the Hodge decomposition

$$Z = \overset{\mathrm{v}}{\nabla}u + \overset{\mathrm{v}}{\delta}\beta$$

for some $u \in C^\infty(SM)$ and $\beta \in \mathcal{Z}^2$, where $\mathcal{Z}^2$ is the set of smooth 2-forms β_x on each S_xM varying smoothly with respect to x, and $\overset{\mathrm{v}}{\delta}\beta$ is the vertical codifferential of β identified with an element of $\mathcal{Z}$. In fact one can take $\beta \in \mathcal{Z}^{2,d}$, where $\mathcal{Z}^{2,d}$ is the set of 2-forms $\beta \in \mathcal{Z}^2$ so that β_x is exact for each x.

The following result is a consequence of [Bo09].

Lemma A.4. *Let (M, g) be a Riemannian manifold with $d = dim(M) \geq 4$. The eigenvalues of Δ_1 are $\{\lambda_m, \mu_m\}_{m=1}^{\infty}$ where $\lambda_m = m(m+d-2)$ and $\mu_m = \lambda_m + d - 3$, and one has the L^2-orthogonal decomposition*

$$\mathcal{Z} = \bigoplus_{m=1}^{\infty} \left[\Omega_m^{1,d} \oplus \Omega_m^{1,\delta}\right]$$

where $\Omega_m^{1,d} = \{Z \in \mathcal{Z}\,;\, \Delta_1 Z = \lambda_m Z\}$ and $\Omega_m^{1,\delta} = \{Z \in \mathcal{Z}\,;\, \Delta_1 Z = \mu_m Z\}$. Moreover, one has

$$\Omega_m^{1,d} = \overset{\mathrm{v}}{\nabla}\Omega_m,$$
$$\Omega_m^{1,\delta} = \overset{\mathrm{v}}{\delta}\Omega_m^{2,d}$$

where $\Omega_m^{2,d} = \{\beta \in \mathcal{Z}^{2,d}\,;\, \Delta_2 \beta = \mu_m \beta\}$ with Δ_2 corresponding to the Hodge Laplacian on 2-forms on each S_xM. In particular, any $Z \in \mathcal{Z}$ has the L^2-orthogonal expansion

$$Z = \sum_{m=1}^{\infty} (\overset{\mathrm{v}}{\nabla} u_m + \overset{\mathrm{v}}{\delta}\beta_m)$$

where $u_m \in \Omega_m$ and $\beta_m \in \Omega_m^{2,d}$.

Next we see how multiplication by $A \in \Omega_1$ affects Fourier decompositions. If $Z, W \in \mathcal{Z}$ we write $Z \wedge W$ for the element of $\mathcal{Z}^2$ obtained as the wedge product of the 1-forms corresponding to Z and W.

Lemma A.5. *Let $d = dim(M) \geq 4$ and let $A \in \Omega_1$. The operator*

$$A_0 : C^\infty(SM) \to \mathcal{Z}^{2,d}, \quad A_0 u = \overset{\mathrm{v}}{\nabla} A \wedge \overset{\mathrm{v}}{\nabla} u$$

satisfies $A_0 : \Omega_m \to \Omega_m^{2,d}$ for any $m \geq 1$. For any $u \in C^\infty(SM)$ one has

$$A\overset{\mathrm{v}}{\nabla} u = \sum_{l=1}^{\infty} \overset{\mathrm{v}}{\nabla}\Big[\left(1 - \frac{1}{l}\right) A_+ u_{l-1} + \left(1 + \frac{1}{l+d-2}\right) A_- u_{l+1}\Big] + \sum_{l=1}^{\infty} \overset{\mathrm{v}}{\delta}\Big[\frac{1}{\mu_l} A_0 u_l\Big].$$

The proof uses the following lemma.

Lemma A.6. *If u and v are 1-forms on a Riemannian manifold (M, g) and if $\Delta = d\delta + \delta d$ is the Hodge Laplacian, then*

$$\Delta(u \wedge v) = (\Delta u) \wedge v - 2 \sum_{j=1}^{\dim(M)} \nabla_{E_j} u \wedge \nabla_{E_j} v + u \wedge (\Delta v) + 2R(u^\sharp, v^\sharp, \cdot\,, \cdot)$$

where $\{E_1, \ldots, E_n\}$ is any local orthonormal frame, ∇ is the Levi-Civita connection and R is the Riemann curvature tensor.

Proof. Follows either by relating Δ to the connection Laplacian via a Weitzenbock identity, or by computations in normal coordinates. $\square$

Proof of Lemma A.5. It is enough to fix x and to work with differential forms on S^{d-1}. Thus we are reduced to the following: if $A(v) = a_j v^j$ is a scalar function in S^{d-1} with $a_j \in \mathbb{C}$, and if d, δ and $\Delta = d\delta + \delta d$ are the corresponding operators on S^{d-1}, we need to show that the operator

$$A_0 : C^\infty(S^{d-1}) \to C^\infty(S^{d-1}, \Lambda^2), \quad A_0(u) = dA \wedge du$$

maps $\Omega_m(S^{d-1})$ to $\Omega_m^{2,d}(S^{d-1})$, and for any $u \in C^\infty(S^{d-1})$ we have

$$A\,du = \sum_{l=1}^\infty d\Big[\left(1 - \frac{1}{l}\right) A_+ u_{l-1} + \left(1 + \frac{1}{l+d-2}\right) A_- u_{l+1} \Big] + \sum_{l=1}^\infty \delta\Big[\frac{1}{\mu_l} A_0 u_l\Big]. \tag{A.3}$$

Since $A_0 u = d(A\,du)$, A_0 maps into the set of exact 2-forms. Let $u \in \Omega_m$. By Lemma A.6, for any local orthonormal frame $\{E_j\}$ of $T(S^{d-1})$ we have

$$\begin{aligned}
\Delta(A_0 u) &= \Delta(dA \wedge du) \\
&= (\Delta dA) \wedge du - 2\sum_{j=1}^{d-1} \nabla_{E_j} dA \wedge \nabla_{E_j} du + dA \wedge (\Delta du) + 2R(dA^\sharp, du^\sharp, \,\cdot\,, \,\cdot\,) \\
&= (\lambda_m + d - 1) dA \wedge du - 2\sum_{j=1}^{d-1} \nabla_{E_j} dA \wedge \nabla_{E_j} du + 2R(dA^\sharp, du^\sharp, \,\cdot\,, \,\cdot\,)
\end{aligned}$$

using that $u \in \Omega_m$ and $A \in \Omega_1$. Now if $v \in S^{d-1}$ and $w \in T_v S^{d-1}$ with $|w| = 1$, and if $\gamma(t)$ is the geodesic on S^{d-1} with $\dot\gamma(0) = w$, one has

$$\nabla dA|_v(w, w) = \frac{d^2}{dt^2} A(\gamma(t))\Big|_{t=0} = \frac{d^2}{dt^2}(a_j \gamma^j(t))\Big|_{t=0} = -A(v)$$

using that geodesics are great circles. Thus $\nabla_{E_j} dA|_v(w) = -A(v)\langle E_j, w\rangle$, which gives that $\nabla_{E_j} dA|_v = -A(v) E_j^\flat$ and

$$\sum_{j=1}^{d-1} \nabla_{E_j} dA \wedge \nabla_{E_j} du = -A(v) \sum_{j=1}^{d-1} E_j^\flat \wedge \nabla_{E_j} du = -A(v) d(du) = 0.$$

Also, on the sphere we have $R(u^\sharp, v^\sharp, \,\cdot\,, \,\cdot\,) = -u \wedge v$. These facts imply that

$$\Delta(A_0 u) = (\lambda_m + d - 3) dA \wedge du = \mu_m A_0 u.$$

Thus A_0 maps Ω_m to $\Omega_m^{2,d}$.

We next show (A.3) for $u \in C^\infty(S^{d-1})$. We begin with the Hodge decomposition

$$A\,du = d\alpha + \delta\beta$$

where the function $\alpha \in C^\infty(S^{d-1})$ and the exact 2-form $\beta \in C^\infty(S^{d-1}, \Lambda^2)$ are determined from

$$\begin{aligned}
\Delta\alpha &= \delta(A du), \\
\Delta\beta &= d(A du).
\end{aligned}$$

To solve for the Fourier components of α and β, we expand the right hand sides of the above equations in Fourier series. One has

$$\begin{aligned}\delta(Adu) &= A(\delta du) - \langle dA, du\rangle = A\Delta u + \frac{1}{2}(\Delta(Au) - (\Delta A)u - A(\Delta u))\\ &= \frac{1}{2}(\Delta(Au) + A\Delta u - (\Delta A)u).\end{aligned}$$

Expanding u in spherical harmonics and using that $\Delta A = (d-1)A$ we get

$$\delta(Adu) = \frac{1}{2}\sum_{l=1}^{\infty}\left[(\lambda_l - (d-1))(A_+u_{l-1} + A_-u_{l+1}) + \lambda_{l-1}A_+u_{l-1} + \lambda_{l+1}A_-u_{l+1}\right].$$

It follows that for $l \geq 1$,

$$\alpha_l = \frac{1}{2\lambda_l}\left[(\lambda_l + \lambda_{l-1} - (d-1))A_+u_{l-1} + (\lambda_l + \lambda_{l+1} - (d-1))A_-u_{l+1}\right]$$

For the second equation, we have

$$d(Adu) = dA \wedge du = A_0 u.$$

Since A_0 maps Ω_l to $\Omega_l^{2,d}$ for $l \geq 1$, one has

$$\beta = \sum_{l=1}^{\infty}\frac{1}{\mu_l}A_0 u_l.$$

Substituting α and β in the Hodge decomposition $Adu = d\alpha + \delta\beta$ proves (A.3). □

Proof of Lemma A.1. Since $A\overset{\mathrm{v}}{\nabla}u_0 = 0$, it is enough to study $A\overset{\mathrm{v}}{\nabla}u_m$ for $m \geq 1$. For $d = 3$, Lemma A.3 shows that $B(u_m) = *\overset{\mathrm{v}}{\nabla}\frac{1}{\lambda_m}A_0u_m$ where $A_0 : \Omega_m \to \Omega_m$. Consequently for $m \neq l$

$$(B(u_m), B(w_l)) = \frac{1}{\lambda_m\lambda_l}(-\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{v}}{\nabla}A_0u_m, A_0w_l) = \frac{1}{\lambda_l}(A_0u_m, A_0w_l) = 0.$$

Finally let $d \geq 4$. By Lemma A.5 we have $B(u_m) = \overset{\mathrm{v}}{\delta}\frac{1}{\mu_m}A_0u_m$ where $A_0 : \Omega_m \to \Omega_m^{2,d}$. On the space $\Omega_m^{2,d}$ corresponding to exact 2-forms, the Hodge Laplacian Δ_2 is given by $d\delta$ and has eigenvalue μ_m. Hence for $m \neq l$

$$(B(u_m), B(w_l)) = \frac{1}{\mu_m\mu_l}(\Delta_2 A_0u_m, A_0w_l) = \frac{1}{\mu_l}(A_0u_m, A_0w_l) = 0$$

using that the eigenspaces for Δ_2 having different eigenvalues are L^2-orthogonal. This concludes the proof. □

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