FIXED ANGLE INVERSE SCATTERING IN THE PRESENCE OF A RIEMANNIAN METRIC

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ABSTRACT. We consider a fixed angle inverse scattering problem in the presence of a known Riemannian metric. First, assuming a no caustics condition, we study the direct problem by utilizing the progressing wave expansion. Under a symmetry assumption on the metric, we obtain uniqueness and stability results in the inverse scattering problem for a potential with data generated by two incident waves from opposite directions. Further, similar results are given using one measurement provided the potential also satisfies a symmetry assumption. This work extends the results of [23, 24] from the Euclidean case to certain Riemannian metrics.

Keywords: inverse medium problem, fixed angle scattering, Carleman estimates, Riemannian metric

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1. INTRODUCTION

We study the fixed angle inverse scattering problem in the presence of a Riemannian metric. In Euclidean space, this problem corresponds to determining a potential $q \in C_c^\infty(\mathbb{R}^n)$ from the knowledge of the scattering amplitude $a_q(\,\cdot\,, \omega, \,\cdot\,)$ for a fixed direction $\omega \in S^{n-1}$. Equivalently, one would like to determine q by measuring $U|_{\partial B \times \mathbb{R}}$ where U solves the wave equation

$$\begin{cases} (\partial_t^2 - \Delta + q)U(x, t) = 0 \text{ in } \mathbb{R}^{n+1} & \text{(1.1a)} \\ U|_{\{t \ll 0\}} = \delta(t - x \cdot \omega), & \text{(1.1b)} \end{cases}$$

and where B is a ball containing the support of q.

In the recent articles [24] and [23], it was proved that measurements corresponding to *two* fixed directions $\pm\omega \in S^{n-1}$ uniquely determine a potential $q \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$. Here $C_c^\infty(\mathbb{R}^n; \mathbb{R})$ stands for the set of real-valued C^∞-smooth functions with compact supports. The objective of this work is to investigate to what extent it is possible to replace the Euclidean metric on $\mathbb{R}^n$ by a known Riemannian metric. This corresponds to determining a potential q from two scattering measurements in the presence of a known nonconstant sound speed.

We establish some notations in order to state the main theorems. Denote by B the open unit ball in $\mathbb{R}^n$. Let x be Cartesian coordinates in $\mathbb{R}^n$, and let g be a smooth Riemannian metric on $\mathbb{R}^n$ so that

$$g_{jk}(x) = \delta_{jk} \text{ for } |x| \geq 1. \tag{1.2}$$

Let Δ_g be the Laplace-Beltrami operator, given in coordinates by

$$\Delta_g u = |g|^{-1/2} \partial_j(|g|^{1/2} g^{jk} \partial_k u)$$

where $(g^{jk}) = (g_{jk})^{-1}$ and $|g| = \det(g_{jk})$. Here and throughout this article, we use the Einstein summation convention where a repeated index in upper and lower position is summed.

Given $q \in C_c^\infty(\mathbb{R}^n)$ supported in $\overline{B}$, let $U_q^\pm = U_q^\pm(x,t)$ be the unique solution of

$$\begin{cases} (\partial_t^2 - \Delta_g + q)U_q^\pm = 0 \text{ in } \mathbb{R}^{n+1}, & (1.3a) \\ U_q^\pm|_{\{t<-1\}} = \delta(t \mp x_n). & (1.3b) \end{cases}$$

The assumption (1.2) ensures that $\delta(t \mp x_n)$ solves the free equation $(\partial_t^2 - \Delta_g)U = 0$ for $t < -1$. Then $U_q^\pm$ is the solution related to the incoming plane wave $\delta(t \mp x_n)$ propagating in direction $\pm e_n$. We will make an *absence of caustics* assumption (here $B_\pm = B \cap \{\pm x_n \geq 0\}$):

There are $\omega_\pm \in C^\infty(\overline{B})$ with $|d\omega_\pm|_g = 1$ in B, $\omega_\pm = x_n$ to first order on $\partial B_\pm$.

This assumption implies that the solutions $U_q^\pm$ have the following explicit representations in $\overline{B} \times \mathbb{R}$ (see Lemma 2.1): one has

$$U_q^\pm(x,t) = \delta(t \mp \omega_\pm(x)) + u_q^\pm(x,t)H(t \mp \omega_\pm(x))$$

where $H(t)$ is the Heaviside function, and $u_q^\pm$ is smooth in $\{t \geq \pm\omega_\pm(x)\}$ and vanishes in $\{t < \pm\omega_\pm(x)\}$. Thus measuring $U_q^\pm|_{\partial B \times \mathbb{R}}$ corresponds to measuring $u_q^\pm|_{\partial B \times \mathbb{R}}$, and the latter quantity is well defined pointwise.

We wish to give an analogue of the results in [24] and [23] stating that measurements from two fixed directions $\pm e_n$ uniquely determine the potential q. The method relies on putting the solutions u_q^+ in $\{t \geq \omega_+(x)\}$ and u_q^- in $\{t \geq -\omega_-(x)\}$ together by applying the reflection $t \mapsto -t$ to the latter solution. For this, we need the interfaces $\{t = \omega_+(x)\}$ and $\{t = \omega_-(x)\}$ to match, i.e. we need $\omega_+ = \omega_-$. This is guaranteed by the following condition:

$$\begin{cases} \text{There is } \omega \in C^\infty(\overline{B}) \text{ with } |d\omega|_g = 1 \text{ in } B \\ \text{and } \omega = x_n \text{ to infinite order on } \partial B. \end{cases} \qquad (1.4)$$

This condition combines an absence of caustics assumption and a symmetry assumption for the metric g. In fact, by Lemma A.1 each of the following two conditions is equivalent with (1.4):

$$\begin{cases} \mathbb{R}^n \text{ is smoothly parametrized by } g\text{-geodesics starting from } \{x_n = -1\} \text{ in} \\ \text{direction } e_n, \text{ and any such geodesic meets } \{x_n = 1\} \text{ pointing in direction } e_n. \end{cases} \qquad (1.4a)$$

$$\begin{cases} \text{There is a global coordinate system } y = (y', y_n) \text{ in } \mathbb{R}^n \text{ with} \\ \partial_{y_n} = \partial_{x_n} \text{ outside } B, \text{ so that } g(y) = \begin{pmatrix} g_0(y) & 0 \\ 0 & 1 \end{pmatrix}. \end{cases} \qquad (1.4b)$$

Finally, the method is based on Carleman estimates for the wave equation, which are available in particular under the following assumption (see the discussion after Lemma 3.4 as well as Lemma 3.5):

$(\overline{B}, g)$ admits a smooth strictly convex function with no critical point in $\overline{B}$. (1.5)

Our first main theorem states that under these assumptions it is possible to determine an arbitrary potential q from two measurements.

Theorem 1.1. *Let g be a smooth Riemannian metric on $\mathbb{R}^n$, $n \geq 2$, satisfying (1.2), (1.4) and (1.5). There is $T > 0$ such that for any $q_1, q_2 \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$ supported in $\overline{B}$, if one has*

$$u_{q_1}^\pm|_{\partial B \times [-1,T]} = u_{q_2}^\pm|_{\partial B \times [-1,T]},$$

then $q_1 = q_2$.

It follows from (1.4b) that, for example, any metric of the form

$$g(x) = \begin{pmatrix} I_{n-1} + h(x) & 0 \\ 0 & 1 \end{pmatrix},$$

where $h \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^{(n-1)\times(n-1)})$ is supported in $\overline{B}$ and $\|h\|_{C^2}$ is sufficiently small, satisfies (1.2), (1.4) and (1.5).

Moreover, if (y', y_n) are as in (1.4b) and if g and q satisfy the symmetry properties

$$g(y', -y_n) = g(y', y_n), \tag{1.6}$$

$$q(y', -y_n) = q(y', y_n), \tag{1.7}$$

then q is uniquely determined by a single measurement $u_q^+|_{\partial B \times [-1,T]}$. This follows immediately from Theorem 1.1 since $u_q^-(y', y_n, t) = u_q^+(y', -y_n, t)$ (it is enough to note that $U = U_q^+(y', -y_n, t)$ solves $(\partial_t^2 - \Delta_g + q)U = 0$ in $\mathbb{R}^{n+1}$ with $U = \delta(t + x_n)$ for $t < -1$, using (1.4b), (1.6) and (1.7)).

Theorem 1.2. *Let g be a smooth Riemannian metric on $\mathbb{R}^n$, $n \geq 2$, satisfying* (1.2), (1.4), (1.5) *and* (1.6). *There is $T > 0$ such that for any $q_1, q_2 \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$ supported in $\overline{B}$ and satisfying* (1.7), *if one has*

$$u_{q_1}^+|_{\partial B \times [-1,T]} = u_{q_2}^+|_{\partial B \times [-1,T]},$$

then $q_1 = q_2$.

Theorems 1.1 and 1.2 come with Lipschitz stability estimates (see Section 4 for the precise statement). As in [23] it is also possible to prove uniqueness and stability results in the single measurement case when the odd part of $q_1 - q_2$ with respect to the reflection $(y', y_n) \mapsto (y', -y_n)$ is small compared to $q_1 - q_2$. We omit the details.

1.1. Discussion and connection to the existing results.
The fixed angle inverse scattering problem is a formally determined nonlinear inverse problem in any dimension $n \geq 1$. It can be studied either in the frequency domain, as the problem of determining q from the scattering amplitude $a_q(\,\cdot\,, \omega, \,\cdot\,)$ for the Schrödinger operator $-\Delta + q$ with a fixed direction $\omega \in S^{n-1}$, or as the problem of recovering q from boundary or scattering measurements of the solution U_q of the wave equation. The equivalence of these two problems is proved in detail in [23] (see also [16,17,19] for the case of odd dimensions).

The one-dimensional case is classical and has been thoroughly studied, see [7,15]. When $n \geq 2$ there are results on uniqueness for small or generic potentials [1,27], recovery of singularities [20,26], and recovery of the zero potential [2]. Recently it was proved in [23,24] that measurements corresponding to two opposite fixed angles uniquely determine a general smooth compactly supported potential. The problem with one measurement is still open in general, but uniqueness was proved in [23,24] for potentials satisfying certain symmetry conditions or horizontal control conditions (analogous to angularly controlled potentials in backscattering, see [25]). One also has Lipschitz stability estimates for the wave equation version of the problem. We also mention the work [21] which studies the fixed angle problem with first order coefficients.

All the above results consider the Euclidean metric. In this paper we extend the approach of [23, 24] to Riemannian metrics satisfying the conditions (1.2), (1.4) and (1.5). The argument follows the Bukhgeim-Klibanov method, see [6] and [11], on the use of Carleman estimates for formally determined inverse problems. Further information about this method and its variants may be found in [3–5,12–14,28]. We also refer to [29] for inverse scattering problems for metrics.

We now briefly describe the argument. The idea is that if $u_{q_1}|_{\partial B \times [-1,T]} = u_{q_2}|_{\partial B \times [-1,T]}$, then $\tilde{u} = u_{q_1} - u_{q_2}$ solves the wave equation

$$(\partial_t^2 - \Delta_g + q_1)\tilde{u} = -(q_1 - q_2)(x)u_2(x,t) \tag{1.8}$$

in a modified space-time cylinder Q_+ with zero lateral boundary values. By looking at the exterior problem, also $\partial_\nu \tilde{u}$ vanishes at the lateral boundary.

Now if the right hand side of (1.8) were zero, one could use the unique continuation principle for the wave equation (proved by using Carleman estimates) to conclude that $\tilde{u} = 0$ and hence $q_1 = q_2$. However, the right hand side source is not zero, but it has the special form $-\tilde{q}(x)u_2(x,t)$ where $\tilde{q} = q_1 - q_2$. This source is of the same form as in the Bukhgeim-Klibanov method. We then follow this method and apply a Carleman estimate with pseudoconvex weight that is large at the bottom of the cylinder Q_+, which effectively converts the source term in (1.8) into a boundary term at the bottom of Q_+. (In effect, this means that we still treat the problem as some kind of unique continuation problem, even if the right hand side source does not vanish.) The final issue is that the Carleman estimate has other boundary terms at the bottom of Q_+ with unfavorable signs. To deal with this, we use a second measurement from direction $-e_n$ and do a reflection argument which will cancel the boundary terms and allow us to conclude that $\tilde{u} = 0$ and thus $q_1 = q_2$.

In order to implement the above approach in the Riemannian setting, we need the metric to satisfy the three conditions (1.2), (1.4) and (1.5). We now describe the role of these assumptions in more detail.

- The condition (1.2) guarantees that the plane waves $\delta(t \mp x_n)$ solve the wave equation for $t \ll 0$, and that geodesics outside B are straight lines.
- The assumption (1.4) imposes both a symmetry condition and an absence of caustics condition on the metric, which may be seen from the equivalent conditions (1.4a) and (1.4b).
- The assumption (1.5) on the existence of a strictly convex function in $(\overline{B}, g)$ ensures that Carleman estimates with pseudoconvex weights are available for the wave equation. This assumption is discussed in [22, Lemma 2.1] when $(\overline{B}, g)$ has strictly convex boundary: in particular (1.5) is valid when the sectional curvature does not change sign or there are no focal points. Clearly (1.5) is stable under small perturbations of the metric g.

This article is organized as follows. Section 1 is the introduction and states the main results. Section 2 introduces some notations and formulates the solution of the direct problem. In Section 3, some energy estimates are given. Section 4 is devoted to the recovery of the potential using two measurements. The equivalence of the conditions (1.4), (1.4a) and (1.4b) is proved in Appendix A.

2. The analysis of the forward problem

In this section, we focus on the direct problem (1.3). The existence of a solution is shown by the progressing wave expansion, and uniqueness is guaranteed by standard energy estimates. We also present some basic properties of the solution which will be useful in solving the inverse problem.

2.1. **Notation and preliminaries.** We adopt the conventional notation for the Dirac delta distribution δ and its derivative δ', i.e. $(\delta, \varphi) = \varphi(0)$, and $(\delta', \varphi) = -\varphi'(0)$. We denote the divergence operator as $\tilde{\delta}$, i.e. $\tilde{\delta}(\alpha_k \, \mathrm{d}x^k) := |g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \alpha_k)$, and the Laplace-Beltrami operator Δ_g is defined as $\Delta_g u := \tilde{\delta}(\mathrm{d}u) = |g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \partial_k u)$, as already mentioned in Section 1. When $j \geq 0$, we define a one-dimensional distribution H_j as

$$H_j(s) = \begin{cases} s^j, & s \geq 0, \\ 0, & s < 0. \end{cases}$$

We also write $H_{-1}(s) := \delta(s)$, $H_{-2}(s) := \delta'(s)$ and $H_{-3}(s) := \delta''(s)$. One should note that

$$\dot{H}_j(s) = C_j H_{j-1}(s),$$

where $C_j = j$ when $j \geq 1$ and $C_j = 1$ when $j \leq 0$.

The g-inner product is denoted as $\langle du, dv \rangle := g^{jk}(\partial_j u)(\partial_k v)$ and $\langle \partial_j, \partial_k \rangle|_p := g_{jk}(p)$. The following equalities hold for sufficiently smooth functions:

$$\begin{cases} \Delta_g(uv) = \Delta_g u \cdot v + 2\langle du, dv \rangle + u \cdot \Delta_g v, & \text{(2.1a)} \\ \tilde{\delta}(udv) = u \cdot \Delta_g v + \langle du, dv \rangle, & \text{(2.1b)} \\ d\big(H_j(t - v(x))\big) = -dv\dot{H}_j(t - v), & \text{(2.1c)} \\ \Delta_g\big(H_j(t - v(x))\big) = -(\Delta_g v)\dot{H}_j(t - v) + \langle dv, dv \rangle \ddot{H}_j(t - v). & \text{(2.1d)} \end{cases}$$

2.2. Progressing wave expansion.

Consider an incoming plane wave $\delta(t - x_n)$, and the corresponding solution of the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_g + q)U & = 0 \text{ in } \mathbb{R}^{n+1}, \\ U|_{\{t \leq -1\}} & = \delta(t - x_n). \end{cases} \tag{2.2}$$

If the underlying metric g is Euclidean, the solution of this equation will have the form $U(x,t) = \delta(t - x_n) + u(x,t)H_0(t - x_n)$. However, when g is general Riemannian metric, the shape of the incident wave will be perturbed when passing through the support of q. We can represent the solution of (2.2) by using the progressing wave expansion method (cf. [18]).

The progressing wave expansion method assumes that the solution has the form

$$U(x,t) = \sum_{j \geq -1}^{N} a_j(x,t) \cdot H_j(t - \omega(x)) + R_N(x,t), \quad (x,t) \in B \times \mathbb{R}, \tag{2.3}$$

where $\omega(x) = x_n$ when $x_n < -1$ and $a_{-1}(x,t) = 1$ when $t < -1$, and R_N is the remainder term which is more regular than the other terms. We only consider the representation (2.3) for $(x,t) \in B \times \mathbb{R}$. We assume that the coefficients $a_j(x,t)$ satisfy $a_j(x,t) = a_j(x, \omega(x))$, i.e. they live on the hypersurface $\{t = \omega(x)\}$. Note that the first term $H_{-1}(t - \omega(x))$ in (2.3) equals to $\delta(t - \omega(x))$. According to (2.1), one can compute

$$(\Box_g + q)\big(U(x,t)\big)$$
$$= \sum_{j \geq -1}^{N} (\Box_g + q)a_j \cdot H_j + \sum_{j \geq -2}^{N-1} 2C_{j+1}[\partial_t a_{j+1} + \langle da_{j+1}, d\omega \rangle + \frac{\Delta_g \omega}{2} a_{j+1}] \cdot H_j$$
$$+ \sum_{j \geq -3}^{N-2} C_{j+2}C_{j+1}a_{j+2}[1 - \langle d\omega, d\omega \rangle]H_j + (\Box_g + q)R_N. \tag{2.4}$$

Combining (2.4) and the fact $(\Box_g + q)\big(U(x,t)\big) = 0$, and analyzing the smoothness of different terms, we see that ω should satisfy the eikonal equation

$$\langle d\omega, d\omega \rangle = 1 \text{ in } B. \tag{2.5}$$

Recall that one should have $\omega(x) = x_n$ when $x_n < -1$. Thus we require that

$$\omega(x) = x_n \text{ to infinite order on } \partial B \cap \{x_n < 0\}. \tag{2.6}$$

Let us assume that there exists $\omega \in C^\infty(\overline{B})$ satisfying (2.5)-(2.6) (this is true under a suitable no caustics assumption, cf. condition (1.4)). Then (2.4) becomes

$$(\Box_g + q)\big(U(x,t)\big)$$

$$= \sum_{j \geq -1}^{N-1} \left\{ 2C_{j+1}[\partial_t a_{j+1} + \langle \mathrm{d}a_{j+1}, \mathrm{d}\omega \rangle + \frac{\Delta_g \omega}{2} a_{j+1}] + (\square_g + q)a_j \right\} \cdot H_j$$

$$+ 2C_{-1}[\partial_t a_{-1} + \langle \mathrm{d}a_{-1}, \mathrm{d}\omega \rangle + \frac{\Delta_g \omega}{2} a_{-1}] \cdot H_{-2} + (\square_g + q)a_N \cdot H_N + (\square_g + q)R_N. \quad (2.7)$$

Denote

$$\Gamma_g := \{(x,t) \in \overline{B} \times \mathbb{R} \, ; \, t = \omega(x)\}. \quad (2.8)$$

By analyzing the smoothness of different terms on the right-hand-side of (2.7), the condition $(\square_g + q)U(x,t) = 0$ gives the following transport equations:

$$\begin{cases} \partial_t a_{-1} + \langle \mathrm{d}\omega, \mathrm{d}a_{-1} \rangle + \frac{\Delta_g \omega}{2} a_{-1} = 0, & \text{on} \quad \Gamma_g, \\ \partial_t a_j + \langle \mathrm{d}\omega, \mathrm{d}a_j \rangle + \frac{\Delta_g \omega}{2} a_j = -\frac{(\square_g + q)a_{j-1}}{2j}, & \text{on} \quad \Gamma_g, \ 0 \leq j \leq N. \end{cases} \quad (2.9)$$

Note that since each a_j only lives on Γ_g, it is enough to solve the equations for $j \geq 1$ on Γ_g.

These transport equations in (2.9) are on the hypersurface Γ_g and can be solved in a way similar to solving the traditional transport equation of the form $(\partial_t + \vec{a} \cdot \nabla)\varphi = 0$, i.e. by exploiting the integral curves of the differential operator $X := \partial_t + \langle \mathrm{d}\omega, \mathrm{d}\cdot \rangle$ which is tangential to Γ_g. The remaining transport equations can be solved on each Γ_{g,t_0} for $t_0 > 0$.

For later purposes we also define differential operators X^m ($1 \leq m \leq n$) as follows:

$$X^m|_{(x,t)} := \partial_j \omega(x) h^{jm}(x)\partial_t|_{(x,t)} + h^{mk}(x)\partial_k|_{(x,t)}, \quad (x,t) \in \overline{B} \times \mathbb{R}, \quad (2.10)$$

where h equals to $g^{-1/2}$, i.e. h is the unique matrix satisfying $\sum_m h^{jm}h^{mk} = g^{jk}$. (These operators are not necessary for the proofs, but they are available since we have global coordinates on $\mathbb{R}^n \times \mathbb{R}$ and some computations are easier to carry out using them.) The matrix h is real-valued and symmetric. From the invertibility of h we see that X^m is nonzero everywhere. The notation $X^m|_{(x,t)}$ means the value of the vector field X^m at point (x,t). For simplicity we also write $X^m|_{(x,t)}$ as $X^m_{(x,t)}$. It can be checked that X^m is tangential to Γ_g.

Proposition 2.1. *The vector field X^m defined as in (2.10) is tangential to Γ_g defined in (2.8).*

Proof. Let $H(x,t) = t - \omega(x)$, then H is the defining function of the hypersurface Γ_g. The conclusion can be reached by the following computation,

$$X^m|_{(x,\omega(x))}H = \partial_j \omega(x) h^{jm}(x)\partial_t|_{(x,t)}H + h^{mk}(x)\partial_k|_{(x,t)}H$$

$$= \partial_j \omega(x) h^{jm}(x) - h^{mk}(x)\partial_k \omega(x) = 0.$$

The proof is complete. $\qquad\square$

The vector field $X = \partial_t + \langle \mathrm{d}\omega, \mathrm{d}\cdot \rangle$ is a linear combination of $\{X^m\}_{m=1}^n$,

$$X|_{(x,t)} := \partial_k \omega(x) h^{km}(x) X^m|_{(x,t)} = \partial_t|_{(x,t)} + g^{jk}(x)\partial_j \omega(x)\partial_k|_{(x,t)}. \quad (2.11)$$

Thus X is also tangential to Γ_g. Without obscurity we may also denote the restriction of X to Γ_g as X. Hence, there exist integral curves of X in Γ_g and we denote them as

$$\gamma_{(x,\omega(x))} \colon t \in \mathbb{R} \ \mapsto \ \gamma_{(x,\omega(x))}(t) \in \Gamma_g \subset \mathbb{R}^{n+1}, \quad (2.12)$$

where $\gamma_{(x,\omega(x))}(0) = (x,\omega(x))$ and $(\gamma_{(x,\omega(x))})_*(\frac{\mathrm{d}}{\mathrm{d}s}|_s) = X|_{\gamma_{(x,\omega(x))}(s)}$. The subscript "$*$" and superscript "$*$" stand for the push-forward and pull-back operation, respectively. In what follows, we may omit the subscript $(x,\omega(x))$, so $\gamma(0) = (x,\omega(x))$ and $\gamma_*(\frac{\mathrm{d}}{\mathrm{d}s}|_s) = X|_{\gamma(s)}$ for short. Now for any smooth enough function f on Γ_g we have

$$X|_{\gamma(s)}f = \gamma_*(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_s)f = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_s (f \circ \gamma). \quad (2.13)$$

With the help of (2.11) and (2.13), we can rewrite (2.9) as

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s}(a_{-1}\circ\gamma)+\dfrac{(\Delta_g\omega)\circ\gamma}{2}(a_{-1}\circ\gamma)=0, \quad \text{on} \quad \Gamma_g & \text{(2.14a)} \\[2mm]
\dfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s}(a_j\circ\gamma)+\dfrac{(\Delta_g\omega)\circ\gamma}{2}(a_j\circ\gamma)=-\dfrac{[(\Box_g+q)a_{j-1}]\circ\gamma}{2}, \quad \text{on} \quad \Gamma_g & \text{(2.14b)}
\end{cases}
$$

for $0\le j\le N$, where γ denotes $\gamma_{(x,\omega(x))}$ and $\gamma(0)=(x,\omega(x))$. Now (2.14a) can be solved and solution is

$$
\begin{aligned}
a_{-1}\circ\gamma(s) &= a_{-1}\circ\gamma(-\infty)\, e^{-\frac{1}{2}\int_{-\infty}^{s}\Delta_g\omega(\gamma(\tau))\,\mathrm{d}\tau} \\
&= e^{-\frac{1}{2}\int_{-\infty}^{s}\Delta_g\omega(\gamma(\tau))\,\mathrm{d}\tau}.
\end{aligned}
\tag{2.15}
$$

We used the fact $a_{-1}\circ\gamma(-\infty)=1$. Note that ω is independent of time t, and we used $\omega(\gamma(\tau))$, i.e. $\omega(x,\omega(x))$, with slight ambiguity to represent $\omega(x)$ where $\gamma(\tau)=(x,\omega(x))\in\mathbb{R}^n\times\mathbb{R}$. We use a similar convention for $q(\gamma(\tau))$. We can solve every a_j from (2.14b) in a similar way as we solve a_{-1} in (2.15). We obtained the existence of the solution.

2.3. **Representation of the solution.** The form (2.3) is not the most convenient way to use to study the inverse problem. Now we are ready to give the formulation of the solution of the direct problem.

Lemma 2.1. *There exists a unique solution of* (2.2). *In the set $B\times\mathbb{R}$ the solution can be represented as*

$$
U(x,t)=a_{-1}(x)\cdot\delta(t-\omega(x))+u(x,t)\cdot H(t-\omega(x))
\tag{2.16}
$$

where a_{-1} is given by (2.15). *Moreover, u satisfies*

$$
u\circ\gamma(s)=-\frac{1}{2}(a_{-1}\circ\gamma)(s)\int_{-\infty}^{s}\Big[\frac{\Box_g a_{-1}(\gamma(\tau))}{a_{-1}(\gamma(\tau))}+q(\gamma(\tau))\Big]\,\mathrm{d}\tau,
\tag{2.17}
$$

$$
X|_{\gamma(s)}\left(\frac{u}{a_{-1}}\right)=-\frac{1}{2}\left(\frac{\Box_g a_{-1}(\gamma(s))}{a_{-1}(\gamma(s))}+q(\gamma(s))\right),
\tag{2.18}
$$

where the γ is defined in (2.12).

Proof. The distribution $U(x,t)=a_{-1}(x)\delta(t-x_n)+V(x,t)$ solves (2.2) if and only if V solves

$$
(\partial_t^2-\Delta_g+q)V=-(\partial_t^2-\Delta_g+q)\big(a_{-1}(x)\delta(t-x_n)\big)
$$

with $V=0$ for $t<-1$. The right hand side is supported in $\{t\ge -1\}$, and hence the existence of a unique solution V is guaranteed by [10, Theorem 23.2.4]. With the help of (2.14a), from (2.16) we obtain

$$
(\Box_g+q)\big(U(x,t)\big)=(\Box_g+q)u\cdot H+[2\partial_t u+2\langle \mathrm{d}\omega,\mathrm{d}u\rangle+\Delta_g\omega u+(\Box_g+q)a_{-1}]\cdot\delta. \tag{2.19}
$$

Combining (2.19) with (2.11) and (2.13), and analyzing the smoothness of different terms on the right-hand-side of (2.19), the condition $(\Box_g+q)U(x,t)=0$ gives

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s}(u\circ\gamma)+\dfrac{(\Delta_g\omega)\circ\gamma}{2}(u\circ\gamma)=-\dfrac{[(\Box_g+q)a_{-1}]\circ\gamma}{2}, & \text{(2.20a)} \\[2mm]
(\Box_g+q)u=0, \quad \text{in} \quad t>\omega(x). & \text{(2.20b)}
\end{cases}
$$

Denote $u_\gamma:=u\circ\gamma$, thus (2.20a) gives

$$
\frac{\mathrm{d}u_\gamma}{\mathrm{d}s}(s)+\frac{1}{2}\Delta_g\omega(\gamma(s))\,u_\gamma(s)=-\frac{1}{2}(\Box_g+q)a_{-1}(\gamma(s)).
\tag{2.21}
$$

Solving (2.21) with the help of (2.15) and the initial condition $u(x,t) = 0$ when $t \ll -1$, we arrive at

$$u_\gamma(s) = -\frac{1}{2}(a_{-1} \circ \gamma)(s) \int_{-\infty}^s \left[\frac{\Box_g a_{-1}(\gamma(\tau))}{a_{-1}(\gamma(\tau))} + q(\gamma(\tau)) \right] d\tau,$$

and hence

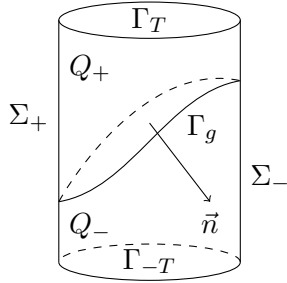$$X|_{\gamma(s)} \left(\frac{u}{a_{-1}} \right) = -\frac{1}{2} \left(\frac{\Box_g a_{-1}(\gamma(s))}{a_{-1}(\gamma(s))} + q(\gamma(s)) \right).$$

The proof is complete. $\qquad\square$

3. Some energy estimates

In this section we prove several energy estimates, which extend corresponding results in [23, 24] to the case of Riemannian metrics. Before that, we define some domains which we work on, and then present some preliminary lemmas.

3.1. **Geometry and preliminary lemmas.** The underlying geometry and related domains are listed below.

$$Q := B \times [-T,T], \quad \Sigma := \partial B \times [T,T],$$
$$Q_\pm := Q \cap \{(p,t)\,;\, \pm(t - \omega(p)) > 0\},$$
$$\Sigma_\pm := \Sigma \cap \{(p,t)\,;\, \pm(t - \omega(p)) > 0\},$$
$$\Gamma_g = Q \cap \{(p,t)\,;\, t = \omega(p)\},$$
$$\Gamma_{\pm T} := Q \cap \{t = \pm T\}, \quad \Gamma_\tau := Q \cap \{t = \tau\},$$
$$\vec{n} := (\nabla\omega(x), -1)/\sqrt{|\nabla\omega(x)|^2 + 1}.$$

The precise value of $T > 0$ shall be determined later. We will also consider a Riemannian metric $\tilde{g} = g \oplus 1$ on $\mathbb{R}^n \times \mathbb{R}$, and we denote by g' the metric on Γ_g induced by $\tilde{g}$. Before we show the energy estimates, we first present some lemmas which are used in the proof of these energy estimates.

Lemma 3.1. *For smooth enough functions α and β, we have three different representations of $\sum_{m=1}^n (X^m\alpha)(X^m\beta)$,*

$$\sum_{m=1}^n (X^m\alpha)(X^m\beta) = \langle d\alpha, d\beta \rangle + \alpha_t \beta_t + \omega_j g^{jk}(\alpha_k \beta_t + \beta_k \alpha_t), \tag{3.1}$$

$$\sum_{m=1}^n (X^m\alpha)(X^m\beta) = \langle d\alpha, d\beta \rangle - \alpha_t \beta_t + \alpha_t(X\beta) + \beta_t(X\alpha), \tag{3.2}$$

$$\sum_{m=1}^n (X^m\alpha)(X^m\beta) = (X\alpha)(X\beta) + \langle d\alpha - \langle d\omega, d\alpha \rangle d\omega, d\beta - \langle d\omega, d\beta \rangle d\omega \rangle. \tag{3.3}$$

Proof. We compute the left-hand-side of (3.1) as follows,

$$\sum_m (X^m\alpha)(X^m\beta) = \sum_m (\omega_j h^{jm}\alpha_t + h^{mk}\alpha_k)(\omega_j h^{jm}\beta_t + h^{mk}\beta_k)$$
$$= \omega_j g^{jk}\omega_k \alpha_t\beta_t + \omega_j g^{jk}\alpha_t\beta_k + \omega_j g^{jk}\beta_t\alpha_k + \alpha_j g^{jk}\beta_k$$
$$= \alpha_t\beta_t + \langle d\alpha, d\beta \rangle + \omega_j g^{jk}(\alpha_k\beta_t + \beta_k\alpha_t).$$

Note that we used the fact (2.5). The equation (3.1) is proved. Then from (3.1), we can obtain (3.2) as follows,

$$\langle d\alpha, d\beta \rangle - \alpha_t\beta_t = \sum_m (X^m\alpha)(X^m\beta) - 2\alpha_t\beta_t - \omega_j g^{jk}(\alpha_k\beta_t + \beta_k\alpha_t)$$

$$= \sum_m (X^m\alpha)(X^m\beta) - \alpha_t(\beta_t + \omega_j g^{jk}\beta_k) - \beta_t(\alpha_t + \omega_j g^{jk}\alpha_k)$$

$$= \sum_m (X^m\alpha)(X^m\beta) - \alpha_t(X\beta) - \beta_t(X\alpha).$$

For (3.3), by (3.1) and noting that $\langle d\omega, d\omega \rangle = 1$, we have

$$\sum_{m=1}^n (X^m\alpha)(X^m\beta) = \langle d\alpha, d\beta \rangle + (X\alpha)(X\beta) - \langle d\omega, d\alpha \rangle \langle d\omega, d\beta \rangle,$$

$$= (X\alpha)(X\beta) + \langle d\alpha - \langle d\omega, d\alpha \rangle d\omega, d\beta - \langle d\omega, d\beta \rangle d\omega \rangle.$$

The proof is complete. $\square$

Lemma 3.2. *It holds*

$$\sum_m |X^m\alpha|^2 \le C \langle d\alpha, d\overline{\alpha} \rangle_{g'}, \tag{3.4}$$

where the g' is the induced Riemannian metric over the hypersurface Γ_g, and the constant C is independent of α.

Proof. For a fixed m and a fixed point $p \in \Gamma_g$, we know $X_p^m \in T_p\Gamma_g$, thus we can find an local coordinates $\{y^1, y^2, \cdots, y^{n-1}\}$ such that

$$\begin{cases} \partial_{y_1}|_p = X_p^m \\ g'(\partial_{y^i}|_p, \partial_{y^j}|_p) = g'_{ij}(p) = \delta_{ij} \end{cases} \Rightarrow \quad g'(dy^j|_p, dy^k|_p) = g'^{jk}(p) = \delta^{jk}.$$

We see that, with particularly chosen real number C, $\{CX^m|_p, \partial_{y^j}|_p ; j = 2, \cdots, n-1\}$ forms a orthonormal basis of $T_p\Gamma_g$ w.r.t. metric g'. The exterior-derivative of α at point p on Γ_g is $d\alpha|_p = X_p^m\alpha + \sum_{j=2}^{n-1}(\partial_{y^j}\alpha)|_p dy^j$, so

$$|X_p^m\alpha|^2 \le |X_p^m\alpha|^2 + \sum_{j=2}^{n-1}|\partial_{y^j}\alpha|^2 = |X_p^m\alpha|^2 + \sum_{j,k=2}^{n-1}\partial_{y^j}\alpha \cdot g'^{jk}(p) \cdot \overline{\partial_{y^k}\alpha} \le C\langle d\alpha|_p, \overline{d\alpha}|_p \rangle_{g'}.$$

We arrive at (3.4). $\square$

We use the following convention for integrations and Sobolev norms on $(\mathbb{R}^n \times \mathbb{R}_t, \tilde{g})$ and on (Γ_g, g'). Below, we assume that $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$ and $\mathcal{Q} \subset \Gamma_g$.

$$\int_{\mathcal{P}} f \, dV_g := \int_{\mathcal{P}} f(x,t)|\tilde{g}(x,t)|^{1/2} \, dx \, dt, \qquad \int_{\mathcal{Q}} f \, dS_{g'} := \int_{\mathcal{Q}} f(x)|g'(x)|^{1/2} \, dS,$$

$$\|f\|_{L^2(\mathcal{P};g)} := \left(\int_{\mathcal{P}} |f|^2 \, dV_g\right)^{1/2}, \qquad \|f\|_{L^2(\mathcal{Q};g')} := \left(\int_{\mathcal{Q}} |f|^2 \, dS_{g'}\right)^{1/2},$$

$$\|f\|_{H^1(\mathcal{P};g)} := \left(\int_{\mathcal{P}} |f|^2 + \langle df, \overline{df} \rangle \, dV_g\right)^{1/2}, \quad \|f\|_{H^1(\mathcal{Q};g')} := \left(\int_{\mathcal{Q}} |f|^2 + \langle df, \overline{df} \rangle_{g'} \, dS_{g'}\right)^{1/2}.$$

Here the volume form dV_g corresponds to the space-time tensor $\tilde{g} := g \oplus 1$, and dS is the volume form on Γ_g induced by the usual Euclidean metric.

Lemma 3.3. *For any non-negative function f and any bounded hypersurface $\mathcal{Q} \subset \Gamma_g$, we have*

$$\int_{\mathcal{Q}} f(p) \, dS_{g'} \lesssim \int_{\mathcal{Q}} f(p) \, dS \lesssim \int_{\mathcal{Q}} f(p) \, dS_{g'}. \tag{3.5}$$

Proof. Because $\mathcal{Q}$ is bounded, we can find finite many local charts (U_i, φ_i) and partition of unity $\{\chi_i\}$ such that

$$\begin{cases} \operatorname{supp} \chi_i \subset U_i, \quad \sum_i \chi_i = 1, \quad \overline{\mathcal{Q}} \subset \cup_i U_i, \\ \{\partial_{(\varphi_i)^j}|_p\}_{j=1}^{n-1} \text{ forms a positve basis of } T_p\mathcal{Q}, \text{ under the metric } g', \\ \{\partial_{(\varphi_i)^j}|_p\}_{j=1}^{n-1} \text{ forms a orthonormal positve basis of } T_p\mathcal{Q}, \text{ under the Euclidean metric.} \end{cases}$$

In these U_i, the volume form is $\mathrm{d}S_g = |g'|^{1/2} \mathrm{d}(\varphi_i)^1 \wedge \cdots \wedge \mathrm{d}(\varphi_i)^{n-1}$ where the jk-th component of the matrix g' is $(g')^{jk}|_p = \langle \mathrm{d}(\varphi_i)^j, \mathrm{d}(\varphi_i)^k \rangle$. Thus, $|g|/\lambda_1 \leq |g'| \leq |g|/\lambda_2$ where the λ_1 and λ_2 are the largest and smallest eigenvalues of g, respectively. If we denote the supremum and infimum of this $|g'|$ on U_i as $C_{i,1}$ and $C_{i,2}$, then $C_{i,1}, C_{i,2} > 0$ and

$$\begin{aligned} \int_{\mathcal{Q}} f(p)\, \mathrm{d}S_{g'} &= \sum_i \int_{\mathcal{Q}} f(p)\chi_i(p)|g'|^{1/2}\, \mathrm{d}(\varphi_i)^1 \wedge \cdots \mathrm{d}\wedge(\varphi_i)^{n-1} \\ &\leq \sum_i C_{i,1} \int_{\mathcal{Q}} f(p)\chi_i(p)\, \mathrm{d}(\varphi_i)^1 \wedge \cdots \mathrm{d}\wedge(\varphi_i)^{n-1} \\ &\lesssim \int_{\mathcal{Q}} f(p) \sum_i \chi_i(p)\, \mathrm{d}(\varphi_i)^1 \wedge \cdots \mathrm{d}\wedge(\varphi_i)^{n-1} \\ &= \int_{\mathcal{Q}} f(p) \sum_i \chi_i(p)\, \mathrm{d}S = \int_{\mathcal{Q}} f(p)\, \mathrm{d}S, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathcal{Q}} f(p)\, \mathrm{d}S_{g'} &= \sum_i \int_{\mathcal{Q}} f(p)\chi_i(p)|g'|^{1/2}\, \mathrm{d}(\varphi_i)^1 \wedge \cdots \mathrm{d}\wedge(\varphi_i)^{n-1} \\ &\geq \sum_i C_{i,2} \int_{\mathcal{Q}} f(p)\chi_i(p)\, \mathrm{d}(\varphi_i)^1 \wedge \cdots \mathrm{d}\wedge(\varphi_i)^{n-1} \\ &\gtrsim \int_{\mathcal{Q}} f(p) \sum_i \chi_i(p)\, \mathrm{d}(\varphi_i)^1 \wedge \cdots \mathrm{d}\wedge(\varphi_i)^{n-1} \\ &= \int_{\mathcal{Q}} f(p) \sum_i \chi_i(p)\, \mathrm{d}S = \int_{\mathcal{Q}} f(p)\, \mathrm{d}S. \end{aligned}$$

The proof is complete. $\qquad\square$

3.2. **Energy estimates.** We consider a parameter $\sigma \geq 1$. All constants in this article will be independent of σ. Throughout the paper, we use the notation $\mathcal{X} = \mathcal{X}_\sigma$ defined as follows,

$$\mathcal{X}f := \frac{\sigma^2|f|^2 + \sum_{m=1}^n |X^m f|^2}{\sqrt{|\nabla\omega(x)|^2 + 1}}, \tag{3.6}$$

for any function for which the right-hand-side of (3.6) can be defined. We first give two energy estimates, which are straightforward extensions of [24, Lemmas 3.4 and 3.5] to the Riemannian case.

Theorem 3.1. *For $\sigma = 1$ we have*

$$\int_{\Gamma_\tau} \left(|\alpha|^2 + |\alpha_t|^2 + \langle \mathrm{d}\alpha, \overline{\mathrm{d}\alpha} \rangle\right) |g|^{1/2}\, \mathrm{d}S \leq C(\|(\Box_g + q)\alpha\|^2_{L^2(Q_+;g)} + \int_{\Gamma_g} (\mathcal{X}_1\alpha)|g|^{1/2}\, \mathrm{d}S$$

$$+ \|\alpha\|^2_{H^1(\Sigma_+;g')} + \|\partial_{\nu,g}\alpha\|^2_{L^2(\Sigma_+;g')}) \tag{3.7}$$

where the constant C depends on $\|q\|_{L^\infty}$, and $\partial_{\nu,g}\alpha := \nu_j g^{jk}(x)\partial_k\alpha$.

Proof. Denote $Q_\tau := \{(x,t) \,;\, |x| \leq 1, \omega(x) \leq t \leq \tau\}$. The outer normal of Γ_g is $\frac{(\nabla\omega(x),-1)}{\sqrt{|\nabla\omega(x)|^2+1}}$ pointing into $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \,;\, t < \omega(x)\}$. Let $\Gamma_{g,\tau} := \Gamma_g \cap \{t \leq \tau\}$ and $\Sigma_{+,\tau} := \Sigma_+ \cap \{t \leq \tau\}$.

Let $\sigma \geq 1$ (later we will take $\sigma = 1$). We have

$$\overline{\alpha_t}(\square_g\alpha + \sigma^2\alpha) + \alpha_t(\overline{\square_g\alpha + \sigma^2\alpha}) = \partial_t(\sigma^2|\alpha|^2 + |\alpha_t|^2 + \langle d\alpha, \overline{d\alpha}\rangle) - \tilde{\delta}(\overline{\alpha_t}\,d\alpha) - \tilde{\delta}(\alpha_t\,\overline{d\alpha}). \quad (3.8)$$

The term

$$f_\sigma(\tau) := \int_{\Gamma_\tau} \big(\sigma^2|\alpha|^2 + |\alpha_t|^2 + \langle d\alpha, \overline{d\alpha}\rangle\big)|g|^{1/2}\,dS$$

is the quantity that we wish to estimate. Integrating (3.8) over Q_τ, we have

$$\int_{Q_\tau} \overline{\alpha_t}(\square_g\alpha + \sigma^2\alpha) + \alpha_t(\overline{\square_g\alpha + \sigma^2\alpha})\,dV_g$$

$$= \int_{Q_\tau} \partial_t(\sigma^2|\alpha|^2 + |\alpha_t|^2 + \langle d\alpha, \overline{d\alpha}\rangle) - \tilde{\delta}(\overline{\alpha_t}\,d\alpha) - \tilde{\delta}(\alpha_t\,\overline{d\alpha})\,dV_g$$

$$= \int_{\Gamma_\tau} (\sigma^2|\alpha|^2 + |\alpha_t|^2 + \langle d\alpha, \overline{d\alpha}\rangle)|g|^{1/2}\,dS - \int_{\Gamma_{g,\tau}} \frac{\sigma^2|\alpha|^2 + |\alpha_t|^2 + \langle d\alpha, \overline{d\alpha}\rangle}{\sqrt{|\nabla\omega(x)|^2+1}}|g|^{1/2}\,dS$$

$$\quad - \int_{\Gamma_{g,\tau} \cup \Sigma_{+,\tau}} \nu_j g^{jk}(\overline{\alpha_t}\alpha_k + \alpha_t\overline{\alpha_k})|g|^{1/2}\,dS$$

$$= f_\sigma(\tau) - \int_{\Gamma_{g,\tau}} \frac{\sigma^2|\alpha|^2 + |\alpha_t|^2 + \langle d\alpha, \overline{d\alpha}\rangle + (\partial_j\omega)g^{jk}(\overline{\alpha_t}\alpha_k + \alpha_t\overline{\alpha_k})}{\sqrt{|\nabla\omega(x)|^2+1}}|g|^{1/2}\,dS$$

$$\quad - \int_{\Sigma_{+,\tau}} [(\partial_{\nu,g}\alpha)\overline{\alpha_t} + (\overline{\partial_{\nu,g}\alpha})\alpha_t]|g|^{1/2}\,dS. \quad (3.9)$$

We now let $\sigma = 1$. Recall the definition of X^m in (2.10). Combining (3.9) with (3.1) and (3.6), we can estimate $f(\tau)$ as

$$f_1(\tau) = \int_{Q_\tau} \overline{\alpha_t}(\square_g\alpha + \alpha) + \alpha_t(\overline{\square_g\alpha + \alpha})\,dV_g + \int_{\Gamma_{g,\tau}} (\mathcal{X}_1\alpha)|g|^{1/2}\,dS$$

$$\quad + \int_{\Sigma_{+,\tau}} [(\partial_{\nu,g}\alpha)\overline{\alpha_t} + (\overline{\partial_{\nu,g}\alpha})\alpha_t]|g|^{1/2}\,dS$$

$$\leq \int_{Q_\tau} (|\alpha_t|^2 + |\square_g\alpha + \alpha|^2)\,dV_g + \int_{\Gamma_{g,\tau}} (\mathcal{X}_1\alpha)|g|^{1/2}\,dS$$

$$\quad + \int_{\Sigma_{+,\tau}} (|\alpha_t|^2 + |\partial_{\nu,g}\alpha|^2)|g|^{1/2}\,dS$$

$$\leq \int_{Q_\tau} [|\alpha_t|^2 + 2(1 + \|q\|_{L^\infty})^2|\alpha|^2 + 2|(\square_g + q)\alpha|^2]\,dV_g + \int_{\Gamma_{g,\tau}} (\mathcal{X}_1\alpha)|g|^{1/2}\,dS$$

$$\quad + \int_{\Sigma_{+,\tau}} (|\alpha_t|^2 + |\partial_{\nu,g}\alpha|^2)|g|^{1/2}\,dS$$

$$\leq C\int_{-1}^{\tau} f_1(t)\,dt + \int_{Q_\tau} |(\square_g + q)\alpha|^2\,dV_g + \int_{\Gamma_{g,\tau}} (\mathcal{X}_1\alpha)|g|^{1/2}\,dS$$

$$\quad + \int_{\Sigma_{+,\tau}} (|\alpha_t|^2 + |\partial_{\nu,g}\alpha|^2)|g|^{1/2}\,dS,$$

and the Grönwall's inequality gives

$$f_1(\tau) \lesssim \int_{Q_\tau} |(\square_g + q)\alpha|^2\,dV_g + \int_{\Gamma_{g,\tau}} (\mathcal{X}_1\alpha)|g|^{1/2}\,dS + \int_{\Sigma_{+,\tau}} (|\alpha_t|^2 + |\partial_{\nu,g}\alpha|^2)|g|^{1/2}\,dS. \quad (3.10)$$

Now combining (3.10) with (3.5), we arrive at (3.7). $\qquad\square$

Theorem 3.2. *For any functions* $f, \varphi, \alpha \in C^2(\overline{Q_+})$, *we have*

$$\int_{\Gamma_g} e^{2\sigma\varphi}(\mathcal{X}(f\alpha))|g|^{1/2}\,\mathrm{d}S$$
$$\leq C_{q,f}\big[\|e^{\sigma\varphi}(\Box_g + q)\alpha\|_{L^2(Q_+;g)}^2 + \sigma\|e^{\sigma\varphi}\nabla_{x,t}\alpha\|_{L^2(Q_+;g)}^2 + \sigma^3\|e^{\sigma\varphi}\alpha\|_{L^2(Q_+;g)}^2$$
$$+ \|e^{\sigma\varphi}\partial_\nu\alpha\|_{L^2(\Sigma_+)}^2 + \|e^{\sigma\varphi}\alpha_t\|_{L^2(\Sigma_+)}^2 + \sigma^2\|e^{\sigma\varphi}\alpha\|_{L^2(\Sigma_+)}^2\big]. \tag{3.11}$$

where the constant $C_{q,f}$ *depends on* $\|q\|_{L^\infty}$ *and* f *and the metric* g.

Proof. We first focus on the case where $f = 1$. From (3.9) and (3.1), we can have

$$\int_{Q_\tau} \overline{\beta_t}(\Box_g\beta + \sigma^2\beta) + \beta_t(\overline{\Box_g\beta + \sigma^2\beta})\,\mathrm{d}V_g$$
$$= f_\sigma(\tau) - \int_{\Gamma_{g,\tau}} (\mathcal{X}\beta)|g|^{1/2}\,\mathrm{d}S - \int_{\Sigma_{+,\tau}} [(\partial_{\nu,g}\beta)\overline{\beta_t} + (\overline{\partial_{\nu,g}\beta})\beta_t]|g|^{1/2}\,\mathrm{d}S,$$

where $f_\sigma(\tau) = \int_{\Gamma_\tau} \left(\sigma^2|\beta|^2 + |\beta_t|^2 + (\mathrm{d}\beta, \overline{\mathrm{d}\beta})_g\right)|g|^{1/2}\,\mathrm{d}S$, thus

$$\int_{\Gamma_{g,\tau}} (\mathcal{X}\beta)|g|^{1/2}\,\mathrm{d}S = f_\sigma(\tau) - \int_{Q_\tau} \overline{\beta_t}(\Box_g\beta + \sigma^2\beta) + \beta_t(\overline{\Box_g\beta + \sigma^2\beta})\,\mathrm{d}V_g$$
$$- \int_{\Sigma_{+,\tau}} [(\partial_{\nu,g}\beta)\overline{\beta_t} + (\overline{\partial_{\nu,g}\beta})\beta_t]|g|^{1/2}\,\mathrm{d}S. \tag{3.12}$$

Now let $\beta = e^{\sigma\varphi}\alpha$, then

$$\begin{cases} X^m\beta = e^{\sigma\varphi}(X^m + \sigma X^m\varphi)\alpha, \\ \partial_t\beta = e^{\sigma\varphi}(\alpha_t + \sigma\varphi_t\alpha) \\ \partial_t^2\beta = e^{\sigma\varphi}\{\alpha_{tt} + 2\sigma\varphi_t\alpha_t + [\sigma\varphi_{tt} + (\sigma\varphi_t)^2]\alpha\} \\ \Delta_g\beta = \Delta_g(e^{\sigma\varphi}) \cdot \alpha + 2\langle\mathrm{d}(e^{\sigma\varphi}), \mathrm{d}\alpha\rangle + e^{\sigma\varphi} \cdot \Delta_g\alpha \\ \Delta_g(e^{\sigma\varphi}) = e^{\sigma\varphi}[\sigma\Delta_g\varphi + \sigma^2\langle\mathrm{d}\varphi, \mathrm{d}\varphi\rangle] \\ \langle\mathrm{d}(e^{\sigma\varphi}), \mathrm{d}\alpha\rangle = \sigma e^{\sigma\varphi}\langle\mathrm{d}\varphi, \mathrm{d}\alpha\rangle \end{cases} \tag{3.13}$$

so

$$(\Box_g + q)\beta = e^{\sigma\varphi}\{(\Box_g + q)\alpha + 2\sigma[\varphi_t\alpha_t - \langle\mathrm{d}\varphi, \mathrm{d}\alpha\rangle] + [\sigma\Box_g\varphi + \sigma^2\varphi_t^2 - \sigma^2\langle\mathrm{d}\varphi, \mathrm{d}\varphi\rangle]\alpha\}$$
$$= e^{\sigma\varphi}\{(\Box_g + q)\alpha + 2\sigma\langle\mathrm{d}\varphi, \mathrm{d}\alpha\rangle_{\hat{g}} + [\sigma\Box_g\varphi + \sigma^2\langle\mathrm{d}\varphi, \mathrm{d}\varphi\rangle_{\hat{g}}]\alpha\},$$

where $\hat{g} = (\mathrm{d}t)^2 - g$.

It can be checked that

$$|\langle\mathrm{d}F, \mathrm{d}G\rangle| \leq \sqrt{\langle\mathrm{d}F, \mathrm{d}\overline{F}\rangle} \cdot \sqrt{\langle\mathrm{d}G, \mathrm{d}\overline{G}\rangle} \leq C_g|\nabla_x F| \cdot |\nabla_x G|$$

for some constant C_g and for any functions $F, G \in C^1(\mathbb{R}^n; \mathbb{C})$. When σ is large enough, i.e. $\sigma^2 > 2\|q\|_{L^\infty}$, we compute the integral of $\overline{\beta_t}(\Box_g\beta + \sigma^2\beta)$ as follows,

$$\int_{Q_\tau} \overline{\beta_t}(\Box_g\beta + \sigma^2\beta)\,\mathrm{d}V_g$$
$$\lesssim \int_{Q_\tau} \overline{\beta_t} \cdot (\Box_g + q)\beta + (\sigma^2 - q)\overline{\beta_t}\beta\,\mathrm{d}V_g$$
$$= \int_{Q_\tau} \overline{\beta_t} \cdot e^{\sigma\varphi}\{(\Box_g + q)\alpha + 2\sigma\langle\mathrm{d}\varphi, \mathrm{d}\alpha\rangle_{\hat{g}} + [\sigma\Box_g\varphi + \sigma^2\langle\mathrm{d}\varphi, \mathrm{d}\varphi\rangle_{vg}]\alpha\} + (\sigma^2 - q)\overline{\beta_t}\beta\,\mathrm{d}V_g$$

$$\lesssim \int_{Q_\tau} |\beta_t| \cdot e^{\sigma\varphi}[|(\Box_g + q)\alpha| + |\nabla_{x,t}\varphi| \cdot \sigma |\nabla_{x,t}\alpha| + (|\nabla_{x,t}\varphi| + |\Box_g\varphi|) \cdot \sigma^2 |\alpha|]$$
$$+ \sigma|\beta_t| \cdot \sigma|\beta| \, \mathrm{d}V_g$$

$$\lesssim C_\varphi \int_{Q_\tau} (|\beta_t|^2 + e^{2\sigma\varphi}|(\Box_g + q)\alpha|^2) + (\sigma|\beta_t|^2 + \sigma e^{2\sigma\varphi}|\nabla_{x,t}\alpha|^2)$$
$$+ (\sigma|\beta_t|^2 + \sigma^3 e^{2\sigma\varphi}|\alpha|^2) + (\sigma|\beta_t|^2 + \sigma^3 |\beta|^2) \, \mathrm{d}V_g$$

$$\lesssim \int_{Q_\tau} e^{2\sigma\varphi}\big(|(\Box_g + q)\alpha|^2 + \sigma|\nabla_{x,t}\alpha|^2 + \sigma^3 \alpha^2\big) + \sigma|\beta_t|^2 + \sigma^3 |\beta|^2 \, \mathrm{d}V_g$$

$$\lesssim \int_{Q_\tau} e^{2\sigma\varphi}\big(|(\Box_g + q)\alpha|^2 + \sigma|\nabla_{x,t}\alpha|^2 + \sigma^3 \alpha^2\big) + \sigma e^{2\sigma\varphi}(|\alpha_t|^2 + \sigma^2|\alpha|^2) + \sigma^3 e^{2\sigma\varphi}|\alpha|^2 \, \mathrm{d}V_g$$

$$\lesssim \|e^{\sigma\varphi}(\Box_g + q)\alpha\|^2_{L^2(Q_\tau;g)} + \sigma\|e^{\sigma\varphi}\nabla_{x,t}\alpha\|^2_{L^2(Q_\tau;g)} + \sigma^3\|e^{\sigma\varphi}\alpha\|^2_{L^2(Q_\tau;g)}. \tag{3.14}$$

Similarly, we have

$$\int_{Q_\tau} \beta_t(\overline{\Box_g\beta + \sigma^2\beta}) \, \mathrm{d}V_g$$
$$\leq C_\varphi(\|e^{\sigma\varphi}(\Box_g + q)\alpha\|^2_{L^2(Q_\tau;g)} + \sigma\|e^{\sigma\varphi}\nabla_{x,t}\alpha\|^2_{L^2(Q_\tau;g)} + \sigma^3\|e^{\sigma\varphi}\alpha\|^2_{L^2(Q_\tau;g)}). \tag{3.15}$$

Also, the $f_\sigma(\tau)$ can be estimated as

$$f_\sigma(\tau) = \int_{\Gamma_\tau} \big(\sigma^2|\beta|^2 + |\beta_t|^2 + \langle \mathrm{d}\beta, \overline{\mathrm{d}\beta}\rangle\big)|g|^{1/2} \, \mathrm{d}S$$
$$\leq C_\varphi \int_{\Gamma_\tau} \sigma^2 e^{2\sigma\varphi}|\alpha|^2 + e^{2\sigma\varphi}(|\nabla_{x,t}\alpha|^2 + \sigma^2|\alpha|^2) \, \mathrm{d}S_g$$
$$\leq C_\varphi(\|e^{\sigma\varphi}\nabla_{x,t}\alpha\|^2_{L^2(\Gamma_\tau;g)} + \sigma^2\|e^{\sigma\varphi}\alpha\|^2_{L^2(\Gamma_\tau;g)}). \tag{3.16}$$

Note the metric g goes back to the usual Euclidean metric on the boundary $\Sigma_+ \cup \Sigma_-$, so $\partial_{\nu,g}$ goes back to ∂_ν. Then the integral on the boundary $\Sigma_{+,\tau}$ is

$$\int_{\Sigma_{+,\tau}} [(\partial_{\nu,g}\beta)\overline{\beta_t} + (\overline{\partial_{\nu,g}\beta})\beta_t]|g|^{1/2} \, \mathrm{d}S$$

$$\lesssim \int_{\Sigma_{+,\tau}} |\partial_\nu\beta|^2 + |\beta_t|^2 \, \mathrm{d}S_{g'}$$

$$\lesssim \int_{\Sigma_{+,\tau}} e^{2\sigma\varphi}|\partial_\nu\alpha + \sigma\partial_\nu\varphi \cdot \alpha|^2 + e^{2\sigma\varphi}|\alpha_t + \sigma\varphi_t\alpha|^2 \, \mathrm{d}S_{g'}$$

$$\lesssim \int_{\Sigma_{+,\tau}} e^{2\sigma\varphi}|\partial_\nu\alpha|^2 + e^{2\sigma\varphi}|\alpha_t|^2 + \sigma^2 e^{2\sigma\varphi}|\alpha|^2 \, \mathrm{d}S_{g'}$$

$$= \|e^{\sigma\varphi}\partial_\nu\alpha\|^2_{L^2(\Sigma_{+,\tau},g')} + \|e^{\sigma\varphi}\alpha_t\|^2_{L^2(\Sigma_{+,\tau},g')} + \sigma^2\|e^{\sigma\varphi}\alpha\|^2_{L^2(\Sigma_{+,\tau},g')}.$$

The positive-definiteness of the Riemannian metric gives $c|\nabla\omega|^2 \leq \langle \mathrm{d}\omega, \mathrm{d}\omega\rangle = 1 \leq C|\nabla\omega|^2$, so $0 < 1/C \leq |\nabla\omega|^2 \leq 1/c + \infty$. Combining (3.12), (3.14), (3.15) and (3.16) and integrating from $\tau = 1$ to $\tau = T$, we obtain

$$\int_{\Gamma_g} (\mathcal{X}\beta)|g|^{1/2} \, \mathrm{d}S$$

$$\lesssim \int_1^T f_\sigma(\tau) \, \mathrm{d}\tau + \|e^{\sigma\varphi}(\Box_g + q)\alpha\|^2_{L^2(Q_+;g)} + \sigma\|e^{\sigma\varphi}\nabla_{x,t}\alpha\|^2_{L^2(Q_+;g)} + \sigma^3\|e^{\sigma\varphi}\alpha\|^2_{L^2(Q_+;g)}$$

$$+ \|e^{\sigma\varphi}\partial_\nu\alpha\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}\alpha_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}\alpha\|^2_{L^2(\Sigma_+)}$$

$$\lesssim \|e^{\sigma\varphi}(\Box_g + q)\alpha\|^2_{L^2(Q_+;g)} + \sigma\|e^{\sigma\varphi}\nabla_{x,t}\alpha\|^2_{L^2(Q_+;g)} + \sigma^3\|e^{\sigma\varphi}\alpha\|^2_{L^2(Q_+;g)}$$
$$+ \|e^{\sigma\varphi}\partial_\nu\alpha\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}\alpha_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}\alpha\|^2_{L^2(\Sigma_+)}. \tag{3.17}$$

Finally from (3.17) and (3.13), we arrive at

$$\int_{\Gamma_g} e^{2\sigma\varphi}(\mathcal{X}\alpha)|g|^{1/2}\,\mathrm{d}S$$
$$\leq C\big[\|e^{\sigma\varphi}(\Box_g + q)\alpha\|^2_{L^2(Q_+;g)} + \sigma\|e^{\sigma\varphi}\nabla_{x,t}\alpha\|^2_{L^2(Q_+;g)} + \sigma^3\|e^{\sigma\varphi}\alpha\|^2_{L^2(Q_+;g)}$$
$$+ \|e^{\sigma\varphi}\partial_\nu\alpha\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}\alpha_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}\alpha\|^2_{L^2(\Sigma_+)}\big], \tag{3.18}$$

which is (3.11) with $f = 1$.

In (3.18), change α to $f\alpha$, we obtain

$$\int_{\Gamma_g} e^{2\sigma\varphi}(\mathcal{X}(f\alpha))|g|^{1/2}\,\mathrm{d}S$$
$$\leq C\big[\|e^{\sigma\varphi}(\Box_g + q)(f\alpha)\|^2_{L^2(Q_+;g)} + \sigma\|e^{\sigma\varphi}\nabla_{x,t}(f\alpha)\|^2_{L^2(Q_+;g)} + \sigma^3\|e^{\sigma\varphi}(f\alpha)\|^2_{L^2(Q_+;g)}$$
$$+ \|e^{\sigma\varphi}\partial_\nu(f\alpha)\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}(f\alpha)_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}(f\alpha)\|^2_{L^2(\Sigma_+)}\big]. \tag{3.19}$$

For $(\Box_g + q)(f\alpha)$, we can compute

$$|(\Box_g + q)(f\alpha)| = |\Box_g(f\alpha) + fq\alpha| = |(\Box_g f)\alpha - 2\langle \mathrm{d}f, \mathrm{d}\alpha\rangle + 2f_t\alpha_t + f\Box_g\alpha + fq\alpha|$$
$$\leq |f\Box_g\alpha| + 2|\langle \mathrm{d}f, \mathrm{d}\alpha\rangle| + 2|f_t||\alpha_t| + |(\Box_g f)\alpha| + |fq\alpha|$$
$$\leq \|f\|_{L^\infty(Q_+)}|\Box_g\alpha| + C_g|\nabla_{x,t}f||\nabla_{x,t}\alpha| + \|f\|_{H^2(Q_+)}|\alpha| + \|fq\|_{L^\infty(Q_+)}|\alpha|,$$

so

$$\|e^{\sigma\varphi}(\Box_g + q)(f\alpha)\|_{L^2(Q_+;g)}$$
$$\leq C\big[\|e^{\sigma\varphi}(\Box_g + q)\alpha\|_{L^2(Q_+;g)} + \|e^{\sigma\varphi}\nabla_{x,t}\alpha\|_{L^2(Q_+;g)} + \|e^{\sigma\varphi}\alpha\|_{L^2(Q_+;g)}\big].$$

We can estimate $\nabla_{x,t}(f\alpha)$, $\partial_\nu(f\alpha)$ and $(f\alpha)_t$ in similar manners. Combining these estimates with (3.19), we finally arrive at (3.11). $\qquad\square$

To continue, we need to borrow a Carleman estimate with boundary term from [23, Theorem A.7]. To make the paper self-contained, we present this theorem below, followed by the arguments that how we modify the estimate to fit in our Riemannian settings.

Lemma 3.4. *Suppose Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with a Lipschitz boundary, and $P(x, D)$ is a second order differential operator on $\overline{\Omega}$ with bounded coefficients whose principal symbol $p(x, \xi)$ has real C^1 coefficients. If φ is a smooth function on $\overline{\Omega}$ with $\nabla\varphi$ never zero in $\overline{\Omega}$ and φ is strongly pseudoconvex with respect to $P(x, D)$ on $\overline{\Omega}$, then for large enough σ and for all $u \in C^2(\overline{\Omega})$ one has*

$$\sigma\int_\Omega e^{2\sigma\varphi}(|\nabla v|^2 + \sigma^2|v|^2)\,\mathrm{d}x + \sigma\int_{\partial\Omega}\nu^j E_j\,\mathrm{d}S \leq \int_\Omega e^{2\sigma\varphi}|Pv|^2\,\mathrm{d}x$$

with the constant independent of σ and u. Here $v = (v_1, \ldots, v_n)$ is the outward unit normal to $\partial\Omega$,

$$E_j = \nu^j A(x, \nabla\alpha, \sigma\alpha)\frac{\partial B}{\partial\xi_j}(x) - \frac{\partial A}{\partial\xi_j}(x, \nabla_x\alpha, \sigma\alpha)[B(x, \nabla\alpha) + \alpha\mathbb{G}],$$

$\alpha = e^{\sigma\varphi}v$, $\mathbb{G}$ some real valued function independent of λ, σ, v and

$$A(x, \xi, \sigma) = p(x, \xi) - \sigma^2 p(x, \nabla\varphi), \quad B(x, \xi) = \{p, \varphi\}(x, \xi).$$

Remark 3.1. The E_j in the lemma are constructed in [23, Theorem A.7] and would seem to depend on the domains $Q_\pm$. However, the E_j depend on $\mathbb{G}$ which itself depends on a function h which satisfies the algebraic identity [23, eq. (A.25)]. We can construct the h so that the algebraic identity is satisfied on Q rather than Q_+ and Q_- separately. Then the $\mathbb{G}$ in Lemma 3.4 will be the same for Q_+ and Q_-.

Proof of Lemma 3.4. See Theorem A.7 in [23]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In our setting, the corresponding "Ω" is Q_+, the outer normal ν of Γ_g pointing into Q_- is $\nu(x) = (\nabla\omega(x), -1)/\sqrt{|\nabla\omega(x)|^2 + 1}$, the operator $\Box_g$ we are interested in is

$$(\Box_g + q)f \,:\, v \,\mapsto\, (\Box_g + q)(fv) \qquad (3.20)$$

where $f \in C^2(\overline{Q_+})$ is nonzero, the principal symbol is $p(x, t; \xi, \tau) = f(x, t)[-\tau^2 + g^{jk}(x)\xi_j\xi_k]$, and the corresponding A and B of Lemma 3.4 are

$$A(x, t; \xi, \tau; \sigma) = f(x, t)[-\tau^2 + g^{jk}\xi_j\xi_k - \sigma^2(-\varphi_t^2 + g^{jk}\varphi_j\varphi_k)],$$

$$B(x, t; \xi, \tau) = f(x, t)(2g^{jk}\xi_k, -2\tau) \cdot (\varphi_j, \varphi_t) = f(x, t)(2g^{jk}\xi_k\varphi_j - 2\tau\varphi_t).$$

Now we show that the $\nu^j E_j$ in Lemma 3.4 equals $e^{\sigma\varphi}F(\sigma v, X^1 v, \cdots, X^n v)$ for some quadratic form F with smooth coefficients depending on x and t. According to Lemma 3.4 and Lemma 3.1, we can compute the corresponding $\nu^j E_j$ as

$$\sqrt{|\nabla\omega(x)|^2 + 1}(\nu^j E_j)/f^2$$

$$= \nu^j A(x, t; \nabla_x\alpha, \alpha_t; \sigma\alpha)\frac{\partial B}{\partial \xi_j}(x, t) - \nu^j\frac{\partial A}{\partial \xi_j}(x, t; \nabla_x\alpha, \alpha_t; \sigma\alpha)[B(x, t; \nabla_x\alpha, \alpha_t) + g\alpha]$$

$$\quad + \nu^{n+1}A(x, t; \nabla_x\alpha, \alpha_t; \sigma\alpha)\frac{\partial B}{\partial \tau}(x, t) - \nu^{n+1}\frac{\partial A}{\partial \tau}(x, t; \nabla_x\alpha, \alpha_t; \sigma\alpha)[B(x, t; \nabla_x\alpha, \alpha_t) + g\alpha]$$

$$= \omega_j[-\alpha_t^2 + g^{jk}\alpha_j\alpha_k - \sigma^2\alpha^2(g^{jk}\varphi_j\varphi_k - \varphi_t^2)](2g^{jk}\varphi_k)$$

$$\quad - \omega_j 2g^{jk}\alpha_k(2g^{jk}\alpha_k\varphi_j - 2\alpha_t\varphi_t + g\alpha)$$

$$\quad + (-1)[-\alpha_t^2 + g^{jk}\alpha_j\alpha_k - \sigma^2\alpha^2(g^{jk}\varphi_j\varphi_k - \varphi_t^2)](-2\varphi_t)$$

$$\quad - (-1)(-2\alpha_t)(2g^{jk}\alpha_k\varphi_j - 2\alpha_t\varphi_t + g\alpha)$$

$$= 2(X\varphi)[-\alpha_t^2 + g^{jk}\alpha_j\alpha_k - \sigma^2\alpha^2(g^{jk}\varphi_j\varphi_k - \varphi_t^2)] - 2(X\alpha)(2g^{jk}\alpha_k\varphi_j - 2\alpha_t\varphi_t + g\alpha)$$

$$= 2(X\varphi)[\sum_m(X^m\alpha)(X^m\alpha) - 2\alpha_t(X\alpha) - \sigma^2\alpha^2(g^{jk}\varphi_j\varphi_k - \varphi_t^2)]$$

$$\quad - 4(X\alpha)[\sum_m(X^m\alpha)(X^m\varphi) - \alpha_t(X\varphi) - \varphi_t(X\alpha) + g\alpha/2]$$

$$= 2(X\varphi)[\sum_m(X^m\alpha)^2 - \sigma^2\alpha^2(g^{jk}\varphi_j\varphi_k - \varphi_t^2)] - 4(X\alpha)[\sum_m(X^m\alpha)(X^m\varphi) - \varphi_t(X\alpha) + g\alpha/2]$$

$$= \tilde{F}(\sigma\alpha, X^1\alpha, \cdots, X^n\alpha),$$

where $\tilde{F}$ is some quadratic form in its variables with smooth coefficients depending on x and t. Due to $\alpha = e^{\sigma\varphi}v$, and noticing that $X\alpha = e^{\sigma\varphi}(Xv + (X\varphi)\sigma v)$, we can find another form F such that

$$\nu^j E_j = \frac{f^2(x, t)\tilde{F}(\sigma\alpha, X^1\alpha, \cdots, X^n\alpha)}{\sqrt{|\nabla\omega(x)|^2 + 1}} = e^{2\sigma\varphi}F(\sigma v, X^1 v, \cdots, X^n v). \qquad (3.21)$$

Here, the F is the desired quadratic form with smooth coefficients depending on x and t. Later, we shall represent $F(\sigma v, X^1 v, \cdots, X^n v)$ by $F(\sigma v, X^m v)$ for simplicity. One should pay special attention to the fact that these differential operators X^m in (3.21) are all

tangential to Γ_g, and later we shall dominate $\int_{\Gamma_g} \nu^j E_j \, dS$ by norms of some one-forms over the hypersurface Γ_g.

The following result, which is a Riemannian counterpart of [24, Lemma 3.1], constructs a suitable strongly pseudoconvex function for $\square_g$ starting from a strictly convex function in $(\overline{B}, g)$.

Lemma 3.5. *Assume $\tilde{\psi}$ is a positive function and is strictly convex for all $x \in B$ in terms of the metric g, i.e. the Hessian matrix $\tilde{\psi}_{jk} - \tilde{\psi}_i \Gamma^i_{jk}$ is positive definite. Let $\psi(x) = \tilde{\psi}(x) + (\sup_B \tilde{\psi} - 2 \inf_B \tilde{\psi})$ and let $\phi(x,t) = e^{\iota \psi(x)} - (t - \omega(x))^2/2$. Then when ι and λ are large enough, $\varphi = e^{\lambda \phi}$ is strongly pseudo-convex for $\square_q$ near $\overline{Q}$. Moreover, when T is large enough, the smallest value of φ on Γ_g is strictly larger than the largest value of φ on $\Gamma_T \cup \Gamma_{-T}$, i.e.*

$$\inf_{\Gamma_g} \varphi > \sup_{\Gamma_T \cup \Gamma_{-T}} \varphi, \tag{3.22}$$

and the function

$$h(\sigma) := \sup_{x \in \overline{B}} \int_{-T}^{T} e^{2\sigma(\varphi(x,t) - \varphi(x,\omega(x)))} \, dt \tag{3.23}$$

satisfies $\lim_{\sigma \to +\infty} h(\sigma) = 0$.

Proof. According to Proposition A.3 and A.5 in [23], we only need to show that the level surfaces of ϕ are pseudo-convex in Q_+. That is to say, we need to show

$$\{p, \{p, \phi\}\}(x, \xi) > 0 \quad \text{whenever } x \in Q_+, \xi \in \mathbb{R}^n \backslash \{0\}, p(x, \xi) = 0, \{p, \phi\}(x, \xi) = 0.$$

From $p = 0$ we obtain $g^{jk} \xi_j \xi_k = \tau^2$. From $\{p, \phi\} = 0$ we obtain $\iota \psi_i \dot{x}^i = e^{-\iota\psi}(t - \omega)(\dot{t} - \omega_i \dot{x}^i)$. Denote the integral curve of the Hamiltonian of p as $(x(s), t(s), \xi(s), \tau(s))$. They satisfy $\dot{x}^j(s) = 2g^{jk} \xi_k$ and $\dot{t}(s) = -2\tau$, so

$$(\dot{t})^2 = 4g^{\ell m} \xi_\ell \xi_m = 4g^{\ell j} g_{jk} g^{km} \xi_\ell \xi_m = (2g^{\ell j} \xi_\ell) g_{jk} (2g^{km} \xi_m) = g_{jk} \dot{x}^j \dot{x}^k.$$

It is known that the projection of this integral curve, i.e. $(x(s), t(s))$, is a geodesic in terms of $-(dt)^2 + g$, so

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad \text{and} \quad \ddot{t} = 0.$$

Combining these analysis, we can compute $\{p, \{p, \phi\}\}$ as follows,

$$\begin{aligned}
\{p, \{p, \phi\}\} &= \frac{d^2}{ds^2}\big(\phi(x(s), t(s))\big) = \phi_i \ddot{x}^i + \phi_{jk} \dot{x}^j \dot{x}^k + \phi_{tt}(\dot{t})^2 + \phi_t \ddot{t} \\
&= (\phi_{jk} - \phi_i \Gamma^i_{jk}) \dot{x}^j \dot{x}^k - (\dot{t})^2 \\
&= \big\{[(\iota\psi_{jk} + \iota^2 \psi_j \psi_k)e^{\iota\psi} + \omega_{jk}(t - \omega) - \omega_j \omega_k] - [\iota\psi_i e^{\iota\psi} + \omega_i(t - \omega)]\Gamma^i_{jk}\big\} \dot{x}^j \dot{x}^k \\
&\quad - (\dot{t})^2 \\
&= \big[\iota e^{\iota\psi}(\psi_{jk} - \psi_i \Gamma^i_{jk}) + (t - \omega)(\omega_{jk} - \omega_i \Gamma^i_{jk}) - \omega_j \omega_k\big] \dot{x}^j \dot{x}^k - (\dot{t})^2 \\
&\quad + e^{-\psi}(t - \omega)^2(\dot{t} - \omega_i \dot{x}^i)^2 \\
&\geq \big[\iota e^{\iota\psi}(\psi_{jk} - \psi_i \Gamma^i_{jk}) + (t - \omega)(\omega_{jk} - \omega_i \Gamma^i_{jk}) - \omega_j \omega_k\big] \dot{x}^j \dot{x}^k - (\dot{t})^2 \\
&= \big[\iota e^{\iota\psi}(\psi_{jk} - \psi_i \Gamma^i_{jk}) + (t - \omega)(\omega_{jk} - \omega_i \Gamma^i_{jk}) - \omega_j \omega_k - g_{jk}\big] \dot{x}^j \dot{x}^k \\
&\geq e^{\iota\psi}\big[\iota(\psi_{jk} - \psi_i \Gamma^i_{jk}) - e^{-\iota\psi}|T \pm \omega||\omega_{jk} - \omega_i \Gamma^i_{jk}| - e^{-\iota\psi}(\omega_j \omega_k + g_{jk})\big] \dot{x}^j \dot{x}^k.
\end{aligned} \tag{3.24}$$

Note that the condition $\tilde{\psi}_{jk} - \tilde{\psi}_i \Gamma^i_{jk}$ being positive definite implies $\psi_{jk} - \psi_i \Gamma^i_{jk}$ is also positive definite. We shall set a value for T such that $e^{-\iota\psi(x)}|T \pm \omega(x)|$ is uniformly bounded (in terms of x) by a constant, e.g.,

$$e^{-\iota\psi(x)}|T \pm \omega(x)| < 2, \tag{3.25}$$

for the purpose of guaranteeing $\{p, \{p, \phi\}\} > 0$ by taking ι to be large enough. But before that, we first investigate the requirement (3.22). We set

$$T := \sup_{x \in \overline{B}} \left(e^{\iota\psi(x)} - \inf_{y \in \overline{B}} e^{\iota\psi(y)} \right)^{1/2} + \sup_{x \in \overline{B}} |\omega(x)| + 1, \tag{3.26}$$

then $T - |\omega(x)| > 0$ and so $0 < T - |\omega(x)| \le T \pm \omega(x)$. Hence,

$$\inf_{x \in \overline{B}} (T \pm \omega(x))^2 \ge \inf_{x \in \overline{B}} (T - |\omega(x)|)^2 = (T - \sup_{x \in \overline{B}} |\omega(x)|)^2$$

$$= \left(\sup_{x \in \overline{B}} \left(e^{\iota\psi(x)} - \inf_{y \in \overline{B}} e^{\iota\psi(y)} \right)^{1/2} + 1 \right)^2$$

$$> \sup_{x \in \overline{B}} e^{\iota\psi(x)} - \inf_{y \in \overline{B}} e^{\iota\psi(y)}.$$

Therefore, one can have

$$\sup_{x \in \overline{B}} [e^{\iota\psi(x)} - (T \pm \omega(x))^2] \le \sup_{x \in \overline{B}} e^{\iota\psi(x)} - \inf_{x \in \overline{B}} (T \pm \omega(x))^2$$

$$< \sup_{x \in \overline{B}} e^{\iota\psi(x)} - \left(\sup_{x \in \overline{B}} e^{\iota\psi(x)} - \inf_{y \in \overline{B}} e^{\iota\psi(y)} \right)$$

$$= \inf_{y \in \overline{B}} e^{\iota\psi(y)},$$

which implies (3.22).

The condition $\lim_{\sigma \to +\infty} h(\sigma) = 0$ holds by (3.22), (3.26) and using Lebesgue dominant convergence theorem in (3.23) and taking the σ to the infinity.

Now we go back to (3.25). Recall

$$\psi(x) = \tilde{\psi}(x) + (\sup_B \tilde{\psi} - 2\inf_B \tilde{\psi}) = (\tilde{\psi}(x) - \inf_B \tilde{\psi}) + (\sup_B \tilde{\psi} - \inf_B \tilde{\psi}),$$

which implies $\sup_B \psi \le 2\inf_B \psi$. Combining this with (3.26), we can estimate $e^{-\iota\psi(x)}|T \pm \omega(x)|$ as follows,

$$e^{-\iota\psi(x)}|T \pm \omega(x)|$$

$$= e^{-\iota\psi(x)} |\sup_{x \in \overline{B}} \left(e^{\iota\psi(x)} - \inf_{y \in \overline{B}} e^{\iota\psi(y)} \right)^{1/2} + \sup_{x \in \overline{B}} |\omega(x)| + 1 \pm \omega(x)|$$

$$\le \sup_{z \in \overline{B}} \left(e^{\iota(\psi(z) - 2\psi(x))} - \inf_{y \in \overline{B}} e^{\iota(\psi(y) - 2\psi(x))} \right)^{1/2} + e^{-\iota\psi(x)}(\sup_{\overline{B}} |\omega| - \inf_{\overline{B}} |\omega| + 1)$$

$$\le 1 + e^{-\iota\psi(x)}(\sup_{\overline{B}} |\omega| - \inf_{\overline{B}} |\omega| + 1)$$

$$\le 2.$$

The last inequality holds when ι is large enough. Formula (3.25) is proved. By the positive-definiteness of $\psi_{jk} - \psi_i \Gamma^i_{jk}$, we can conclude $\{p, \{p, \phi\}\}$ is strictly greater than 0 when ι is large enough.

The proof is complete.   □

The following theorem is analogous to [23, Proposition 3.2] and will be used in the solution of the inverse problem.

Theorem 3.3. *Assume that g satisfies the assumption (1.5), and that $v_\pm \in H^2(Q_\pm)$ satisfy*

$$(\square_g + q_{1,\pm})v_\pm = -X\left(v_\pm/a_{-1,\pm}\right)|_{(x,\omega(x))} \cdot u_{2,\pm} \quad in \quad Q_\pm, \tag{3.27}$$

where $a_{-1,\pm} \in C^\infty(\Gamma_g)$ and $\|q_{1,\pm}\|_{L^\infty(B)} \le M$, $\|u_{2,\pm}\|_{L^\infty(Q_\pm)} \le M$ for some $M > 0$. Then, if T and σ are large enough but fixed, one has

$$\sum_\pm \int_{\Gamma_g} \mathcal{X}(v_\pm/a_{-1,\pm})\,|g|^{1/2}\,\mathrm{d}S$$

$$\lesssim \int_{\Gamma_g} \mathcal{X}(v_+/a_{-1,+} - v_-/a_{-1,-})\,\mathrm{d}S + \sum_\pm (\|v_\pm\|^2_{H^1(\Sigma_+;g')} + \|\partial_{\nu,g}v_\pm\|^2_{L^2(\Sigma_+;g')})$$

with constants depending on M.

Proof of Theorem 3.3. The assumption (3.27) implies

$$\begin{aligned}
\|(\square_g + q_{1,\pm})v_\pm\|_{L^2(Q_\pm)} &= \|-X|_{\gamma(s)}(v_\pm/a_{-1,\pm}) \cdot u_{2,\pm}\|_{L^2(Q_\pm)} \\
&\le \|u_{2,\pm}\|_{L^\infty(Q_+)} \cdot \|X|_{\gamma(s)}(v_\pm/a_{-1,\pm})\|_{L^2(Q_\pm)} \\
&\lesssim \|X(v_\pm/a_{-1,\pm})\|_{L^2(\Gamma_g)} \\
&\lesssim \sum_m \|X^m(v_\pm/a_{-1,\pm})\|_{L^2(\Gamma_g)}.
\end{aligned} \tag{3.28}$$

Likewise, we have

$$\|e^{\sigma\varphi}(\square_g + q_{1,\pm})v_\pm\|^2_{L^2(Q_\pm)} \lesssim h(\sigma) \sum_m \|e^{\sigma\varphi}X^m(v_\pm/a_{-1,\pm})\|^2_{L^2(\Gamma_g)}, \tag{3.29}$$

where φ is strongly pseudoconvex for $\square_g$ near Q, and $h(\sigma) := \sup_{x\in B} \int_{-T}^T e^{2\sigma[\varphi(x,t)-\varphi(x,\omega(x))]}\,\mathrm{d}t$. Thanks to the assumption (1.5), Lemma 3.5 can be applied and we can find such a strongly pseudoconvex function φ.

Combining (3.28) and Theorem 3.1, we can have

$$\int_{\Gamma_T} \left((\mathrm{d}(v/a_{-1}), \overline{\mathrm{d}(v/a_{-1})})_g + |(v/a_{-1})_t|^2 + \sigma^2|v/a_{-1}|^2\right)|g|^{1/2}\,\mathrm{d}S$$

$$\lesssim \int_{\Gamma_g} (\mathcal{X}(v/a_{-1}))|g|^{1/2}\,\mathrm{d}S + \|v/a_{-1}\|^2_{H^1(\Sigma_+;g')} + \|\partial_{\nu,g}(v/a_{-1})\|^2_{L^2(\Sigma_+;g')}.$$

If we take into consideration the term $e^{\sigma\varphi}$, we can continue

$$\int_{\Gamma_T} e^{2\sigma\varphi}\left((\mathrm{d}(v/a_{-1}), \overline{\mathrm{d}(v/a_{-1})})_g + |(v/a_{-1})_t|^2 + \sigma^2|v/a_{-1}|^2\right)|g|^{1/2}\,\mathrm{d}S$$

$$\lesssim e^{-2\sigma\delta} \int_{\Gamma_g} e^{2\sigma\varphi}(\mathcal{X}(v/a_{-1}))|g|^{1/2}\,\mathrm{d}S + e^{C\sigma}(\|v/a_{-1}\|^2_{H^1(\Sigma_+;g')} + \|\partial_{\nu,g}(v/a_{-1})\|^2_{L^2(\Sigma_+;g')}), \tag{3.30}$$

where the $\delta := \sup_{(x,t)\in\Gamma_T} \varphi(x,t) - \inf_{(x,t)\in\Gamma_g} \varphi(x,t) < 0$ and the constant $C = 2\sup_{x\in\Gamma_T} \varphi(x)$.

Theorem 3.2 gives

$$\int_{\Gamma_g} e^{2\sigma\varphi}(\mathcal{X}(v/a_{-1}))|g|^{1/2}\,\mathrm{d}S - C\sigma\|e^{\sigma\varphi}\nabla_{x,t}v\|^2_{L^2(Q_+;g)} - C\sigma^3\|e^{\sigma\varphi}v\|^2_{L^2(Q_+;g)}$$

$$\lesssim C(\|e^{\sigma\varphi}(\square_g + q)v\|^2_{L^2(Q_+;g)} + \|\partial_\nu v\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Sigma_+)}) \tag{3.31}$$

for some constant C.

By Lemma 3.4 and (3.21), for sufficiently large σ we have

$$\sigma\|e^{\sigma\varphi}\nabla_{x,t}v\|^2_{L^2(Q_+)} + \sigma^3\|e^{\sigma\varphi}v\|^2_{L^2(Q_+)} + \sigma\int_{\partial Q_+}e^{2\sigma\varphi}F(\sigma(\frac{v}{a_{-1}}),X^m(\frac{v}{a_{-1}}))\,\mathrm{d}S$$

$$\lesssim \sigma\|e^{\sigma\varphi}\nabla_{x,t}(\frac{v}{a_{-1}})\|^2_{L^2(Q_+)} + \sigma^3\|e^{\sigma\varphi}(\frac{v}{a_{-1}})\|^2_{L^2(Q_+)} + \sigma\int_{\partial Q_+}e^{2\sigma\varphi}F(\sigma(\frac{v}{a_{-1}}),X^m(\frac{v}{a_{-1}}))\,\mathrm{d}S$$

$$\lesssim \|e^{\sigma\varphi}\mathcal{L}(\frac{v}{a_{-1}})\|^2_{L^2(Q_+)} = \|e^{\sigma\varphi}(\Box_g + q)v\|^2_{L^2(Q_+)}, \tag{3.32}$$

where $\mathcal{L} = (\Box_g + q)a_{-1}$ is as the operator (3.20) where we set f to be a_{-1}.

Summing up (3.32) and (3.31) and dividing the boundary integral on ∂Q_+ into integrals on Γ_g and on $\Sigma_+ \cup \Gamma_T$, and combining with (3.29), we obtain

$$\int_{\Gamma_g}e^{2\sigma\varphi}(\mathcal{X}(v/a_{-1}))|g|^{1/2}\,\mathrm{d}S + \sigma\int_{\Gamma_g}e^{2\sigma\varphi}F(\sigma(v/a_{-1}),X^m(v/a_{-1}))\,\mathrm{d}S$$

$$\lesssim \|e^{\sigma\varphi}(\Box_g+q)v\|^2_{L^2(Q_+;g)} + \|\partial_\nu v\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Sigma_+)}$$

$$\qquad + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Gamma_T)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Gamma_T)}$$

$$\lesssim h(\sigma)\sum_m\|e^{\sigma\varphi}X^m(v/a_{-1})\|^2_{L^2(\Gamma_g)} + \|\partial_\nu v\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Sigma_+)}$$

$$\qquad + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Gamma_T)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Gamma_T)}$$

$$\lesssim h(\sigma)\|e^{\sigma\varphi}\mathcal{X}(v/a_{-1})\|^2_{L^2(\Gamma_g)} + \|\partial_\nu v\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Sigma_+)}$$

$$\qquad + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Gamma_T)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Gamma_T)}. \tag{3.33}$$

According to Lemma 3.5, as σ goes to infinity, $h(\sigma)$ goes to zero, so the term containing $h(\sigma)$ on the right-hand-side of (3.33) can be absorbed by the corresponding term on the left-hand-side, and then gives us

$$\int_{\Gamma_g}e^{2\sigma\varphi}(\mathcal{X}(v/a_{-1}))|g|^{1/2}\,\mathrm{d}S + \sigma\int_{\Gamma_g}e^{2\sigma\varphi}F(\sigma(v/a_{-1}),X^m(v/a_{-1}))\,\mathrm{d}S$$

$$\lesssim \|\partial_\nu v\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Gamma_T)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Gamma_T)}. \tag{3.34}$$

Combining (3.30) and (3.34), we obtain

$$\int_{\Gamma_g}e^{2\sigma\varphi}(\mathcal{X}(v/a_{-1}))|g|^{1/2}\,\mathrm{d}S + \sigma\int_{\Gamma_g}e^{2\sigma\varphi}F(\sigma(v/a_{-1}),X^m(v/a_{-1}))\,\mathrm{d}S$$

$$\lesssim \|\partial_\nu v\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Sigma_+)}$$

$$\qquad + e^{-2\sigma\delta}\int_{\Gamma_g}e^{2\sigma\varphi}(\mathcal{X}v)|g|^{1/2}\,\mathrm{d}S + e^{C\sigma}(\|v\|^2_{H^1(\Sigma_+;g')} + \|\partial_{\nu,g}v\|^2_{L^2(\Sigma_+;g')}).$$

Again, the term containing $e^{-2\sigma\delta}$ on the right-hand-side can be absorbed to the left-hand-side when σ is large enough, so

$$\int_{\Gamma_g}e^{2\sigma\varphi}(\mathcal{X}(v_-/a_{-1,-}))|g|^{1/2}\,\mathrm{d}S + \sigma\int_{\Gamma_g}e^{2\sigma\varphi}F(\sigma(v/a_{-1}),X^m(v/a_{-1}))\,\mathrm{d}S$$

$$\lesssim \|\partial_\nu v\|^2_{L^2(\Sigma_+)} + \|e^{\sigma\varphi}v_t\|^2_{L^2(\Sigma_+)} + \sigma^2\|e^{\sigma\varphi}v\|^2_{L^2(\Sigma_+)}$$

$$\qquad + e^{C\sigma}(\|v\|^2_{H^1(\Sigma_+;g')} + \|\partial_{\nu,g}v\|^2_{L^2(\Sigma_+;g')})$$

$$\lesssim \sigma^2 e^{C\sigma}(\|v\|^2_{H^1(\Sigma_+;g')} + \|\partial_{\nu,g}v\|^2_{L^2(\Sigma_+;g')}), \tag{3.35}$$

where the constant $C = 2\sup\{\varphi(x)\,;\,x \in \Gamma_T \cup \Sigma_+\}$.

Similarly, for v_- we have

$$\int_{\Gamma_g} e^{2\sigma\varphi}(\mathcal{X}(v_-/a_{-1,-}))|g|^{1/2}\,\mathrm{d}S - \sigma\int_{\Gamma_g} e^{2\sigma\varphi}F(\sigma(v_-/a_{-1,-}), X^m(v_-/a_{-1,-}))\,\mathrm{d}S$$
$$\lesssim \sigma^2 e^{C\sigma}(\|v_-\|^2_{H^1(\Sigma_-;g')} + \|\partial_{\nu,g}v_-\|^2_{L^2(\Sigma_-;g')}). \tag{3.36}$$

Note that the quadratic form F satisfies

$$|F(a_1, a_2, \cdots) - F(b_1, b_2, \cdots)|$$
$$\lesssim |\sum_{i,j}(a_i - b_i + b_i)(a_j - b_j + b_j) - \sum_{i,j} b_i b_j|$$
$$= |\sum_{i,j}(a_i - b_i)(a_j - b_j) + 2\sum_{i,j}(a_i - b_i)b_j|$$
$$\leq \sum_i (a_i - b_i)^2 + 2\big[(1+M)\sum_i(a_i - b_i)^2\big]^{1/2} \cdot \big[\sum_j b_j^2/(1+M)\big]^{1/2}$$
$$\leq (2+M)\sum_i (a_i - b_i)^2 + \sum_j b_j^2/(1+M),$$

for any positive number M. Summing up (3.35) and (3.36), we arrive at

$$\sum_{\pm}\int_{\Gamma_g} e^{2\sigma\varphi}(\mathcal{X}(v_{\pm}/a_{-1,\pm}))|g|^{1/2}\,\mathrm{d}S$$
$$\lesssim \sigma\int_{\Gamma_g} e^{2\sigma\varphi}[(2+M)\mathcal{X}((v_+/a_{-1,+}) - (v_-/a_{-1,-})) + \frac{\mathcal{X}(v_-/a_{-1,-})}{1+M}]\,\mathrm{d}S$$
$$+ \sigma^2 e^{C\sigma}\sum_{\pm}(\|v_{\pm}\|^2_{H^1(\Sigma_+;g')} + \|\partial_{\nu,g}v_{\pm}\|^2_{L^2(\Sigma_+;g')}).$$

Let σ be large enough so that all the arguments hold and then fix $M = \sigma^2$, thus the term $\sigma\int_{\Gamma_g} e^{2\sigma\varphi}\frac{\mathcal{X}v_-}{1+M}\,\mathrm{d}S$ can be absorbed by the left-hand-side and then it gives us

$$\sum_{\pm}\int_{\Gamma_g}(\mathcal{X}(v_{\pm}/a_{-1,\pm}))|g|^{1/2}\,\mathrm{d}S$$
$$\lesssim \int_{\Gamma_g}\mathcal{X}((v_+/a_{-1,+}) - (v_-/a_{-1,-}))\,\mathrm{d}S + \sum_{\pm}(\|v_{\pm}\|^2_{H^1(\Sigma_+;g')} + \|\partial_{\nu,g}v_{\pm}\|^2_{L^2(\Sigma_+;g')}).$$

The proof is complete. $\qquad\square$

4. POTENTIAL RECOVERY WITH TWO MEASUREMENTS

In this section, we consider the recovery of the potential q utilizing scattering data generated by two incident plane waves coming from two opposite directions; we are devoted to the proof of Theorem 1.1. Without loss of generality, we assume these two directions are e_n and $-e_n$. We denote these geometric-related quantities associated with e_n and $-e_n$ as ω_i, γ_i, X_i^m, X_i and $\mathcal{X}_i$ for $i = 1, 2$, and denote the two solutions coming from e_n and $-e_n$ as

$$U(x,t) = a_{-1}(x)\delta(t - \omega_1(x)) + u(x,t)H(t - \omega_1(x)),$$
$$V(x,t) = a'_{-1}(x)\delta(t - \omega_2(x)) + v(x,t)H(t - \omega_2(x)),$$

respectively. Note that a_{-1} and a'_{-1} are not necessarily the same. According to the assumption (1.4), we can assume that ω_1 and ω_2 satisfy $\omega_1(x) = -\omega_2(x)$, and

$$\text{if } \gamma_1(s) = (x,t), \text{ then } \gamma_2(-s) = (x,-t). \tag{4.1}$$

Moreover,

$$X_i^m = (\partial_j \omega_i) h^{jm} \partial_t + h^{mk} \partial_k \quad \text{and} \quad X_i = \partial_t + (\partial_j \omega_i) g^{jk} \partial_k, \quad i = 1, 2. \tag{4.2}$$

The following result extends [24, Theorem 1.2] to the Riemannian case.

Theorem 4.1. *Assume g satisfies the assumptions* (1.4) *and* (1.5). *Let T be large enough such that* (3.22) *is satisfied. There exists a positive constant C depending only on T, with the following property: if q_1, $q_2 \in C^\infty(\mathbb{R}^n; \mathbb{R})$ are supported in B and u_1, u_2, v_1 and v_2 are the corresponding solutions of* (1.3) *and* (2.16), *then*

$$\|q_1 - q_2\|_{L^2(B)}^2 \le C(\|u_1 - u_2\|_{H^1(\Gamma_g \cap \Sigma)}^2 + \|u_1 - u_2\|_{H^1(\Sigma; g')}^2 + \|v_1 - v_2\|_{H^1(\Sigma; g')}^2). \tag{4.3}$$

Proof of Theorem 1.1. Theorem 4.1 provides a stability estimate. The condition

$$u_{q_1}^\pm|_{\partial B \times [-1, T]} = u_{q_2}^\pm|_{\partial B \times [-1, T]},$$

implies $u_1 - u_2 = 0$ and $v_1 - v_2 = 0$ in (4.3), and hence $q_1 - q_2 = \tilde{q} = 0$. □

Proof of Theorem 4.1. Assume there are two potentials q_1 and q_2, and we denote as u_1 and v_1 (resp. u_2 and v_2) the solutions of (1.3) and (2.16) corresponding to q_1 (resp. q_2). Denote $\tilde{q}(x) = (q_1 - q_2)(x)$, $\tilde{u}(x, t) = (u_1 - u_2)(x, t)$ and $\tilde{v}(x, t) = -(v_1 - v_2)(x, -t)$, then it can be checked that

$$\Box_g \tilde{v}(x, t) = -(\Box_g(v_1 - v_2))(x, -t), \quad X_1 \tilde{v}(y, z, z) = -(X_2(v_1 - v_2))(y, z, -z). \tag{4.4}$$

where $x = (y, z)$ with $y \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$. Therefore,

$$\begin{cases} (\Box_g + q_1)\tilde{u}(x, t) = -\tilde{q}(x) u_2(x, t), & t > \omega_1(x), \\ (\Box_g + q_1)\tilde{v}(x, t) = \tilde{q}(x) v_2(x, -t), & t < -\omega_2(x) = \omega_1(x). \end{cases} \tag{4.5}$$

According to (4.1) and the definition of $\tilde{v}$, we can conclude

$$\tilde{v}(\gamma_1(s)) = -(v_1 - v_2)(\gamma_2(-s)).$$

Combining this with formulae (2.20), we arrive at

$$\begin{cases} \tilde{u}(\gamma_1(s)) = -\dfrac{1}{2} a_{-1}(\gamma_1(s)) \displaystyle\int_{-\infty}^{s} \tilde{q}(\gamma_1(\tau)) \, d\tau, & \text{so} \quad X_1|_{(x, \omega_1(x))}(\tilde{u}/a_{-1}) = -\tilde{q}(x)/2, \\ \tilde{v}(\gamma_1(s)) = \dfrac{1}{2} a'_{-1}(\gamma_1(s)) \displaystyle\int_{s}^{+\infty} \tilde{q}(\gamma_1(\tau)) \, d\tau, & \text{so} \quad X_1|_{(x, \omega_1(x))}(\tilde{v}/a'_{-1}) = -\tilde{q}(x)/2. \end{cases} \tag{4.6}$$

By applying Theorem 3.3 to (4.5) and (4.6), we obtain

$$\int_{\Gamma_g} [\mathcal{X}_1(\tilde{u}/a_{-1}) + \mathcal{X}_1(\tilde{v}/a'_{-1})] \, |g|^{1/2} \, dS$$

$$\lesssim \int_{\Gamma_g} \mathcal{X}_1(\tilde{u}/a_{-1} - \tilde{v}/a'_{-1}) \, dS + \|\tilde{u}/a_{-1}\|_{H^1(\Sigma_+; g')}^2 + \|\tilde{v}/a'_{-1}\|_{H^1(\Sigma_+; g')}^2$$

$$+ \|\partial_{\nu, g}(\tilde{u}/a_{-1})\|_{L^2(\Sigma_+; g')}^2 + \|\partial_{\nu, g}(\tilde{v}/a'_{-1})\|_{L^2(\Sigma_+; g')}^2. \tag{4.7}$$

Now we analyze these terms involving $\mathcal{X}_1$ in (4.7). First, we analyze the left-hand-side of (4.7). For any function $\varphi \in C^1(\mathbb{R}_x^n \times \mathbb{R}_t)$, by (3.1), (3.6) and (4.2), we have

$$\sqrt{|\nabla \omega_1|^2 + 1}(\mathcal{X}_1 \varphi) = \sigma^2 |\varphi|^2 + |X_1 \varphi|^2 + |d\varphi - d\omega_1 \langle d\omega_1, d\varphi \rangle|_g^2 \ge |X_1 \varphi|^2. \tag{4.8}$$

Replacing φ with $\tilde{u}/a_{-1}$ and with $\tilde{v}/a'_{-1}$ respectively in (4.8), we obtain

$$\sqrt{|\nabla \omega_1|^2 + 1} \mathcal{X}_1|_{(x, \omega_1(x))}(\tilde{u}/a_{-1}) \ge \left| X_1|_{(x, \omega_1(x))}(\tilde{u}/a_{-1}) \right|^2 = |\tilde{q}(x)|^2/4, \tag{4.9}$$

$$\sqrt{|\nabla \omega_1|^2 + 1} \mathcal{X}_1|_{(x, \omega_1(x))}(\tilde{v}/a'_{-1}) \ge \left| X_1|_{(x, \omega_1(x))}(\tilde{v}/a'_{-1}) \right|^2 = |\tilde{q}(x)|^2/4. \tag{4.10}$$

Then we investigate the term $\mathcal{X}_1(\tilde{u}/a_{-1} - \tilde{v}/a'_{-1})$ on the right-hand-side of (4.7). Define a mapping $F\colon \Gamma_g \to \Gamma_g \cap \Sigma$ in the following way. For any $(x, \omega_1(x)) \in \Gamma_g$, there exists a unique $s \in \mathbb{R}$ such that $\gamma_{(x,\omega_1(x))}(s) \in \Sigma$, and we define $F(x, \omega_1(x)) := \gamma_{(x,\omega_1(x))}(s)$. It means that F sends $(x, \omega_1(x))$ to the intersection between Σ and the integral curve γ of X_1 passing the point $(x, \omega_1(x))$. We denote as F^a the a-th component $(a = 1, \cdots, n+1)$ of F. It can be checked that

$$\begin{cases} \sum_{a=1}^{n} F^a(x, \omega_1(x))^2 = 1, & \text{(4.11a)} \\ F^n(x, \omega_1(x)) = F^{n+1}(x, \omega_1(x)). & \text{(4.11b)} \end{cases}$$

We show that $X_1 F = 0$. Note that $\{\gamma_{(x,\omega_1(x))}(t)\,;\, t \in \mathbb{R}\} \subset \Gamma_g$. According to the definition of F, for any fixed $(x, \omega_1(x))$, there exists an integral curve $\bar{\gamma}$ of X_1 and a number s such that $\bar{\gamma}(0) = (x, \omega_1(x))$ and $\bar{\gamma}(s) = F(x, \omega_1(x))$, and moreover, $\bar{\gamma}(s) = F(\bar{\gamma}(t))$ for any t. Hence,

$$X_1 F(x, \omega_1(x)) = X_1|_{(x,\omega_1(x))} F = \frac{\mathrm{d}}{\mathrm{d}t}\big(F(\bar{\gamma}(t))\big)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\big(\bar{\gamma}(s)\big)|_{t=0} = 0.$$

From $X_1 F = 0$ and (2.11), we can obtain

$$\partial_t F^a(x, \omega_1(x)) + \partial_j \omega_1 g^{jk} \partial_k F^a(x, \omega_1(x)) = 0. \tag{4.12}$$

Keeping in mind that the potential is assumed to be supported in $B(0,1)$, then formula (4.6) gives

$$\frac{\tilde{u}}{a_{-1}}(x, \omega_1(x)) - \frac{\tilde{v}}{a'_{-1}}(x, \omega_1(x)) = -\frac{1}{2}\int_{-\infty}^{+\infty} \tilde{q}(\gamma_1(\tau))\,\mathrm{d}\tau = \frac{\tilde{u}}{a_{-1}}(F(x, \omega_1(x))). \tag{4.13}$$

Recall that $X_1^m = (\partial_j \omega_1) h^{jm} \partial_t + h^{mk} \partial_k$, so for any $\varphi \in C^1(\mathbb{R}_x^n \times \mathbb{R}_t)$, its can checked that

$$X_1^m|_{(x,\omega_1(x))}\varphi = X_1^m \varphi(x, \omega_1(x)) = h^{mk}\partial_k\big(\varphi(x, \omega_1(x))\big). \tag{4.14}$$

Therefore, by combining (4.12), (4.13) and (4.14), one can compute

$$X_1^m|_{(x,\omega_1(x))}(\tilde{u}/a_{-1} - \tilde{v}/a'_{-1}) = h^{mk}\partial_k\Big(\frac{\tilde{u}}{a_{-1}} \circ F(x, \omega_1(x))\Big)$$

$$= h^{mk}\sum_{a=1}^{n+1} \partial_a\big(\tilde{u}/a_{-1}\big)(F(x, \omega_1(x))) \cdot \big(\partial_k F^a + \partial_t F^a(\partial_k \omega_1)\big)$$

$$= \sum_{a=1}^{n+1} h^{mk}\big(\partial_k F^a(x, \omega_1(x)) - (\partial_j \omega_1)g^{j\ell}(\partial_\ell F^a(x, \omega_1(x)))(\partial_k \omega_1)\big)\partial_a|_{F(x,\omega_1(x))}(\tilde{u}/a_{-1})$$

$$=: Y^m|_{F(x,\omega_1(x))}(\tilde{u}/a_{-1}), \tag{4.15}$$

where $Y^m|_{F(x,\omega_1(x))}$ is the value at $F(x, \omega_1(x))$ of a vector field.

Combining (3.6), (4.13) and (4.15), we arrive at

$$\sqrt{|\nabla \omega_1|^2 + 1}\,\mathcal{X}_1|_{(x,\omega_1(x))}(\tilde{u}/a_{-1} - \tilde{v}/a'_{-1})$$

$$= \sigma^2\Big|\frac{\tilde{u}}{a_{-1}}(F(x, \omega_1(x)))\Big|^2 + \sum_{m=1}^{n} \big|\,Y^m|_{F(x,\omega_1(x))}(\tilde{u}/a_{-1})\big|^2. \tag{4.16}$$

We make a claim as follow,

$$\textbf{Claim: } Y^m|_{F(x,\omega_1(x))} \text{ is tangential to } \Gamma_g \cap \Sigma, \tag{4.17}$$

and this can be justified as follows. Denoting $H_1(x,t) = t - \omega_1(x)$ and $H_2(y,z,t) = |x|^2/2 - 1/2$, it is clear that $\Gamma_g \cap \Sigma$ is the intersection of $\{(x,t)\,;\, H_1(x,t) = 0\}$ and $\{(x,t)\,;\, H_2(x,t) = $

$0\}$. First, formula (4.11a) implies that the n-dimensional vector $\left(F^a(x, \omega_1(x))\right)_{a=1}^n$ belongs to ∂B, so by (1.4) we have

$$\partial_a|_{F^a(x,\omega_1(x))} H_1 = \partial_a|_{F^a(x,\omega_1(x))}(t-z) = \begin{cases} 0, & a < n, \\ -1, & a = n, \\ 1, & a = n+1. \end{cases}$$

Therefore,

$$\begin{aligned} Y^m|_{F(x,\omega_1(x))} H_1 &= \left[h^{mk}\left(\partial_k F^{n+1}(x, \omega_1(x)) - (\partial_j \omega_1) g^{j\ell}(\partial_\ell F^{n+1}(x, \omega_1(x)))(\partial_k \omega_1)\right)\right] \\ &\quad - \left[h^{mk}\left(\partial_k F^n(x, \omega_1(x)) - (\partial_j \omega_1) g^{j\ell}(\partial_\ell F^n(x, \omega_1(x)))(\partial_k \omega_1)\right)\right] \\ &= 0. \end{aligned} \tag{4.18}$$

Note that in (4.18) we used (4.11b). Second, we have

$$\partial_a|_{F^a(x,\omega_1(x))} H_2 = \partial_a|_{F^a(x,\omega_1(x))}(|x|^2/2 - 1/2) = \begin{cases} F^a(x, \omega_1(x)), & a \le n, \\ 0, & a = n+1. \end{cases}$$

Hence $Y^m|_{F(x,\omega_1(x))} H_2$ can be computed as follows,

$$\begin{aligned} Y^m|_{F(x,\omega_1(x))} H_2 &= \sum_{a=1}^n h^{mk}\left[\partial_k F^a(x, \omega_1(x)) - (\partial_j \omega_1) g^{j\ell}(\partial_\ell F^a(x, \omega_1(x)))(\partial_k \omega_1)\right] F^a(x, \omega(x)) \\ &= \frac{1}{2} h^{mk}\left[\partial_k\left(\sum_{a=1}^n F^a(x, \omega_1(x))^2\right) - (\partial_j \omega_1) g^{j\ell}\partial_\ell\left(\sum_{a=1}^n F^a(x, \omega_1(x))^2\right)(\partial_k \omega_1)\right] \\ &= \frac{1}{2} h^{mk}\left[\partial_k(1) - (\partial_j \omega_1) g^{j\ell}\partial_\ell(1)(\partial_k \omega_1)\right] = 0. \end{aligned} \tag{4.19}$$

Note that in (4.19) we used (4.11a). Formulae (4.18) and (4.19) imply that $Y_m|_{F(w,\omega_1(x))}$ is tangential to both $\{H_1 = 0\}$ and $\{H_2 = 0\}$, and hence tangential to $\{H_1 = 0\} \cap \{H_2 = 0\}$ which is $\Gamma_g \cap \Sigma$.

Combining the claim (4.17) with (4.16), we can obtain

$$\int_{\Gamma_g} \mathcal{X}_1(\tilde{u}/a_{-1} - \tilde{v}/a'_{-1}) \, \mathrm{d}S \lesssim \int_{\Gamma_g} \sigma^2 \left|\frac{\tilde{u}}{a_{-1}}(F(x, \omega_1(x)))\right|^2 + \sum_{m=1}^n \left| Y^m|_{F(x,\omega_1(x))}(\tilde{u}/a_{-1})\right|^2 \mathrm{d}S$$

$$\lesssim \sigma^2 \|\tilde{u}/a_{-1}\|_{L^2(\Gamma_g \cap \Sigma)}^2 + \|\tilde{u}/a_{-1}\|_{H^1(\Gamma_g \cap \Sigma)}^2. \tag{4.20}$$

Now we combine (4.7), (4.9), (4.10) and (4.20), and we obtain

$$\begin{aligned} &\|\tilde{q}\|_{L^2(B(0,1))}^2 \\ &\lesssim \int_{\Gamma_g} [\mathcal{X}_1(\tilde{u}/a_{-1}) + \mathcal{X}_1(\tilde{v}/a'_{-1})] \, |g|^{1/2} \, \mathrm{d}S \\ &\lesssim \sigma^2 \|\tilde{u}/a_{-1}\|_{L^2(\Gamma_g \cap \Sigma)}^2 + \|\tilde{u}/a_{-1}\|_{H^1(\Gamma_g \cap \Sigma)}^2 + \|\tilde{u}/a_{-1}\|_{H^1(\Sigma_+;g')}^2 + \|\tilde{v}/a'_{-1}\|_{H^1(\Sigma_-;g')}^2 \\ &\quad + \|\partial_{\nu,g}(\tilde{u}/a_{-1})\|_{L^2(\Sigma_+;g')}^2 + \|\partial_{\nu,g}(\tilde{v}/a'_{-1})\|_{L^2(\Sigma_-;g')}^2 \\ &\lesssim \sigma^2 \|\tilde{u}\|_{L^2(\Gamma_g \cap \Sigma)}^2 + \|\tilde{u}\|_{H^1(\Gamma_g \cap \Sigma)}^2 + \|\tilde{u}\|_{H^1(\Sigma_+;g')}^2 + \|\tilde{v}\|_{H^1(\Sigma_-;g')}^2 \\ &\quad + \|\partial_{\nu,g}\tilde{u}\|_{L^2(\Sigma_+;g')}^2 + \|\partial_{\nu,g}\tilde{v}\|_{L^2(\Sigma_-;g')}^2. \end{aligned}$$

Applying [24, Lemma 3.3], we can conclude the proof. $\qquad\square$

Appendix A. Equivalence of conditions

We prove the equivalence of three conditions appearing in the introduction. The required arguments are standard (one could also use the language of Lagrangian manifolds as in [10, Section 6.4]). We give the details for completeness.

Lemma A.1. *Let g be a smooth Riemannian metric in $\mathbb{R}^n$ satisfying (1.2). Then the conditions (1.4), (1.4a), and (1.4b) are equivalent.*

Proof. We first prove that (1.4) implies (1.4a). First extend ω smoothly to $\mathbb{R}^n$ so that $\omega(x) = x_n$ outside B. If γ is any unit speed g-geodesic, we claim that

$$d\omega(\gamma(t_0)) = \dot{\gamma}(t_0)^\flat \text{ for some } t_0 \implies d\omega(\gamma(t)) = \dot{\gamma}(t)^\flat \text{ for all } t \in \mathbb{R}. \tag{A.1}$$

To prove this, write $p(x, \xi) = \frac{1}{2}(|\xi|_g^2 - 1)$ where $(x, \xi) \in T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, let $X(t)$ solve $\dot{X}(t) = \nabla_\xi p(X(t), \nabla\omega(X(t)))$ with $X(t_0) = \gamma(t_0)$ and write $\Xi(t) = \nabla\omega(X(t))$. We compute

$$\dot{\Xi}_j(t) = \partial_{jk}\omega(X(t))\dot{X}_k(t) = \partial_{jk}\omega(X(t))\partial_{\xi_k}p(X(t), \nabla\omega(X(t))).$$

But (1.4) gives that $p(x, \nabla\omega(x)) = 0$, and differentiating this equation yields

$$\partial_{x_j}p(x, \nabla\omega(x)) + \partial_{\xi_k}p(x, \nabla\omega(x))\partial_{jk}\omega(x) = 0.$$

Combining the last two equations shows that $(X(t), \Xi(t))$ satisfies the Hamilton equations

$$\dot{X}(t) = \nabla_\xi p(X(t), \Xi(t)), \qquad \dot{\Xi}(t) = -\nabla_x p(X(t), \Xi(t)).$$

Note also that $(X(t_0), \Xi(t_0)) = (\gamma(t_0), \dot{\gamma}(t_0)^\flat)$, using that $d\omega(\gamma(t_0)) = \dot{\gamma}(t_0)^\flat$. Since solutions are unique, $(X(t), \Xi(t))$ must agree with the null bicharacteristic $(\gamma(t), \dot{\gamma}(t)^\flat)$ for p. This shows (A.1). By integrating, we also see that

$$\omega(\gamma(t_2)) - \omega(\gamma(t_1)) = \int_{t_1}^{t_2} d\omega|_{\gamma(t)}(\dot{\gamma}(t))\, dt = \int_{t_1}^{t_2} \langle d\omega, d\omega \rangle\, dt = t_2 - t_1. \tag{A.2}$$

For any $x' \in \mathbb{R}^{n-1}$ let $\gamma_{x'}(t)$ be the unit speed g-geodesic with $\gamma_{x'}(-1) = (x', -1)$ and $\dot{\gamma}_{x'}(-1) = e_n$. Then $d\omega(\gamma_{x'}(-1)) = \dot{\gamma}_{x'}(-1)^\flat$ since $\omega = x_n$ outside B, so (A.1) gives that $d\omega(\gamma_{x'}(t)) = \dot{\gamma}_{x'}(t)^\flat$ for all t. Consider the map

$$F : \mathbb{R}^n \to \mathbb{R}^n, \quad F(x', t) = \gamma_{x'}(t).$$

This map is smooth, and it is the identity map in $\mathbb{R}^n \setminus \{(x', t) : |x'| \leq 1, t \geq -1\}$. It is injective, since if $\gamma_{x'}(t) = \gamma_{\tilde{x}'}(\tilde{t})$ then by (A.1)

$$\dot{\gamma}_{x'}(t)^\flat = d\omega(\gamma_{x'}(t)) = d\omega(\gamma_{\tilde{x}'}(\tilde{t})) = \dot{\gamma}_{\tilde{x}'}(\tilde{t})^\flat.$$

Thus the two geodesics must be the same, i.e. $\gamma_{x'}(t+s) = \gamma_{\tilde{x}'}(\tilde{t}+s)$. By taking $s \ll -1$ we obtain $x' = \tilde{x}'$, and (A.2) then gives $t = \tilde{t}$. Finally, we show that F is surjective. For any y with $|y'| \leq 1$ and $y_n > -1$, let γ be the geodesic with $\gamma(0) = y$ and $\dot{\gamma}(0)^\flat = d\omega(y)$. We denote $A := \inf_{\{x \in \mathbb{R}^n; x_n > -1\}} \omega(x)$ for short, and it can be checked that A is finite. If $\gamma_n(s) > -1$ for all $s \leq 0$, then $\omega(\gamma(s)) \geq A$ for all $s \leq 0$ (using that ω is smooth in $\overline{B}$ and $\omega = x_n$ outside B). Now (A.2) implies that

$$s = \omega(\gamma(s)) - \omega(\gamma(0)) \geq A - \omega(\gamma(0)) \text{ for all } s \leq 0.$$

This is a contradiction, which proves that $y = \gamma_{x'}(t)$ for some x' and t. A similar argument for $y = (x', -1)$ shows that for any $x' \in \mathbb{R}^{n-1}$, the geodesic $\gamma_{x'}(t)$ meets $\{x_n = 1\}$.

We have proved that F is smooth and bijective, i.e. a diffeomorphism. Hence the map $(x', t) \mapsto \gamma_{x'}(t)$ smoothly parametrizes $\mathbb{R}^n$, and any geodesic starting from $\{x_n = -1\}$ in direction e_n meets $\{x_n = 1\}$ pointing in direction e_n. This concludes the proof that (1.4) implies (1.4a).

Now, if (1.4a) is satisfied, then the map $F(x', t) = \gamma_{x'}(t)$ above is a diffeomorphism of $\mathbb{R}^n$. Writing $y' = x'$ and $y_n = t$ gives a global coordinate system $y = (y', y_n)$ in $\mathbb{R}^n$. (These are sometimes called semigeodesic coordinates with respect to $\{x_n = -1\}$.) In particular, $y_n(x) = t$ if $x = \gamma_{z'}(t)$ for some (unique) z' and t, and then $\partial_{y_n}|_x = \dot{\gamma}_{z'}(t)$. The fact that the metric is Euclidean outside B together with (1.4a) ensure that $\partial_{y_n} = \partial_{x_n}$ outside B. Clearly we have $|\partial_{y_n}|_g = 1$, so the metric has the form

$$g(y) = \begin{pmatrix} g_0(y) & * \\ * & 1 \end{pmatrix}.$$

It remains to show that $g(\partial_{y_\alpha}, \partial_{y_n}) = 0$ for $1 \le \alpha \le n - 1$. This is a standard property of semigeodesic coordinates, and it follows from the computation

$$\partial_{y_n}(g(\partial_{y_\alpha}, \partial_{y_n})) = g(\nabla_{\partial_{y_n}} \partial_{y_\alpha}, \partial_{y_n}) + g(\partial_{y_\alpha}, \nabla_{\partial_{y_n}} \partial_{y_n})$$

where ∇ is the Levi-Civita connection for g. The last term vanishes since ∂_{y_n} is the tangent vector of a geodesic. For the first term on the right we use that ∇ is torsion free, i.e. $\nabla_{\partial_{y_n}} \partial_{y_\alpha} = \nabla_{\partial_{y_\alpha}} \partial_{y_n}$, and this implies

$$\partial_{y_n} g(\partial_{y_\alpha}, \partial_{y_n}) = \frac{1}{2} \partial_{y_\alpha}(g(\partial_{y_n}, \partial_{y_n})) = 0.$$

Since $g(\partial_{y_\alpha}, \partial_{y_n})|_{(x', -1)} = e(\partial_{x_\alpha}, \partial_{x_n}) = 0$ where e is the Euclidean metric, we see that $g(\partial_{y_\alpha}, \partial_{y_n}) = 0$ everywhere. This shows (1.4b).

Finally, if (1.4b) holds then we may choose $\omega(y) = y_n$. The function ω is smooth everywhere, and the form of the metric implies that $|d\omega|_g = 1$. We also have $d\omega = dx_n$ outside B, hence $\omega = x_n + C$ outside B for some constant C. Subtracting the constant gives the function ω required in (1.4). $\qquad\square$

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