# Injectivity of geodesic X-ray transform

$(M, g)$ compact non-trapping Riemannian manifold w/ $\partial M$.

$f \in \underset{\text{symmetric}}{\Gamma}(T^*M^{\otimes m}) =: S_m(M)$ , $If(\gamma) = \int_a^b f_{i_1 \dots i_m}(\gamma(t))\dot\gamma^{i_1}(t) \dots \dot\gamma^{i_m}(t)\,dt =: \int_a^b \langle f, \dot\gamma^m(t)\rangle dt$

// for $\gamma : [a,b] \to M$ geodesic between boundary pts.

**Define**: $d : S_m(M) \to S_{m+1}(M)$ ; $d = \sigma \nabla$

symmetrization, covariant derivative

$$P_m(M) := \{ f \in S_m(M) \mid f = dp \text{ for some } p \in S_{m-1}(M) \text{ w/ } p|_{\partial M} = 0 \}$$

$$Z_m(M) := \{ f \in S_m(M) \mid \int_a^b \langle f, \dot\gamma^m(t)\rangle dt = 0 \ \forall \gamma \}$$

**Note**: $\frac{d}{dt}\langle p, \dot\gamma^m\rangle = \frac{\partial p_{i_1 \dots i_m}}{\partial x^j}(\gamma(t))\dot\gamma^j(t)\dot\gamma^{i_1}(t) \dots \dot\gamma^{i_m}(t) + \sum_{k=1}^m p_{i_1 \dots i_m}(\gamma(t))\dot\gamma^{i_1}(t) \dots \ddot\gamma^{i_k}(t) \dots \dot\gamma^{i_m}(t)$

$$= \frac{\partial p_{i_1 \dots i_m}}{\partial x^j}\dot\gamma^j\dot\gamma^{i_1} \dots \dot\gamma^{i_m} - \sum_{k=1}^m p_{i_1 \dots i_m}\dot\gamma^{i_1} \dots \dot\gamma^j\dot\gamma^l\Gamma^{i_k}_{jl} \dots \dot\gamma^{i_m}$$

$$= \underbrace{\left(\frac{\partial p_{i_1 \dots i_m}}{\partial x^j} - \sum_{k=1}^m \Gamma^l_{j i_k} p_{i_1 \dots i_{k-1} l i_k \dots i_m}\right)}_{=(\sigma\nabla p)_{i_1 \dots i_m j}}\dot\gamma^{i_1} \dots \dot\gamma^{i_m}\dot\gamma^j = \langle dp, \dot\gamma^{m+1}\rangle$$

$\Rightarrow \int_a^b \langle dp, \dot\gamma^{m+1}\rangle dt = \langle p, \dot\gamma^m\rangle\Big|_a^b = 0$ $\Rightarrow$ $\boxed{P_m(M) \subset Z_m(M)}$

**Q**: For which $(M, g)$ we have $Z_m(M) = P_m(M)$ ?

**Note**: For $m = 2$ this is the deformation boundary rigidity problem i.e. the linearization of the boundary rigidity problem

**Note**: For $m = 0$ this reduces to much simpler form:

For which $(M, g)$ we have $\int_a^b f(\gamma(t))dt = 0 \ \forall \gamma \Rightarrow f = 0$?

**Terminology**: If there exists a vector field $X \neq 0$ along a geodesic $\gamma : [a,b] \to M$ s.t.

$$D_{\dot\gamma}D_{\dot\gamma}X + sR(\dot\gamma, X)\dot\gamma = 0 \quad , \quad X(a) = 0, \ X(b) = 0,$$

$R$ Riemannian curvature tensor

then $\gamma(a)$ & $\gamma(b)$ are s-conjugate along $\gamma$.

**Theorem 1.1** [Dairbekov 06 : "Integral geometry problem for non-trapping manifolds" IP 22 (2006) 431-445]

Let $m \geq 1$, $n \geq 2$. If $(M, g)$ has no s-conjugate points for

$$s = \frac{m(m+1)}{n + 2m - 1}, \quad \text{then } Z_m(M) = P_m(M)$$

**Theorem 1.3** [Dairbekov 06] Let $(M,g)$ be compact non-trapping Riemannian manifold without conjugate points and let $f\in C^\infty(M)$.
If $If(\gamma):=\int_a^b f(\gamma(t))\,dt=0$ for all geodesics $\gamma:[a,b]\to M$ between boundary points, then $f=0$.

**Pf. Step I:** (Reduction to the [illegible] equation)

**Lemma 2.1:** If $If(\gamma)=0$ for all $\gamma$, then $f|_{\partial M}=0$.

**Corollary 1:** Zero-extension of $f$ into the diff-top. double $N$ of $M$ belongs to $H^1(N)\cap C(N)$.

**Define** $u: SM\to\mathbb{R}$ ; $u(\xi)=\int_{l_\xi}^0 f(\gamma_\xi(t))\,dt$ , independent of $l_\xi$ s.t. $\gamma_\xi(l_\xi)\in N\setminus M$

$K:=\{\xi\in SN \mid \Phi_H^t(\xi)\in S\partial M \text{ for some } t\in\mathbb{R}\}=pr_{SM}((\Phi_H)^{-1}(S\partial M))$

codim 2 in $\mathbb{R}\times SM$

Clearly $K\subset SN$ is closed and of measure zero.

**Lemma 2.3:** (i) $u|_{S(N\setminus M)}=0$

(ii) $u\in H^1(SN)\cap C(SN)\cap C^\infty(SM\setminus K)$

(iii) $Hu=f$

## Pestov's identity

$\pi: TN\to N$ natural projection, $\pi^*TN \xrightarrow{\iota} TN$ pullback bundle

$$\begin{tikzcd}
\pi^*TN \arrow[r, "\iota"] \arrow[d, "q"'] & TN \arrow[d, "\pi"] \\
TN \arrow[r, "\pi"] & N
\end{tikzcd}$$

fiberwise → isomorphism

Sections $\Xi\in\Gamma(\pi^*TN)$ are semibasic vectorfields

**Idea:** Tangent space $T_xM$ is glued to each $\xi\in T_xM$

$\Xi=\Xi^i(x,\xi)\,e_i|_{(x,\xi)}$ , $q(e_i|_{(x,\xi)})=\frac{\partial}{\partial x^i}\big|_x$

Base manifold has dimension $2n$, each fiber has dimension $n$.

**Define:** $\frac{\delta}{\delta x^i}\big|_\xi\in T_\xi TN$ by $\frac{\delta}{\delta x^i}\big|_\xi:=\frac{\partial}{\partial x^i}\big|_\xi-\xi^j\Gamma_{ij}^k\frac{\partial}{\partial\xi^k}\big|_\xi$

$H_\xi TN:=\operatorname{span}\{\frac{\delta}{\delta x^1}\big|_\xi,\dots,\frac{\delta}{\delta x^n}\big|_\xi\}$, $V_\xi TN:=\operatorname{span}\{\frac{\partial}{\partial\xi^1}\big|_\xi,\dots,\frac{\partial}{\partial\xi^n}\big|_\xi\}$

$e_i|_\xi \mathrel{\hat=} \frac{\delta}{\delta x^i}\big|_\xi \mathrel{\hat=} \frac{\partial}{\partial\xi^i}\big|_\xi \mathrel{\hat=} \frac{\partial}{\partial x^i}\big|_x$ , $\pi(\xi)=x$

$\pi^*TN|_\xi \mathrel{\hat=} H_\xi TN \mathrel{\hat=} V_\xi TN \mathrel{\hat=} T_xN$

Horizontal gradient $\overset{h}{\nabla}u = g^{ij}\frac{\delta u}{\delta x^i} e_i$ for $u \in C^\infty(TN)$

Vertical gradient $\overset{v}{\nabla}u = g^{ij}\frac{\partial u}{\partial \xi^i} e_i$ — '' —

Horizontal divergence $\overset{h}{\operatorname{div}}\, X = \frac{\delta X^i}{\delta x^i} + \Gamma^j_{ij} X^i$ for $X \in \Gamma(\pi^* TN)$

Vertical divergence $\overset{v}{\operatorname{div}}\, Z = \frac{\partial Z^i}{\partial \xi^i}$ — '' —

Inherited inner product: $\langle Z, X\rangle = g_{ij} Z^i X^j$ — '' — $(\Rightarrow Hu = \langle \xi, \overset{h}{\nabla}u\rangle)$

<u>Pestov's identity</u>: $2\langle \overset{h}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = |\overset{h}{\nabla}u|^2 + \overset{h}{\operatorname{div}}\, V + \overset{v}{\operatorname{div}}\, W - \langle R(\xi, \overset{v}{\nabla}u)\xi, \overset{v}{\nabla}u\rangle$

$$V = \langle \overset{h}{\nabla}u, \overset{v}{\nabla}u\rangle \xi - \langle \overset{h}{\nabla}u, \xi\rangle \overset{v}{\nabla}u \quad , \quad W = \langle \overset{h}{\nabla}u, \xi\rangle \overset{h}{\nabla}u$$

<u>Gauss-Ostrogradskii formulae:</u> Let $X \in \Gamma(\pi^* TM)$ be r-homogeneous. Then

$$\int_{SM} \overset{h}{\operatorname{div}}\, X \, d\xi dx = (-1)^n \int_{\partial SM} \langle \nu, X\rangle d\xi dx \quad , \quad \int_{SM} \overset{v}{\operatorname{div}}\, X \, d\xi dx = (r+n-1)\int_{SM} \langle \xi, X\rangle d\xi dx.$$

Since $u(\lambda\xi) = \int_{\ell_{\lambda\xi}}^{0} f(\gamma_{\lambda\xi}(t))dt = \frac{1}{\lambda}\int_{\underbrace{\tau_-(\lambda\xi)}_{=\frac{1}{\lambda}\tau_-(\xi)}}^{0} f(\gamma_\xi(\lambda t))\lambda dt \underset{s=\lambda t}{=} \frac{1}{\lambda}\int_{\tau_-(\xi)}^{0} f(\gamma_\xi(s))ds = \frac{1}{\lambda}u(\xi),$

$u|_{S(\partial M)} = 0$

$$\int_{SD} (|\overset{h}{\nabla}u|^2 - \langle R(\xi, \overset{v}{\nabla}u)\xi, \overset{v}{\nabla}u\rangle) d\xi dx + (-1)^n \int_{\partial SD} \langle \nu, V\rangle d\xi dx + (n-2)\int_{SD} \underbrace{\langle \xi, W\rangle}_{=(Hu)^2 = f^2} d\xi dx = 0$$

Denote $\overset{h}{\nabla}{}^{\parallel}u = \overset{h}{\nabla}u - \langle \overset{h}{\nabla}u, \nu\rangle \nu$ on $\partial SD$. Then

$$\langle \nu, V\rangle = \langle \overset{h}{\nabla}{}^{\parallel}u, \overset{v}{\nabla}u\rangle \langle \xi, \nu\rangle - \langle \overset{h}{\nabla}{}^{\parallel}u, \xi\rangle \langle \overset{v}{\nabla}u, \nu\rangle \underset{u|_{\partial SD}=0}{=} 0.$$

<u>Lemma 5.1</u>
<u>+Lemma 5.1:</u> Let $u : SD \to \mathbb{R}$ be such that $u \in H^1(SD)$, $u|_{\partial SM} = 0$, $Hu = f$. Then

Then $\int_{SM} (|\overset{h}{\nabla}u|^2 - \langle R(\xi, \overset{v}{\nabla}u)\xi, \overset{v}{\nabla}u\rangle) d\xi dx = -(n-2)\int_{SM} f^2 d\xi dx.$

— = —

Define $\overset{v}{\nabla}{}^{\parallel}u = \overset{v}{\nabla}u - \langle \overset{v}{\nabla}u, \xi\rangle \xi$. (The $\overset{v}{\nabla}{}^{\parallel}u$ depends only on $u|_{SM}$)

Now $D_\xi(\overset{v}{\nabla}u) = \langle \xi, \overset{v}{\nabla}\overset{v}{\nabla}u\rangle = \overset{v}{\nabla}(Hu) - \overset{h}{\nabla}u = -\overset{h}{\nabla}u$, $\langle \xi, \overset{v}{\nabla}u\rangle = -u$ ← by homogeneity

$\Rightarrow D_\xi(\overset{v}{\nabla}{}^{\parallel}u) = -\overset{h}{\nabla}u + D_\xi(u\xi) = \underline{-\overset{h}{\nabla}u + f\xi}$

$\Rightarrow |D_\xi(\overset{v}{\nabla}{}^{\parallel}u)|^2 = |\overset{h}{\nabla}u|^2 + f^2 - 2f\underbrace{\langle \xi, \overset{h}{\nabla}u\rangle}_{=Hu=f} = |\overset{h}{\nabla}u|^2 - f^2$

$$\Rightarrow \underbrace{\int_{SM} (|D_\xi(\overset{v}{\nabla}{}^{\parallel}u)|^2 - \langle \overset{v}{\nabla}{}^{\parallel}u, R(\xi, \overset{v}{\nabla}{}^{\parallel}u)\xi\rangle) d\xi dx}_{\int_{\partial_+ SM} \left( \underbrace{\int_0^{\tau_+(\eta)} (|D_{\dot\gamma_\eta}\Sigma_\eta|^2 - \langle \Sigma_\eta, R(\dot\gamma_\eta, \Sigma_\eta)\dot\gamma_\eta\rangle) dt}_{\text{Index form } I(\Sigma,\Sigma)>0 \text{ for non-zero } \Sigma(t) \text{ with } \Sigma(a)=\Sigma(b)=0} \right) g(\nu,\eta)\, dy\, d\eta} = -(n-1)\int_{SM} f^2 d\xi dx$$

$, \ \Sigma_\eta(t) := \overset{v}{\nabla}{}^{\parallel}u(\dot\gamma(t))$

$\Longleftrightarrow$ no conjugate points along $\gamma : [a,b] \to M$ $\square$