Geodesic X-ray transform
solvability for the adjoint.

Theorem 1.4.

Let (M, g) be a simple, cpct 2D - RM with bdry. Then the operator $I^*: C^\infty_\alpha(\partial_+ \Omega(M)) \longrightarrow C^\infty(M)$ is surjective.

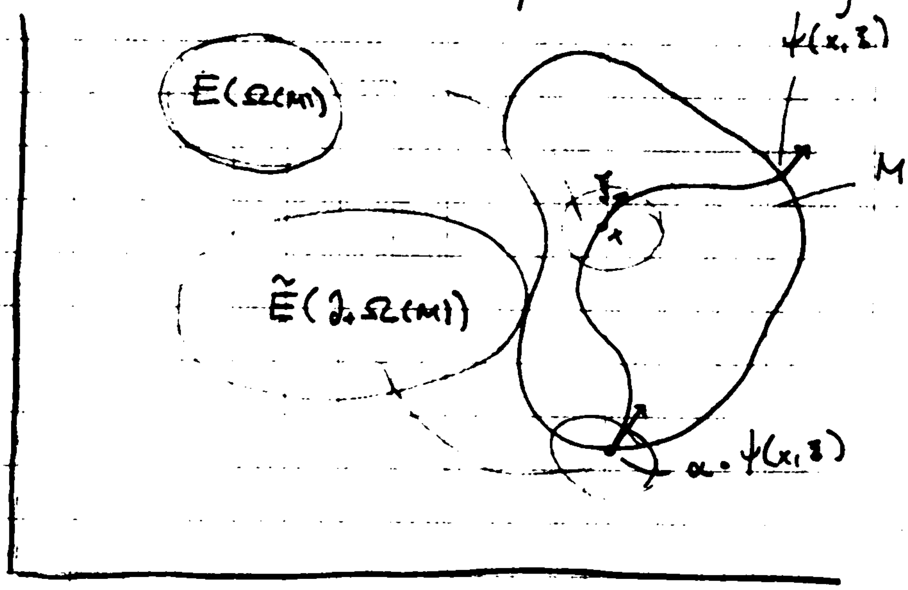
Proof (modulo few things)

The space $C^\infty_\alpha(\partial_+\Omega(M))$ is

$$C^\infty_\alpha(\partial_+\Omega(M)) = \{ w \in C^\infty(\partial_+\Omega(M)) \,;\, w_\psi \in C^\infty(\Omega(M)) \}$$

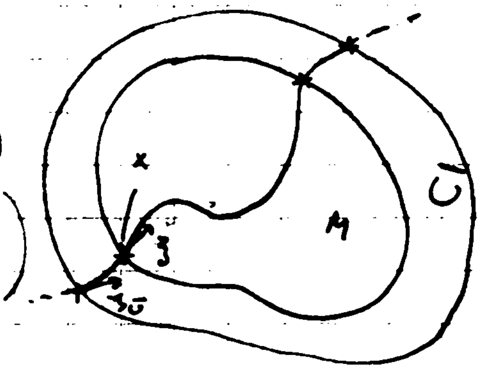
Note:
$w_\psi \notin C^\infty(\Omega(M))$ in general.

$E(X)$ = "generic function space over X"



Let (S, g) be a cpct RM w/out a bdry where (M, g) can be embedded. Let $U \subset S$ any open simple nhood of M.

Let I, $\tilde{I}$ be the geodesic X-ray transforms on M and $\overline{U}$.



i.e.

$$I : E(M) \longrightarrow \tilde{E}(\partial_+\Omega(M))$$

$$\left(\tilde{I} : E(\overline{U}) \longrightarrow \tilde{E}(\partial_+\Omega(\overline{U})) \right)$$

$$(If)(x,\xi) = \int_0^{\tau_M} f \circ \varphi_t(x,\xi)\,dt$$

(resp. $\tilde{I}$)

We need the Theorem 3.1. (which we will prove after this)

Theorem 3.1. Let $s \geq 0$.

For every $h \in H^s(M)$ $\exists f \in H^{s-1}(U)$ s.t.

$$\tilde{I}^*\tilde{I}f \big|_M = h$$

By Sobolev embedding this implies that given $h \in C^\infty(M)$ $\exists f \in C^\infty(\overline{U})$ s.t.

$$\tilde{I}^*\tilde{I}f \big|_M = h.$$

Now we can show the claim. Let $h \in C^\infty(M)$.

Then we can define $u^f \in E(\Omega(U))$

$$u^f(x,\xi) = \int_0^{\tau_{\overline{U}}} f \circ \varphi_t(x,\xi)\,dt, \quad (x,\xi) \in \Omega(U)$$

where $f \in C^\infty(\overline{U})$ and $\tilde{I}^*\tilde{I}f \big|_M = h$.

$f \in C^\infty$

$\Rightarrow \quad u^f \in C^\infty(\Omega(U))$.

Let $\quad w(x,\xi) := \underbrace{u^f(x,\xi) + u^f(x,-\xi)}_{=: 2u_+^f(x,\xi)} \quad , \quad (x,\xi) \in \partial_+\Omega(M)$

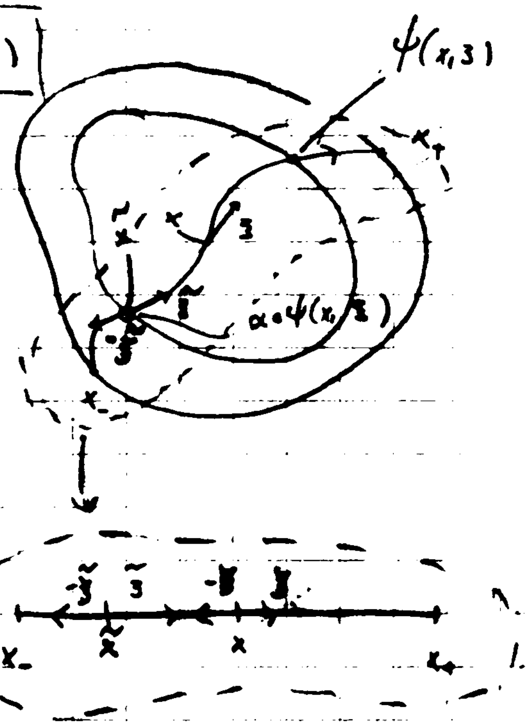
$u^f \in C^\infty(\Omega(U)) \quad \Rightarrow \quad w \in C^\infty(\partial_+\Omega(M))$

We claim that $\quad I^* w = h \quad$ and $\quad w_\psi \in C^\infty(\Omega(M))$.

This gives the claim, since if $w_\psi \in C^\infty(\Omega(M))$

$\Rightarrow \quad w \in C^\infty_\alpha(\partial_+\Omega(M))$.

$\boxed{1) \quad w_\psi = u_+^f | \Omega(M) \in C^\infty(\Omega(M))}$



Given $(x,\xi) \in \Omega(M)$, then

$w_\psi(x,\xi) = u^f(\tilde{x},\tilde{\xi}) + u^f(\tilde{x},-\tilde{\xi})$

By the semigroup prop. of φ_t

$u^f(\tilde{x},\tilde{\xi}) + u^f(\tilde{x},\tilde{\xi})$

$$= \int_{-\tau_U(\tilde{x},\tilde{\xi})}^{\tau_U(\tilde{x},\tilde{\xi})} f\circ\varphi_t(\tilde{x},\tilde{\xi})\,dt$$

$=$ "integral of f over the geodesic"

$$= \int_{-\tau_U(x,\xi)}^{\tau_U(x,\xi)} f\circ\varphi_t(x,\xi)\,dt = u^f(x,\xi) + u^f(x,-\xi)$$

$\therefore \quad w_\psi(x,\xi) = u^f(x,\xi) + u^f(x,-\xi) \in C^\infty(\Omega(M))$.

$\boxed{2) \quad I^* w = h}$

Mikko showed that

$$(I^* w)(x) = \int_{\Omega_x} w_\psi(x,\xi)\,d\Omega_x \quad , \quad x \in M$$

Hence

$$(I^* w)(x) = \int_{\Omega_x} w_\psi(x,\xi)\, d\Omega_x$$

$$\overset{1)}{=} \int_{\Omega_x} \left(u^f(x,\xi) + u^f(x,-\xi) \right) d\Omega_x$$

$$= 2 \int_{\Omega_x} u^f(x,\xi)\, d\Omega_x$$

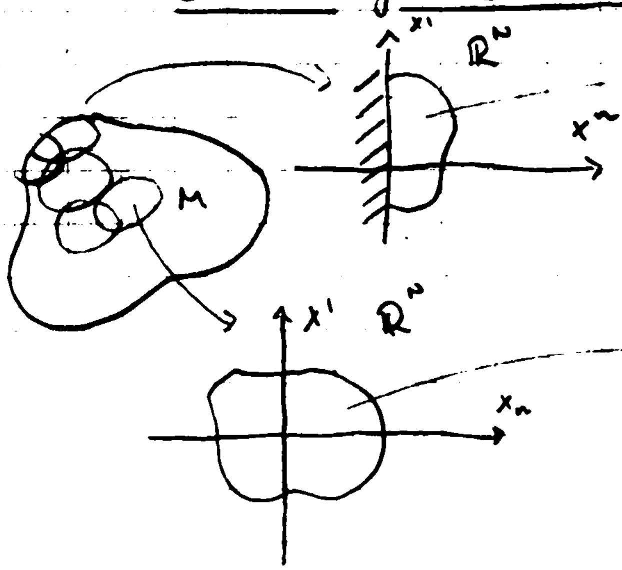
$$\overset{\text{defn}}{=} 2 \int_{\Omega_x} d\Omega_x \int_0^{\tau_U} dt \; f \circ \varphi_t(x,\xi)$$

$$\overset{\text{Mikko}}{=} (\tilde{I}^* \tilde{I} f)(x) = h(x). \qquad \square$$

Now we will proceed to the Theorem 3.1.

Before that we need the concepts of Sobolev spaces H^s and the Fredholm operators.

<u>Sobolev spaces on mfolds with bdry</u>



domain in $\mathbb{R}^N_+ = \{x_n \geq 0\}$

domain in $\mathbb{R}^N$

Definition: The Sobolev space $H^s(\mathbb{R}^N)$ is

$$H^s(\mathbb{R}^N) = \{ f \in \mathcal{S}'(\mathbb{R}^N);\ \|f\|_{H^s(\mathbb{R}^N)} = \|\hat{f}\|_{L^2_s(\mathbb{R}^N)} < \infty \}$$

where $L^2_s(\mathbb{R}^N) = \{ f \in \mathcal{S}'(\mathbb{R}^N);\ f(x)(1+|x|^2)^{s/2} \in L^2(\mathbb{R}^N) \}$.

When $s = k \in \mathbb{N}$, we have that

$$H^k(\mathbb{R}^N) = \{ f \in L^2(\mathbb{R}^N);\ \partial^\alpha f \in L^2(\mathbb{R}^N)\ \forall |\alpha| \leq k \}$$

The latter can easily be used to define

Definition When $k \in \mathbb{N}$ the Sobolev space $H^k(\mathbb{R}^N_+)$ is $H^k(\mathbb{R}^N_+) = \{ f \in L^2(\mathbb{R}^N_+);\ \partial^\alpha f \in L^2(\mathbb{R}^N_+)\ \ \forall |\alpha| \leq k \}$

In order to reduce the manifold case to these we need the restriction mapping r_Ω and the right inverse (the extension mapping) ε_Ω.

Fact 1 There exists a cont. linear onto

$r_{\mathbb{R}^N_+} : H^k(\mathbb{R}^N) \longrightarrow H^k(\mathbb{R}^N_+)$ and continuous

$\varepsilon_{\mathbb{R}^N_+} : H^k(\mathbb{R}^N_+) \longrightarrow H^k(\mathbb{R}^N)$ s.t. $r_{\mathbb{R}^N_+} \varepsilon_{\mathbb{R}^N_+} = id.$

Fact 2 By localising and flattening of the bdry we can define Sobolev space $H^s(S)$ for every s on compact RM w/out bdry and if $M \subset S$ is a submfold with C^∞-bdry then $\forall k \in \mathbb{N}$ we can def. a Sobolev space $H^k(M)$ and there $\exists$ restriction and extension maps.

Fredholm operators and elliptic ΨDOs

Definition

Let E_j, $j=1,2$, be B-spaces & $A: E_1 \to E_2$ cont. linear operator. Operator A is Fredholm, if $\dim \operatorname{Ker} A < \infty$ & $\dim \operatorname{Coker} A < \infty$.

Γ $\operatorname{Ker} A = \{x \,;\, Ax = 0\}$ $\quad \operatorname{Coker} A = E_2 / \operatorname{Im} A$.
$\operatorname{Im} A = \{Ax \,;\, x \in E_1\}$ $\quad$ regardless of topology

The index of A is

$$\operatorname{index} A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A.$$

$A \in \mathcal{L}(E_1, E_2)$ Fredh. $\Leftrightarrow$ $A \in \operatorname{Fred}(E_1, E_2)$.

Fact 1 If $A \in \operatorname{Fred}(E_1, E_2) \Rightarrow \overline{\operatorname{Im} A} = \operatorname{Im} A$.

Γ $A \rightsquigarrow A_1 : E_1/\operatorname{Ker} A \to E_2 \rightsquigarrow \hat{A} : E_1/\operatorname{Ker} A \oplus C \to E_2$
$\hat{A}(x, c) = A_1 x + c$

bij & cont. $\overset{\text{OMT}}{\Rightarrow}$ $\hat{A}$ isomorphism $\Rightarrow \hat{A}(\operatorname{Im} A) = E_1/\operatorname{Ker} A \oplus 0$ closed $\Rightarrow$ claim]

Fact 2 $\dim \operatorname{Coker} A < \infty \Rightarrow \dim \operatorname{Coker} A = \dim \operatorname{Ker} A^*$

Fact 3 Finite rank perturbations $\overset{\text{Lin. alg} \Rightarrow}{}$ $\xrightarrow{\text{easily}}$ cpct perturbations

$A \in \operatorname{Fred}(E_1, E_2)$, $R \in \mathcal{K}(E_1, E_2)$.

$\Rightarrow$ $A + R \in \operatorname{Fred}(E_1, E_2)$ & $\operatorname{index}(A+R) = \operatorname{index} A$

<u>Fact 4</u> Finite rank $\longrightarrow$ cpt operators

If $A \in \mathcal{L}(E_1, E_2)$ & $\exists\, B \in \mathcal{L}(E_2, E_1)$, $R_j \in K(E_j, E_j)$

s.t. $AB = I + R_2$, $BA = I + R_1$ $\Rightarrow$ $A \in \mathrm{Fred}(E_1, E_2)$.

<u>Fact 5</u> $\mathrm{Fred}(E_1, E_2) \overset{\circ}{\subset} \mathcal{L}(E_1, E_2)$

and index : $\mathrm{Fred}(E_1, E_2) \longrightarrow \mathbb{Z}$ is continuous

<u>Fact 6</u> $A \in \mathrm{Fred}(E_1, E_2)$, $B \in \mathrm{Fred}(E_2, E_3)$

$\Rightarrow BA \in \mathrm{Fred}(E_1, E_3)$ & $\text{index } BA = \text{index } A + \text{index } B$

<u>Note</u> $\mathrm{Fred}(E_1, E_2)$ is <u>not</u> an ideal. Ex.

$A = I \in \mathrm{Fred}(\ell^2(\mathbb{N}))$, $B(x_k) = (\frac{1}{k} x_k)$

$AB = BA = B \notin \mathrm{Fred}(\ell^2(\mathbb{N}))$ since $\overline{\mathrm{Im}\, B} \neq \mathrm{Im}\, B$

Now the connection with ΨDO's and our problem.

Proposition

Let M be a cpct mfold w/out a bdry.

Suppose $A \in Op\, S^m_{1,0}(M)$ is elliptic 400

[Petri introduced these, and we don't need the definition here] For every $s \in \mathbb{R}$,

we construct $A_s \in \mathcal{L}(H^s(M), H^{s-m}(M))$

by extending A by continuity.

Then.

(i) $A_s \in \mathrm{Fred}\,(H^s(M), H^{s-m}(M))$

(ii) $\mathrm{Ker}\, A_s \subset C^\infty(M)$ (and is indep. of $s \xrightarrow{\text{notation}} \mathrm{Ker}\, A$)

(iii) $\mathrm{index}\, A_s = \text{constant} =: \mathrm{index}\, A$

$= \dim \mathrm{Ker}\, A - \dim \mathrm{Ker}\, A^*$

(iv) if $R \in S^{m'}_{1,0}(M)$, $m' < m \Rightarrow \mathrm{index}\,(A+R) = \mathrm{index}\, A$.

__Pf__ (i) A_s has parametrix (Petri showed this), $A_s B_s = I + R_1$, $B_s A_s = I + R_2$,

where $R_j \in Op\, S^{m-1}_{1,0}(M)$.

Now $R_j : H^{\tilde{s}}(M) \longrightarrow H^{\tilde{s}-m+1}(M) \xhookrightarrow{\text{cpct}} H^{\tilde{s}-m}(M)$

$\therefore$ A_s Fredholm !

(ii) A elliptic $\Rightarrow$ sing supp Au = sing supp u

$u \in \operatorname{Ker} A \Rightarrow Au \in C^\infty(M)$.

(iii) $A^* \in \operatorname{Op} S^m_{1,0}(M) \overset{(i)\&(ii)}{\Rightarrow} A^*_s \in \operatorname{Fred}(H^s(M), H^{s-m}(M))$

& $\operatorname{Ker} A^*_s = \operatorname{Ker} A^* \subset C^\infty(M)$

$\Rightarrow$ index A_s = index A = dim Ker A − dim Ker A^*

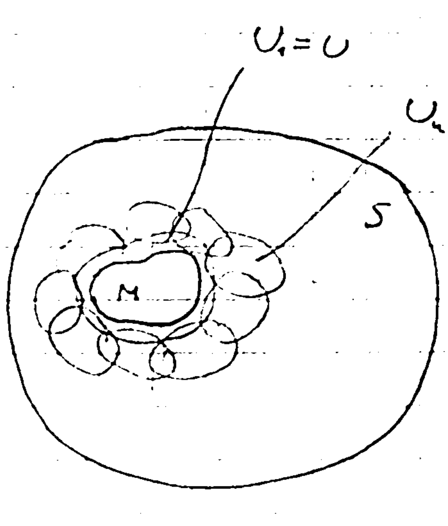
(iv) Since R is a cpct perturbation, this follows from fact 3.

Now we can (almost) show the proof of Theorem 3.1. We will show something less.

<u>Modified Theorem 3.1.</u> Let $M \subset U \subset\subset S$ and suppose U is simple. Provided that $\tilde{I}: L^2(U) \to L^2_\mu(\partial_+ \Omega(U))$ is injective, then the operator

$$r_M \tilde{I}^* \tilde{I}: H^s(U) \to H^{s+1}(M)$$

is surjective, for $s \geq 0$.



<u>Pf.</u> Step 1

We define a ΨDO P on the cpct mfold S that looks like $r_M I^* I$

We cover S with finite cover of open simple charts $\{(U_k, \kappa_k)\}$, so that $U_1 = U$

and $U_n \cap M = \phi$ for other n.

S cpct $\Rightarrow \exists C^\infty$-partition of unity $\{\varphi_n\}$ subordinated to $\{U_n\}$ (i.e. $\varphi_n \geq 0$, supp $\varphi_n \subset U_n$ and $\sum_n \varphi_n(x) = 1 \quad \forall x \in S$) $\overset{U_n \cap M = \phi \, \forall n \neq 1}{\Rightarrow} \varphi_1 |M = 1$

Now we define

$$Pf = \sum_n \varphi_n \, (I_n^* I_n)(f|U_n) \quad , \quad f \in \mathcal{D}'(S)$$

where I_n is the GXRT on the simple mfold $\bar{U}_n$ (and $I_n : L^2(\Omega(\bar{U}_n)) \longrightarrow L^2_\mu(\partial_+ \Omega(\bar{U}_n))$)

Petri showed $I_n^* I_n : C_0^\infty(U_n) \to C^\infty(U_n)$ is an elliptic ΨDO of order -1 and principal symbol $c_n |\xi|^{-1}$ $\overset{\text{glueing the pieces}}{\Longrightarrow}$ P is an elliptic ΨDO

c_n ← dimension

(see Prop. 5.5 and Theor. 6.1. in Mikhail Aleksandrovich Shubin's book "ΨDO's and spectral theory"

Step 2 Since $(S, \tilde{g})$ is a cpct RM w/out a bdry $\overset{\text{step 1}}{\Longrightarrow}$ $P_s \in \text{Fred}(H^s(S), H^{s+1}(S))$ for every $s \in \mathbb{R}$

$\Rightarrow$ 1) index(P_s) = dim Ker P − dim Ker P^*

2) $\overline{\text{Im}\, P_s} = \text{Im}\, P_s \quad \forall s$

Now $P^* \simeq P$: ($s \in \mathbb{R}$. $H^s(S)^* = H^{-s}(S)$,

$P_s : H^s(S) \longrightarrow H^{s+1}(S)$ $\Rightarrow$ P_s^* $H^{-s-1}(S) \to H^{-s}(S)$

$< f, P_s^* g >_{H^s \times H^{-s}} = < P_s f, g >_{H^{s+1} \times H^{-s-1}}$ $f \in H^s$, $g \in H^{-s-1}$

$= \sum_n < \varphi_n (I_n^* I_n)(f | U_n), g >$.

The restriction $f | U_n = \psi_n f$, where

$\{\psi_n\}$ is a family of C_c^∞-fncts subord. to $\{U_n\}$ s.t.

$\psi_n | \operatorname{supp} \varphi_n = 1$

$\Rightarrow < f, P_s^* g > = \sum_n < f, \psi_n I_n^* I_n \varphi_n g >_{H^{s+1} \times H^{-s-1}}$

By choosing $\psi_n \approx \varphi_n$ (i.e. the partition $\{\varphi_n\}$ must be

chosen carefully) we can show $\overset{\text{fist}}{P_{s+1}^*} - P_{-s-1}$ has a

as small op. norm as we wish

$P_{-s-1} \in$ Fred

$\Rightarrow$ $P_s^* \in$ Fred ($H^{-s-1}(S)$, $H^{-s}(S)$) and

index P_s^* = index P_{-s-1} = index P)

Since index $P^* = -$ index P $\Rightarrow$ $\boxed{\text{index } P = 0}$

<u>Step 3</u> The restriction operator ($s \geq 0$)

$r_M : H^s(S) \longrightarrow H^s(M)$ is cont and surjective

Since P_s is Fredholm, we can show that

$r_M P_s$ has closed range

[Sketch. Ass. $v_n = r_M P_s u_n \longrightarrow v_0 \in H^{s+1}(M)$

$w_k := \mathcal{E}_M v_k \in H^{s+1}(S) = \operatorname{Im} P_s \oplus L$ ← finite dim

$\Rightarrow \; w_n = P_s y_n + z_n \quad \Rightarrow r_M P_s u_n = v_n = r_M w_n = r_M P_s y_n + r_M z_n$

finite dim.

$\therefore \quad r_M z_n \in \mathrm{Im}(r_M|L) \cap \mathrm{Im}(r_M P_s) = F$

$\oplus$

$\mathcal{E}_M$ cont $\Rightarrow (w_n)$ bdd $\Rightarrow (P_s y_n), (z_n)$ bdd

L finite dim

$\Rightarrow \quad \tilde{z}_n \longrightarrow z \in L$ (for subseq) $\Rightarrow \quad P_s \tilde{y}_n \longrightarrow P_s y \in \mathrm{Im}\, P_s$

$\oplus$

r_M cont.

$\Rightarrow \quad z = z_0 \;\&\; P_s y = P_s y_0 \quad \Rightarrow \quad r_M \tilde{z}_n \longrightarrow r_M z_0$

$\Rightarrow \quad r_M z_0 \in \overline{F} = F \subset \mathrm{Im}(r_M P_s)$

$\Rightarrow \quad v_0 = r_M P_s y_0 + r_M z_0 \in \mathrm{Im}(r_M P_s) \quad \Rightarrow$ claim $\quad \perp$

Since

$(*) \quad r_M P f = r_M(\varphi_1 I_1^* I_1 (f|U_1)) = r_M \tilde{I}^* \tilde{I} (f|U_k)$

$\Rightarrow \quad \mathrm{Im}(r_M \tilde{I}^* \tilde{I}) = \overline{\mathrm{Im}(r_M \tilde{I}^* \tilde{I})}$.

Step 4

Solvability of $r_M \tilde{I}^* \tilde{I} f = h \in H^{s+1}(M)$

reduces to injectivity of $(r_M \tilde{I}^* \tilde{I})$:

step 3)

$\mathrm{Ker}\left((r_M \tilde{I}^* \tilde{I})^*\right) = \{0\} \Rightarrow H^{s+1}(M) = \overline{\mathrm{Im}(r_M \tilde{I}^* \tilde{I})} = \mathrm{Im}(r_M \tilde{I}^* \tilde{I})$.

Step 5. The action of $(r_M \tilde{I}^* \tilde{I})^*$

Let $f \in H^s(U)$, $u \in (H^{s+1}(M))^*$ and $\tilde{f} \in r_U^{-1}\{f\}$.

$\langle f, (r_M \tilde{I}^* \tilde{I})^* u \rangle_{H^s \times (H^s)^*} = \langle r_M \tilde{I}^* \tilde{I} f, u \rangle_{H^{s+1} \times (H^{s+1})^*}$

$(*)$

$= \langle P_s \tilde{f}, u \rangle = \langle \tilde{f}, P_s^* u \rangle = \langle \chi_\delta \mathcal{E}_U f, P_s^* \rangle$

supp $u \subset M \longrightarrow$ identification with zero extension

$= \langle f, \mathcal{E}_U^* \chi_\delta P_s^* u \rangle$

Here $\chi_\delta \in C_0^\infty(S)$ s.t. $\chi_\delta|U = 1$ and supp $\chi_\delta \subset U + B_\delta$

$\delta \searrow 0 \Rightarrow \quad \langle f, (r_M \tilde{I}^* \tilde{I})^* u \rangle = \langle f, r_U P_s^* u \rangle$

$\therefore \quad (r_M \tilde{I}^* \tilde{I})^* = r_U P_s^*$

<u>Step 6</u> If $u \in \ker \left((r_M \tilde{I}^* \tilde{I})^* \right) \cap \left(H^{1+s}(M) \right)^*$

elliptic

Step 5 $\Rightarrow \quad r_U P_s^* u = 0 \quad \Rightarrow \quad u|U \in C^\infty$

$u \in \left(H^{1+s}(M) \right)^* \Rightarrow \operatorname{supp} u \subset M. \Rightarrow u \in C_0^\infty(U)$

$\Rightarrow \quad 0 = (r_M \tilde{I}^* \tilde{I})^* u = \tilde{I}^* \tilde{I} r_M^* u = \tilde{I}^* \tilde{I} u$

assumed

$\Rightarrow \quad \|\tilde{I} u\|^2_{L^2_\mu(\partial_+ \Omega(\bar{U}))} = 0 \Rightarrow \tilde{I} u = 0 \Rightarrow u = 0.$

to be known