Inverse problems seminar
P. Ola 12.03.07

Ellipticity of the normal operator

Where are we? (And am I also there?)

Geodesic X-ray transform:

$$If(x,\xi) = \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi))\,dt, \quad f \in C^\infty(M),\ (x,\xi) \in \partial_+\Omega(M)$$

Its adjoint:

$$I^*f(x) = \int_{\Omega_x} f_\psi(x,\xi)\,d\Omega_+ \quad (x \in M)$$

$$d\Omega_+ = \sqrt{|g|}\sum_k (-1)^{k+1}\xi^k\, d\xi^1\wedge\dots\wedge\widehat{d\xi^k}\wedge\dots\wedge d\xi^n = \sqrt{|g|}\,dS_x$$

$dS_x \to$ Euclidean vol. form of Ω_x in T_xM

where

$f_\psi = f\circ\alpha\circ\psi$, ψ = "end point map"
α = scattering relation $\partial\Omega_\pm(M) \to \partial\Omega_\mp(M)$

Mikko has explained that $I^*I : L^2(M) \to L^2(M)$, and today I aim to explain why I^*I is an elliptic operator (of order -1) on M so in fact

$$I^*I : L^2_{\cancel{comp}}(M^\circ) \to H^1_{loc}(M)$$

and it also has a parametrics $R : H^1_{comp}(M^\circ) \to L^2(M)$ i.e.

$$(I^*I)R = I + K, \quad K : H^1_{comp} \to H^2_{loc}(M)$$

& also a left "inverse" in suitable sense.

We'll need ΨDO's, so let's recall their basic properties:

Fourier-transf. $\hat f(\xi) = (2\pi)^{-n}\int e^{-i\langle x,\xi\rangle} f(x)\,dx$

$$u(x) = \int e^{i\langle x,\xi\rangle}\hat u(\xi)\,d\xi$$

Then

$$\frac{\partial u(x)}{\partial x_j} = \int e^{i\langle x,\xi\rangle}(i\xi_j)\hat u(\xi)\,d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n) \text{ or } C_0^\infty(\mathbb{R}^n)$$

$$\Rightarrow \quad \partial^\alpha u(x) = \int e^{i\langle x,\xi\rangle}(i\xi)^\alpha\hat u(\xi)\,d\xi,$$

and if $P = p(x,\partial) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha$ then

$$p(x,\partial)u(x) = \int e^{i\langle x,\xi\rangle}\underbrace{p(x,i\xi)}_{\text{symbol of } P\text{; a polynomial in } \xi}\hat u(\xi)\,d\xi$$

This can be generalized as follows:

Def. 1 a) $m \in \mathbb{R}$; a function $a \in C^\infty(\Omega\times\mathbb{R}^n)$ is a symbol of order m if $(D = \frac{1}{i}\partial)$

$$x \in K \subset\subset \Omega \quad |D_x^\beta D_\xi^\alpha a(x,\xi)| \le C^K_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

Class of all symbols of order m is $S^m_{1,0}(\Omega)$

b) to a symbol $a \in S^m_{1,0}$ we adjoin an operator ("quantization rule") by (a ΨDO of order m)

$$a(x,D)u(x) = \int e^{i\langle x,\xi\rangle}a(x,\xi)\hat u(\xi)\,d\xi$$

Class of ops with symbols in S^m is $\mathrm{Op}S^m(\Omega)$

<u>Prop.</u> $A = a(x,D) \in \mathrm{Op}S^m_{1,0} \Rightarrow A : H^s_{comp} \to H^{s-m}_{loc}$

Can one define these ops on manifolds: this can be answered as follows: let $\Omega_1, \Omega_2 \subset\circ \mathbb{R}^n$, $\varphi: \Omega_1 \to \Omega_2$ a diffeo; assume $A = a(x,D) \in Op\, S^m_{1,0}(\Omega_1)$ and let $\tilde{A}u = A(u\circ\chi)\circ\chi^{-1} : C_0^\infty(\Omega_2) \to C^\infty(\Omega_2)$

Then we have

<u>Prop.</u> $\tilde{A} \in Op\, S^m_{1,0}(\Omega_2)$ with symbol $\tilde{a}(y,\xi)$ s.t.

$$\tilde{a}(\varphi(x),\xi) = a(x, {}^t\varphi'(x)\xi) + \text{symbol in } S^{m-1}(\Omega_2)$$

transforms as a cotangent vector!

From this we can draw the following conclusions:

a) One can define a ΨDO of class $Op\, S^m_{1,0}(M)$ on a smooth manifold without boundary by saying that it is an op. in $Op\, S^m_{1,0}$.

b) The symbol is <u>not</u> preserved

c) However the leading order part transforms as an invariantly defined function on $T^*(M)$; this is the <u>principal</u> <u>symbol</u> of $a(x,D) \in Op\, S^m_{1,0}(M)$.

Let's again look at things locally:

Let's compute formally:

$$\langle p(x,D)u, v\rangle = \int v(x)\, p(x,D)u(x)\,dx = \iint e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)v(x)\,dx\ldots$$

$$= (2\pi)^{-n}\iiint e^{i\langle x-y,\xi\rangle} p(x,\xi)\, u(y)\, v(x)\,dx\,dy\,d\xi = \langle K(x,y), v\otimes u\rangle,$$

where

$$K(x,y) = (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle} p(x,\xi)\,d\xi \in \mathcal{D}'(\Omega\times\Omega);$$

one can also prove that

$$(x-y)^\alpha K(x,y) = (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}\underbrace{(D_\xi^\alpha p)(x,\xi)}_{\le C^k(1+|\xi|)^{m-|\alpha|}}d\xi$$

and this is C^k when $m-|\alpha| < -n-j$

i.e. $\operatorname{sing\,supp} K \subset \Delta = \{(x,y)\in\Omega;\ x=y\}$.

i.e. the kernel of a ΨDO is singular only on the diagonal; also knowing the kernel we can compute the symbol:

$$p(x,\xi) = \int K(x,x+y)\, e^{i\langle y,\xi\rangle} d\xi$$

——— End of the mini-course on ΨDO's; well almost!

<u>Remark</u> on geodesics: If $\gamma(x,\xi,\cdot)$ is an geodesic, then actually

$$\gamma(x,\xi,t) = v(x,\xi t)$$

for some $v \in C^\infty T M$; infact ~~let $v(x,y) = \gamma(x, \frac{y}{|y|}, |y|)$ and then it is easy to see geodesic eq's are satisfied~~)

~~two~~ | Geodesic eqn is homogeneous in ξ, t i.e.

If $\gamma(x,\xi,\cdot)$ solves $\dot\gamma(x,\xi,0)=\xi$, ~~then~~ then

$$\gamma_a(x,\xi,t)=\gamma(x,a\xi,t/a) \text{ solves the same eqn! } (a>0)$$

Hence we also denote $\gamma(x,\xi,t)=\gamma(x,\xi t)$.

<u>Remark</u> on simple nbhds: remember (S,g) C^∞-mfld, cpt & without bnd s.t. $(M,g)\hookrightarrow(S,g)$. Then $\exists$ open $U\subset S$, $M\subset U$ s.t. $\overline{U}$ is also simple (no conj. pts, simply connected & $\partial\overline{U}$ strictly convex). Hence, $\forall x\in U$ $\exists$ $D_x^U \Subset T_x(U)$ s.t.

$$\exp_x : D_x^U \to U, \quad (x,\xi)\mapsto\gamma(x,\xi)$$

is diffeo onto U; Let

$$D_x = \exp_x^{-1}(M);$$

then $\exp_x^{-1}: M\to D_x \subset T_xU \cong \mathbb{R}^n$ is diffeo.

Simple $\Rightarrow$ any two pts. can be joined by a unique geodesic.

Now we have

<u>Prop.</u> $V\Subset M$ open; then $I^*I : C_0^\infty(V)\to C^\infty(V)$ is an <u>elliptic</u> ΨDO of order -1 with principal symbol $c_n|\xi|^{-1}$.

Before pf let's recall what elliptic means:

$a(x,D)\in Op\,S^m_{1,0}(\Omega)$ is <u>elliptic</u> if for some R

$$|a(x,\xi)|\geq C_K(1+|\xi|)^m, \quad x\in K\subset\subset\Omega, \quad |\xi|\geq R$$

$\Rightarrow$ doesn't depend on lower order terms!

This depends only on the principal symbol & is invariant

<u>Why important</u>? Well, if $\begin{cases} a(x,D)\in Op\,S^{m_1}_{1,0}(\Omega) \\ b(x,D)\in Op\,S^{m_2}_{1,0}(\Omega)\end{cases}$

and say ~~x,b~~ $b(x,\xi)$ comp. supported in x (less is sufficient) then

$$c(x,D)=b(x,D)a(x,D)\in Op\,S^{m_1+m_2}_{1,0}(\Omega)$$

and

$$c(x,\xi)=a(x,\xi)b(x,\xi)+\text{terms in } S^{m_1+m_2-1}_{1,0}(\Omega)$$

Hence, if $a(x,\xi)$ is elliptic, let $b(x,\xi)\in S^{-m}_{1,0}(\Omega)$ be s.t.

$$b(x,\xi)=\frac{1}{a(x,\xi)}, \quad |\xi|\geq R$$

Then

$$a(x,\xi)b(x,\xi)=1+r(x,\xi), \quad r\in S^{-1}_{1,0}(\Omega)$$

i.e.

$$a(x,D)b(x,D)=I+r(x,D)$$

lower order

$\rightsquigarrow$ $\exists$ parametrices!

<u>Pf. of prop.</u> $f \in C_0^\infty(V)$

Recall
$d\Omega_+ = \sqrt{|g|}\, dS_x$

$$I^* I f(x) = \int_{\Omega_x} d\Omega_+ (If)_\gamma (x,\xi)$$

$$= \int_{\Omega_x} d\Omega_* \int_{-\tau(x,-\xi)}^{\tau(x,\xi)} f(\gamma(x,\xi,t))\,dt = 2 \int_{\Omega_+} d\Omega_* \int_0^{\tilde\tau(x,\xi)} f(\gamma(x,\xi,t))\,dt$$

Let $y = \gamma(x,\xi t)$ $\quad$ ($\overset{\circ}{\sim}$ (ξ,t) polar coordinates cent. at x $\overset{\circ}{}$)

$\Updownarrow$ $\quad t = d_g(x,y)$

Jacobian left-over from $\int d\Omega_+$

Then

$$(I^* I) f(x) = \int_M K(x,y) f(y)\,dy, \quad K(x,y) = 2\, \frac{|(\exp_x^{-1})'(y)|\,\sqrt{|g|}}{t^{n-1} = d_g^{n-1}(x,y)}$$

$$(\,dy = t^{n-1}\,dt\,dS_x\,)$$

Now

$$\gamma(x,\eta) = x + y + O(|\eta|^2)$$

and

$$|\exp_x'(0)| = 1 \qquad \exp_x(0) = x$$

$$(\text{ i.e. } \exp_x^{-1}(x) = 0\ )$$

$$\Rightarrow |(\exp_x^{-1})'(x)| = \frac{1}{|\exp_x'(0)|} = 1$$

$$\cancel{O(|\eta|^3)\ |y|^2_{g(x)} = d^2(x,y)}$$
$$\cancel{= g_{ij}(x)\, y^i y^j + O(|\eta|^3) = G_{ij}(x,y)(x-y)^i(x-y)^j}$$

Taylor-expansion of $y \mapsto d_g^2(x,y)$ $\overset{\text{at } x}{\checkmark}$ gives

$$d_g^2(x,y) = g_{ij}(x)(x-y)^i(x-y)^j + r_{ij}(x,y)(x-y)^i(x-y)^j$$
$$= G_{ij}(x,y)(x-y)^i(x-y)^j$$

$$f(x+y) = \sum_{|\alpha|<k} \partial^\alpha f(x)\, y^\alpha/\alpha!$$
$$+ k \int_0^1 (1-t)^{k-1} \sum_{|\alpha|=k} \partial^\alpha f(x+ty)\, x\, y^\alpha/\alpha!\, dt$$

Then

$$K(x,y) = \underbrace{\frac{2|g(x)|^{1/2}}{(g_{ij}(x)(x-y)^i(x-y)^j)^{\frac{n-1}{2}}}}_{= K_0(x,y)} + \underbrace{R(x,y)}_{\text{singular only on } \Delta};$$

$$|R(x,y)| \le \frac{C}{|x-y|^{-2+n}}$$

$\Rightarrow$ $I^* I$ is a ΨDO with princ. symbol

$\Rightarrow$ smoothing.

$$(2\pi)^{-n} \int K_0(x,x+y)\, e^{i\langle y,\xi\rangle}\, dy$$

actually $= C_n |({}^t g^{ij}(x))\xi|^{1/k - 1}$

$$= (2\pi)^{-n} \int \frac{2|g(x)|^{1/2}}{(g_{ij}(x)\, y^i y^j)^{\frac{n-1}{2}}}\, e^{i\langle y,\xi\rangle}\, dy \;\Big(= \underline{C_n |\xi|^{-1}}. \quad \square$$