Inverse Problems Seminar

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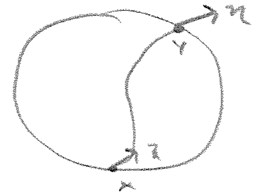
## Scattering relation and Dirichlet-to-Neumann map

Review from last time:

$(M,g)$ nontrapping cpt, $\partial M$ strictly convex

$\varphi_t : (x,\xi) \mapsto (\gamma(t,x,\xi), \dot\gamma(t,x,\xi))$ geodesic flow on $\Omega(M)$

Scattering relation $\alpha : \partial\Omega(M) \to \partial\Omega(M)$,




$$\alpha : (x,\xi) \mapsto \varphi_{\tau^0(x,\xi)}(x,\xi)$$

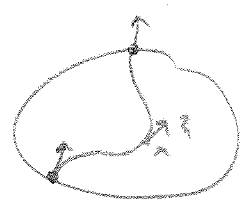
where $\tau^0$ regularised travel time.

$H$ geodesic vector field, $Hu(x,\xi) = \frac{d}{dt} u(\varphi_t(x,\xi))\big|_{t=0}$

Boundary value problem

$$Hu = 0 \quad \text{in } \Omega(M), \quad u|_{\partial_+\Omega(M)} = w$$

$$\Rightarrow \quad u = w_\psi = w \circ \alpha \circ \psi$$




where $\psi$ is end point map.

Geodesic X-ray transform

$$If(x,\xi) = \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi))\, dt, \qquad (x,\xi) \in \partial_+\Omega(M)$$

$$IHf = (f \circ \alpha - f)\big|_{\partial_+\Omega(M)} \tag{1.1}$$

$$I : L^2(M) \longrightarrow L^2_\mu(\partial_+\Omega(M))$$

$$(u,v)_{L^2_\mu(\partial_+\Omega(M))} = \int_{\partial_+\Omega(M)} u v \mu \, d\Sigma^{2n-2}, \qquad \mu = (\xi, \nu)$$

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The adjoint $I^*$ is bounded $L^2_\mu(\partial_+\Omega(M)) \to L^2(M)$, and

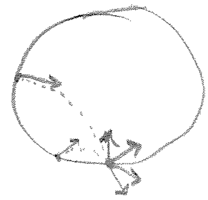
$$(If, w)_{L^2_\mu(\partial_+\Omega(M))} = \int_{\partial_+\Omega(M)} \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi)) \underbrace{w(x,\xi)}_{= w_\psi(\varphi_t(x,\xi))} \mu \, dt \, d\Sigma^{2n-2}$$

$$= \int_{\Omega(M)} f w_\psi \, d\Sigma^{2n-1}$$

$$= \int_M f(x) \Big( \int_{\Omega_x} w_\psi(x,\xi)\, d\Omega_x(\xi) \Big) dV^n(x)$$

so $I^* w(x) = \int_{\Omega_x} w_\psi(x,\xi)\, d\Omega_x$.

Introduce operators of even and odd continuation w.r.t. $\alpha$:



$$A_\pm w(x,\xi) = \begin{cases} w(x,\xi), & (x,\xi) \in \partial_+ \Omega(M) \\ \pm(\alpha^* w)(x,\xi), & (x,\xi) \in \partial_- \Omega(M) \end{cases}$$

Define $L^2_{|\mu|}(\partial\Omega(M))$ with inner product

$$(u,v)_{L^2_{|\mu|}(\partial\Omega(M))} = \int_{\partial\Omega(M)} uv|\mu|\, d\Sigma^{2n-2}, \qquad \mu = (\xi, \nu).$$

Lemma $A_\pm$ bounded $L^2_\mu(\partial_+\Omega(M)) \to L^2_{|\mu|}(\partial\Omega(M))$.

Pf

$$\|A_\pm w\|^2_{L^2_{|\mu|}(\partial\Omega(M))} = \int_{\partial_+\Omega(M)} w^2 \mu\, d\Sigma + \int_{\partial_-\Omega(M)} w(\alpha(x,\xi))^2 (-\mu\, d\Sigma)$$

$$= \int_{\partial_+\Omega(M)} w^2 \mu\, d\Sigma + \int_{\partial_+\Omega(M)} w^2\, \alpha^*(-\mu\, d\Sigma)$$

where $\alpha: \partial_+\Omega(M) \to \partial_-\Omega(M)$ diffeo. Enough to show that

$\alpha^*(-\mu\, d\Sigma) = \mu\, d\Sigma$.

Let $w \in C^\infty(\partial_+\Omega(M))$. Then

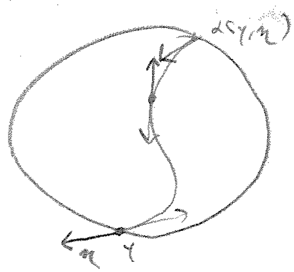
$$\int_{\partial_+\Omega(M)} w\tau\,\mu\, d\Sigma = \int_{\partial_+\Omega(M)} \int_0^{\tau(x,\xi)} w_\psi(\varphi_t(x,\xi))\,\mu\, dt\, d\Sigma = \int_{\Omega(M)} w_\psi\, d\Sigma.$$

If $\tilde u(x,\xi) = u(x,-\xi)$ for $u \in C^0(\Omega(M))$, one has

$$\int_{\Omega(M)} w_\psi\, d\Sigma = \int_{\Omega(M)} \tilde w_\psi\, d\Sigma = \int_{\partial_-\Omega(M)} \int_0^{\tau(y,-\eta)} \tilde w_\psi(\varphi_t(y,-\eta))\,(-\mu)\, dt\, d\Sigma$$

$$= \int_{\partial_-\Omega(M)} \int_0^{\tau(y,-\eta)} w(\alpha(y,\eta))\,(-\mu)\, dt\, d\Sigma$$

$$= \int_{\partial_+\Omega(M)} w\tau\, \alpha^*(-\mu\, d\Sigma).$$




Varying $w$ shows that $\mu\, d\Sigma = \alpha^*(-\mu\, d\Sigma)$ on $\partial_+\Omega(M) \setminus \partial_0\Omega(M)$. □

The adjoint $A_\pm^*: L^2_{|\mu|}(\partial\Omega(M)) \to L^2_\mu(\partial_+\Omega(M))$ is given by

$$(A_\pm w, u)_{L^2_{|\mu|}(\partial\Omega(M))} = \int_{\partial_+\Omega(M)} wu\mu\, d\Sigma \pm \int_{\partial_-\Omega(M)} (w\circ\alpha)u\,(-\mu\, d\Sigma)$$

$$= \int_{\partial_+\Omega(M)} w(u \pm u\circ\alpha)\,\mu\, d\Sigma$$

so $A_\pm^* u = (u \pm u\circ\alpha)|_{\partial_+\Omega(M)}$. Then (1.1) may be written as

$$I\mathcal{H}f = -A_-^* f^0, \qquad f^0 = f|_{\partial\Omega(M)}. \tag{1.2}$$

Define

$$C^\infty_\alpha(\partial_+\Omega(M)) = \{w \in C^\infty(\partial_+\Omega(M))\ ;\ w_\psi \in C^\infty(\Omega(M))\}$$

Then $I^* w \in C^\infty(M)$ whenever $w \in C^\infty_\alpha(\partial_+\Omega(M))$.

<u>Lemma 1.1</u> $C^\infty_\alpha(\partial_+\Omega(M)) = \{w \in C^\infty(\partial_+\Omega(M))\ ;\ A_+ w \in C^\infty(\partial\Omega(M))\}$.

The following is the main result for solvability for $I^*$.

<u>Thm 1.4</u> Let $(M,g)$ be a cpt simple 2D Riemannian mfld with boundary. Then $I^* : C^\infty_\alpha(\partial_+\Omega(M)) \to C^\infty(M)$ is onto.

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Move on to conjugate harmonic functions. If $u$ is a real-valued harmonic function in $\mathbb{D}$, then $v$ is conjugate harmonic to $u$ if

$$\bar\partial(u+iv) = 0$$

$$\Leftrightarrow\ \bar\partial u = -i\bar\partial v, \quad \bar\partial v = i\bar\partial u.$$

Writing $x_1 + ix_2 \leftrightarrow \binom{x_1}{x_2}$, get $\bar\partial u \leftrightarrow \nabla u = \binom{\partial_1 u}{\partial_2 u}$, $-i\bar\partial v \leftrightarrow \nabla_\perp v = \binom{+\partial_2 v}{-\partial_1 v}$.

Then $u$ and $v$ are conjugate harmonic iff

$$\nabla u = +\nabla_\perp v, \quad \nabla v = -\nabla_\perp u.$$

This is an invariant formulation, and can be used to define conjugate harmonic functions in $(T_x M, g(x)) \simeq (\mathbb{R}^2, e)$ by the following lemma (then $\nabla_\perp = \varepsilon\nabla$).

<u>Lemma</u> Let $M$ 2D oriented mfld. Then $\exists!$ 2-tensor field $\varepsilon$ ("multiplication by $-i$") such that $\{v, -\varepsilon v\}$ is a positive ON basis of $T_x M$ whenever $v \in T_x M$, $|v| = 1$.

One has $(\varepsilon v, \varepsilon w) = (v, w)$ and $(\varepsilon v, w) = -(v, \varepsilon w)$.

The Hilbert transform on $\partial\mathbb{D}$ is

$$Hf(z) = \mathrm{p.v.}\int_{\partial\mathbb{D}} \frac{1+\mathrm{Re}(z\bar w)}{-\mathrm{Re}(iz\bar w)} f(w)\, dm(w), \qquad dm(e^{i\theta}) = \frac{1}{2\pi}d\theta.$$

If $x_1 + ix_2 \leftrightarrow \binom{x_1}{x_2}$, then $z\cdot w = \mathrm{Re}(z\bar w)$ and $-iz = \varepsilon z = z_\perp$, so

$$Hf(z) = \mathrm{p.v.}\int_{\partial\mathbb{D}} \frac{1 + z\cdot w}{+z_\perp\cdot w} f(w)\, dm(w).$$

Let now $u \in C^\infty(\Omega(M))$. The Hilbert transform is defined by

$$Hu(x,\xi) = \mathrm{p.v.}\ \frac{1}{2\pi}\int_{\Omega_x} \frac{1+(\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta)\, d\Omega_x(\eta), \qquad \xi \in \Omega_x. \qquad (1.3)$$

$H$ maps even (resp. odd) functions w.r.t. $\xi$ to even (resp. odd) ones. If $H_+$ ($H_-$) is the even (odd) part of $H$, so

$$H_+ u(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{(\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta)\, d\Omega_x(\eta),$$

$$H_- u(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{1}{(\xi_\perp,\eta)} u(x,\eta)\, d\Omega_x(\eta),$$

and $u_+$ ($u_-$) is the even (odd) part of $u$, then $H_+ u = H u_+$, $H_- u = H u_-$

Let $\nabla$ be covariant derivative w.r.t. $g$ and $\nabla_\perp = \varepsilon\nabla$. Write $\mathcal{H}_\perp = (\xi_\perp, \nabla) = -(\xi, \nabla_\perp)$. The following commutator identity connects $H$ and $\mathcal{H}$.

<u>Thm 1.5</u> Let $(M,g)$ 2D Riemannian mfld. For any $u \in C^\infty(\Omega(M))$

$$[H, \mathcal{H}]u = \mathcal{H}_\perp u_0 + (\mathcal{H}_\perp u)_0 \tag{1.4}$$

where $u_0(x) = \frac{1}{2\pi}\int_{\Omega_x} u(x,\xi)\, d\Omega_x$ is the average value.

We now move to show that the scattering relation determines the DN map on 2D simple mflds. Let $(h, h_*)$ be a pair of conjugate harmonic functions on $M$,

$$\nabla h = \nabla_\perp h_*, \quad \nabla h_* = -\nabla_\perp h.$$

The DN map is

$$h_*^0 \mapsto (\nabla h_*, \nu) = (-\nabla_\perp h, \nu) = -(\nabla h, \nu_\perp)|_{\partial M}$$

so enough to determine the map $h_*|_{\partial M} \mapsto h|_{\partial M}$ from scattering relation.

Separating the odd and even parts in (1.4) we get

$$H_+ \mathcal{H} u - \mathcal{H} H_- u = (\mathcal{H}_\perp u)_0, \quad H_- \mathcal{H} u - \mathcal{H} H_+ u = \mathcal{H}_\perp u_0.$$

Take $u = w_\psi$, $w \in C^\infty_\alpha(\partial_+\Omega(M))$. Then

$$2\pi\, \mathcal{H} H_+ w_\psi = -\mathcal{H}_\perp I^* w$$

and using (1.2) we conclude

$$2\pi A_-^* H_+ A_+ w = I \mathcal{H}_\perp I^* w \tag{1.5}$$

since $w_\psi|_{\partial\Omega(M)} = A_+ w$.

choose now $w$ so that $I^* w = h$. Then $I \mathcal{H}_\perp h = I \mathcal{H} h_* = -A_-^* h_*^0$, (5)
we obtain from (1.5)

$$2\pi A_-^* H_+ A_+ w = -A_-^* h_*^0. \quad (1.6)$$

Thm 1.6 Let $M$ 2D simple mfld. Let $w \in C_\alpha^\infty(\partial_+ S(M))$ and let $h_*$ be the harmonic extension of $h_*^0$. Then (1.6) holds iff $h = I^* w$ and $h_*$ are conjugate harmonic functions.
Pf One direction was done above. Assume (1.6) holds and let $h = I^* w$. By (1.2) and (1.5)

$$I \mathcal{H}_\perp h = I \mathcal{H} q$$

where $q \in C^\infty(M)$, $q|_{\partial M} = h_*^0$. Thus ray transform of $\nabla q + \nabla_\perp h$ vanishes, and it is known (Anikonov 1978) that $\nabla q + \nabla_\perp h = \nabla p$ where $p|_{\partial M} = 0$. Then $h$ and $h_* = q - p$ are conjugate harmonic functions and $h_*|_{\partial M} = h_*^0$. □

In summary, one can determine DN map from scattering relation as follows. For given $h_*^0 \in C^\infty(\partial M)$, let $w$ be the solution of (1.6) and let $h = I^* w$. That is,

$$h = I^*(2\pi A_-^* H_+ A_+)^{-1}(-A_-^* h_*^0).$$

Then $h^0 = 2\pi (A_+ w)^0$ and $h_*^0$ are traces of conjugate harmonic functions.

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## APPENDIX: Conjugate harmonic functions

Let $(M, g)$ be a 2D oriented simply connected mfld. If $\Delta_g u = 0$ in $M$, we say that $v \in C^\infty(M)$ is a harmonic conjugate of $u$ if

$$dv = * du$$

where $*$ is Hodge star on 1-forms. The connection to $\nabla_\perp$ is

$$(\nabla_\perp u(p), \xi) = (* du)_p(\xi) \qquad \forall\, p \in M,\ \xi \in T_p M.$$

Given $u$ harmonic, we want to construct $v$. But the 1-form $\omega = * du$ satisfies $d\omega = d * du = * \Delta_g u = 0$, so there is $v$ satisfying $dv = \omega$ by simply connectedness. This gives the conjugate harmonic function $v$, which is uniquely defined up to additive constant.