Inverse Problems Seminar

Mikko Salo, 19.2.2007 TKK U358

Scattering relation and geodesic X-ray transform

We start going through the Pestov-Uhlmann paper. The aim is to prove the following result.

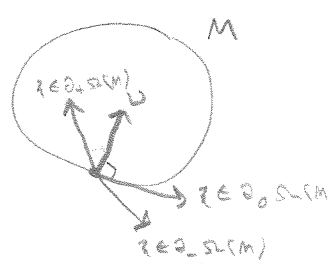
Thm Let (M, g_i), $i=1,2$, be compact simple 2D mflds with boundary. Assume that $\alpha_{g_1} = \alpha_{g_2}$. Then $\Lambda_{g_1} = \Lambda_{g_2}$.

Here Λ_g is the Dirichlet-to-Neumann map, and α_g is the scattering relation. We proceed to define α_g rigorously.

Let (M, g) be a compact n-dimensional mfld with boundary. Assume that (M,g) is embedded in (S, g) where S is compact n-dim. boundaryless mfld.

Let ρ be a defining function for $\partial M \hookrightarrow S$. That is, $\rho : S \to \mathbb{R}$ is smooth, $M = \{\rho > 0\}$, $\partial M = \{\rho = 0\}$, and $|\text{grad}\, \rho| = 1$ near ∂M. (Such ρ exists: near any $p \in \partial M$ let (x', x_n) be semigeodesic coordinates adapted to ∂M. Then locally $\rho(x', x_n) = x_n$.)

Let ν be the unit inner normal to ∂M. One has $\nu = \text{grad}\, \rho$. Define unit sphere bundle



$$\Omega(M) = \bigcup_{x \in M} \Omega_x, \qquad \Omega_x = \{\xi \in T_x M \,;\, |\xi|_g = 1\}.$$

This is $(2n-1)$-dim. compact mfld with boundary $\partial\Omega(M)$. We consider the inner vectors $\partial_+\Omega(M)$ and outer vectors $\partial\Omega_-(M)$,

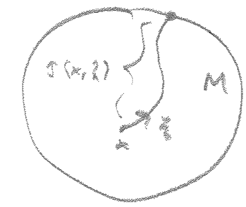
$$\partial_\pm \Omega(M) = \{(x,\xi) \in \partial\Omega(M) \,;\, \pm(\nu(x), \xi) \geq 0\}.$$

The manifolds of inner and outer vectors intersect at the set of tangent vectors $\partial_0\Omega(M)$,

$$\partial_0\Omega(M) = \{(x,\xi) \in \partial\Omega(M) \,;\, (\nu(x), \xi) = 0\}.$$

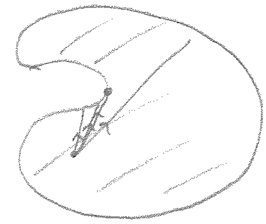
If $(x,\xi) \in \Omega(M)$, let $\gamma(t,x,\xi)$ be the unit speed S-geodesic starting from x in direction ξ. Define travel time $\tau : \Omega(M) \to [0,\infty]$,

$$\tau(x,\xi) = \inf\{t > 0 \,;\, \gamma(t,x,\xi) \in S \setminus M\}.$$



Def. (M,g) is nontrapping if $\tau(x,\xi) < \infty$ for all $(x,\xi) \in \Omega(M)$.

Without convexity conditions on M, τ may not be continuous.



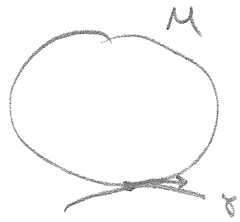
Def. ∂M is strictly convex if

$$\frac{d^2}{dt^2}\left(\rho(\gamma(t))\right)\Big|_{t=0} < 0$$

for any S-geodesic γ with $\dot\gamma(0)$ tangent to ∂M.

For such γ, $\frac{d}{dt}(\rho(\gamma(t)))\big|_{t=0} = d\rho_{\gamma(t)}(\dot\gamma(t))\big|_{t=0} = (\mathrm{grad}\,\rho(\gamma(0)), \dot\gamma(0))$
$= (\nu(\gamma(0)), \dot\gamma(0)) = 0.$

Thus, if ∂M is strictly convex and $\dot\gamma(0)$ tangent to ∂M, then $\rho(\gamma(t)) < 0$ for $t \in (-\varepsilon,\varepsilon) \setminus \{0\}$, i.e. $\gamma(t)$ is in $S \setminus M$ for $t \in (-\varepsilon,\varepsilon)\setminus\{0\}$.



Lemma Let (M,g) nontrapping with ∂M strictly convex. Then $\tau : \Omega(M) \to \mathbb{R}$ is continuous.

Pf Let $(x_0,\xi_0) \in \Omega(M)$ and $t_0 = \tau(x_0,\xi_0) > 0$.

Given ε_0, $\inf_{t\in[0,t_0-\varepsilon_0]} \rho(\gamma(t,x_0,\xi_0)) > 0$

$\Rightarrow \inf_{t\in[0,t_0-\varepsilon_0]} \rho(\gamma(t,x,\xi)) > 0$ for (x,ξ) near (x_0,ξ_0)

$\Rightarrow \gamma(t,x,\xi) \in M$ for $0 \le t \le t_0-\varepsilon_0$ and (x,ξ) near (x_0,ξ_0)

$\Rightarrow \tau(x,\xi) \ge t_0 - \varepsilon_0$ for (x,ξ) near (x_0,ξ_0).

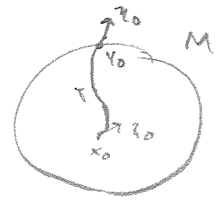
Further, since ∂M strictly convex, one has $\rho(\gamma(t_0+\varepsilon_0, x_0,\xi_0)) < 0$ for $\varepsilon_0 > 0$ small $\Rightarrow \rho(\gamma(t_0+\varepsilon_0,x,\xi)) < 0$ for (x,ξ) near (x_0,ξ_0) $\Rightarrow \tau(x,\xi) \le t_0+\varepsilon_0$ for (x,ξ) near (x_0,ξ_0). This shows the case $t_0 > 0$, and $t_0 = 0$ is similar. $\square$

<u>Lemma</u> Let (M,g) nontrapping and ∂M strictly convex. Then τ is smooth on $\Omega(M) \setminus \partial_0 \Omega(M)$.

<u>Pf</u> Let

$$h(t,x,\xi) = \rho(\gamma(t,x,\xi)), \quad t \in \mathbb{R},\ (x,\xi) \in \Omega(M).$$

Then h smooth. Fix $(x_0,\xi_0) \in \Omega(M) \setminus \partial_0\Omega(M)$, and let $t_0 = \tau(x_0,\xi_0)$. Then $h(t_0,x_0,\xi_0) = 0$, and



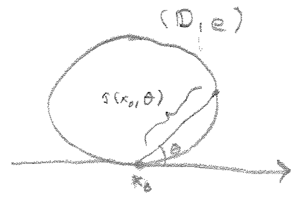
$$\frac{\partial h}{\partial t}(t_0,x_0,\xi_0) = \langle \operatorname{grad} \rho(y_0), \eta_0 \rangle = \langle \nu(y_0), \eta_0 \rangle$$

where $y_0 = \gamma(t_0,x_0,\xi_0)$, $\eta_0 = \dot\gamma(t_0,x_0,\xi_0)$. Now if η_0 were tangent to ∂M, then $\gamma|_{(t_0-\varepsilon,t_0+\varepsilon)}$ would lie outside of M, contradiction. Thus $\frac{\partial h}{\partial t}(t_0,x_0,\xi_0) \neq 0$, and by implicit function theorem $\exists\ t = t(x,\xi)$, smooth near (x_0,ξ_0), so that

$$h(t,x,\xi) = 0 \iff t = t(x,\xi), \quad \text{for } (t,x,\xi) \text{ near } (t_0,x_0,\xi_0).$$

Since $h(\tau(x,\xi),x,\xi) = 0$ and τ is continuous, we get $\tau = t$. $\square$

It is easy to see that τ is not smooth on $\partial_0\Omega(M)$.



$$\tau(x_0,\theta) = \begin{cases} 2\sin\theta, & \theta > 0 \\ 0, & \theta \le 0 \end{cases}$$

We define

$$\tau^0(x,\xi) = \begin{cases} \tau(x,\xi), & (x,\xi) \in \partial_+\Omega(M), \\ -\tau(x,-\xi), & (x,\xi) \in \partial_-\Omega(M). \end{cases}$$

<u>Lemma</u> $\tau^0 : \partial\Omega(M) \to \mathbb{R}$ is smooth when (M,g) nontrapping and ∂M strictly convex.

<u>Pf</u> τ^0 is continuous on $\partial\Omega(M)$ by earlier lemma. As before, define $h(t,x,\xi) = \rho(\gamma(t,x,\xi))$. Then

$$h(0,x,\xi) = 0$$
$$\frac{\partial h}{\partial t}(0,x,\xi) = \langle \nu(x), \xi \rangle$$
$$\frac{\partial^2 h}{\partial t^2}(0,x,\xi) = \frac{d^2}{dt^2}(\rho(\gamma(t,x,\xi)))\Big|_{t=0} = c(x,\xi)$$

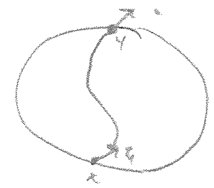
where c is smooth and <0 by strict convexity. Then

$$h(t,x,\xi) = \langle \nu(x), \xi \rangle t + \frac{1}{2} c(x,\xi) t^2 + R(t,x,\xi) t^3$$

with R smooth. Fix $(x_0,\xi_0) \in \partial_0\Omega(M)$ and $t_0 = 0$. Define

$$F(t,x,\xi) = \langle \nu(x), \xi \rangle + \frac{1}{2} c(x,\xi) t + R(t,x,\xi) t^2 = \frac{\rho(\gamma(t,x,\xi))}{t}.$$

Then $\frac{\partial F}{\partial t}(t_0,x_0,\xi_0) \neq 0$, so there is a smooth function $t = t(x,\xi)$ such that $F(t,x,\xi) = 0 \iff t = t(x,\xi)$ for (t,x,ξ) near (t_0,x_0,ξ_0). Since $F(\tau^0(x,\xi),x,\xi) = 0$, one sees that τ^0 is smooth. $\square$



<u>Def.</u> Let (M,g) be nontrapping with ∂M strictly convex. The scattering relation $\alpha : \partial\Omega(M) \to \partial\Omega(M)$ is defined by

$$\alpha(x,\xi) = (\gamma(\tau^0(x,\xi),x,\xi),\ \dot\gamma(\tau^0(x,\xi),x,\xi))$$

By the preceding lemma, α is smooth. One has $\alpha^2 = \mathrm{id}$ by construction, and one obtains diffeomorphisms $\alpha : \partial\Omega(M) \to \partial\Omega(M$
$\alpha : \partial\Omega_\pm(M) \to \partial\Omega_\mp(M)$. Also, $\alpha(x,\xi) = (x,\xi)$ iff $(x,\xi) \in \partial_0\Omega(M)$.

We move on to the geodesic X-ray transform. Let φ_t be the geodesic flow on $\Omega(S)$, so that

$$\varphi_t(x,\xi) = (\gamma(t,x,\xi),\ \dot\gamma(t,x,\xi)).$$

Let $\mathcal{H}$ be the geodesic vector field, which acts on the function $f \in C^\infty(\Omega(S))$ by

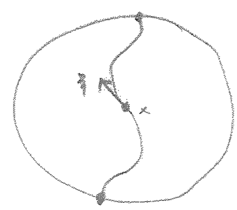
$$\mathcal{H} f(x,\xi) = \frac{d}{dt} f(\gamma(t,x,\xi))\Big|_{t=0}, \qquad (x,\xi) \in \Omega(M).$$

Introduce the "end point map" $\psi : \Omega(M) \to \partial_-\Omega(M)$,

$$\psi(x,\xi) = \varphi_{\tau(x,\xi)}(x,\xi), \qquad (x,\xi) \in \Omega(M).$$

Consider the boundary value problem for the transport equation

$$\mathcal{H} u = 0 \quad \text{in } \Omega(M), \quad u|_{\partial_+\Omega(M)} = w.$$



Since the solution is constant on geodesics, it has the form

$$u = w_\psi = w \circ \alpha \circ \psi.$$

Also consider the inhomogeneous problem

$$\mathcal{H} u = -f, \qquad u|_{\partial_-\Omega(M)} = 0.$$

Since $\varphi_{t+s} = \varphi_t \varphi_s$, a solution u satisfies

$$\frac{d}{dt} u(\varphi_t(x,\xi)) = -f(\varphi_t(x,\xi))$$

and consequently $u = u^f$ where

$$u^f(x,\xi) = \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi))\,dt, \qquad (x,\xi) \in \Omega(M).$$

The trace $If = u^f|_{\partial_+\Omega(M)}$ is the geodesic X-ray transform, i.e.

$$If(x,\xi) = \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi))\,dt, \qquad (x,\xi) \in \Omega(M).$$

By the fundamental theorem of calculus,

$$IXf(x,\xi) = \int_0^{\tau(x,\xi)} \frac{d}{dt} f(\varphi_t(x,\xi))\,dt = f(\alpha(x,\xi)) - f(x,\xi), \quad (x,\xi) \in \partial_+\Omega(M).$$

To study mapping properties of I, we recall the Santaló formula

$$\int_{\Omega(M)} f \, d\Sigma^{2n-1} = \int_{\partial_+\Omega(M)} \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi)) \langle \xi, \nu(x) \rangle \, dt \, d\Sigma^{2n-2}.$$

This is a change-of-variables formula which is valid when (M,g) is nontrapping with smooth boundary, and $f \in C(\Omega(M))$. For a proof, see notes for Juha-Matti's talk in Fall 2006. Here the integrations are w.r.t. the forms

$$d\Sigma^{2n-1} = dV^n \wedge d\Omega_x$$
$$d\Sigma^{2n-2} = dV^{n-1} \wedge d\Omega_x$$

where dV^n (resp. dV^{n-1}) is the volume form of M (resp. ∂M), and

$$d\Omega_x = \sqrt{\det g(x)}\, dS_x$$

where dS_x is the Euclidean volume form of Ω_x in T_xM.

For $(x,\xi) \in \partial\Omega(M)$ let $\mu(x,\xi) = \langle \xi, \nu(x) \rangle$, and $L^2_\mu(\partial_+\Omega(M))$ is the space of functions on $\partial_+\Omega(M)$ with inner product

$$(u,v)_{L^2_\mu(\partial_+\Omega(M))} = \int_{\partial_+\Omega(M)} u v \mu \, d\Sigma^{2n-2}.$$

We consider I applied to functions $f(x,\xi) = f(x)$.

<u>Lemma</u> $I : L^2(M) \to L^2_\mu(\partial_+\Omega(M))$ is bounded.

<u>Pf</u> Since (M,g) is nontrapping,

$$|If(x,\xi)|^2 = \left| \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi))\,dt \right|^2 \leq C \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi))^2 \, dt$$

$$\Rightarrow \int_{\partial_+\Omega(M)} |If(x,\xi)|^2 \mu \, d\Sigma^{2n-2} \leq C \int_{\partial_+\Omega(M)} \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi))^2 \mu \, dt \, d\Sigma^{2n-2}$$

$$= C \int_{\Omega(M)} f^2 \, d\Sigma^{2n-1} = C \int_M f^2 \, dV$$

by the Santaló formula. $\square$

The adjoint I^* is bounded $L^2_\mu(\partial_+\Omega(M)) \to L^2(M)$. For $f \in C^\infty(M)$, $w \in C^\infty(\partial_+\Omega(M))$,

$$(If, w)_{L^2_\mu(\partial_+\Omega(M))} = \int_{\partial_+\Omega(M)} \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi)) w_\varphi(\varphi_t(x,\xi)) \mu \, dt \, d\Sigma^{2n-2}$$

$$= \int_{\Omega(M)} f w_\varphi \, d\Sigma^{2n-1} = \int_M f(x) \left(\int_{\Omega_x} w_\varphi(x,\xi) \, d\Omega_x(\xi) \right) dV^n(x)$$

so $I^* w(x) = \int_{\Omega_x} w_\varphi(x,\xi) \, d\Omega_x$.