Smooth boundaries of riemannian manifolds – obstacles to classical theory

Spse: M smooth, connected, (metrically) complete, riemannian n-manifold with smooth $(n-1)$-dim manifold boundary.

$d(p,q) := \inf_{\gamma} \{ \ell(\gamma) \mid \gamma$ is piecewise C^1 path from p to $q\}$

Can show [R]: $\forall\ p,q \in M$ can be joined by a shortest path in M.

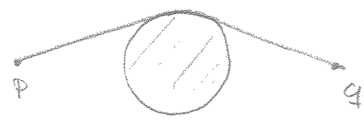
Definition A piecewise C^1 path $\gamma : [0,L] \to M$ is a geodesic between its endpoints if it is parametrized by arclength and it is locally minimizing, that is $\exists$ finite subdivision $0 = t_0 < t_1 < \dots < t_k = L$ such that $\gamma|_{[t_{j-1}, t_j]}$ is a shortest path between points $\gamma(t_{j-1}), \gamma(t_j)$ $\forall\ j = 1, \dots, k$.

Note: "locally minimizing" NOT necessary between endpoints in the same sense as in the classical case ($\partial M = \emptyset$) as seen earlier. (Ex, if $K(\dot\gamma, u) > 0$ then long enough geodesics do not minimize even locally in the variational sense)

1° Regularity of geodesics

Theorem ([AA]-81) If M is C^3-smooth and ∂M C^1-smooth then geodesics are C^1.

<u>Note</u>: Taking M C^∞, ∂M C^∞ does not smoothen the geodesics: $M = \mathbb{R}^2 \setminus B(0,1)$ with euclidean metric.



Q1: How badly does the classical characterization of geodesics $\nabla_{\dot\gamma}\dot\gamma = (\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j)\frac{\partial}{\partial x^k} = 0$ break down?

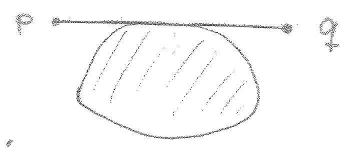
In [AA] it is shown that:

- $\not\exists\ \ddot\gamma$ only at a countably number of points
- $\exists \lim_{t\to t_0^+} \ddot\gamma(t)$ and $\lim_{t\to t_0^-} \ddot\gamma(t)$ $\forall t_0$.

Divide the points $\gamma(t)$ into following classes

a) <u>interior segment</u> $\gamma((t_0, t_0+\varepsilon))$: $\exists\ \ddot\gamma(t)$ and $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ $\forall t \in (t_0, t_0+\varepsilon)$.

<u>Note</u>: interior segment can contain boundary points. At those points the boundary does not behave like an obstacle.



b) <u>boundary segment</u> $\gamma((t_0, t_0+\varepsilon))$:

$\exists\ \ddot\gamma(t)$ and $\nabla_{\dot\gamma(t)}\dot\gamma(t) \neq 0$ in a dense subset of $(t_0, t_0+\varepsilon)$

Then [AA] show that $\nabla_{\dot\gamma}\dot\gamma \perp \partial M$.

c) <u>switch points</u> where the geodesic switches from a boundary segment to an interior segment and vice-versa.

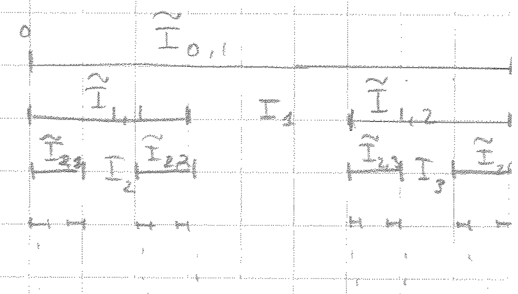
d) <u>Intermittent points</u> are accumulation points of switch points.

Points in class d) main difficulty for variational techniques.

Clearly if $x(t) \in$ b), c), d) then $x(t) \in \partial M$.

Point set d) can be a Cantor set of positive Hausdorff measure.

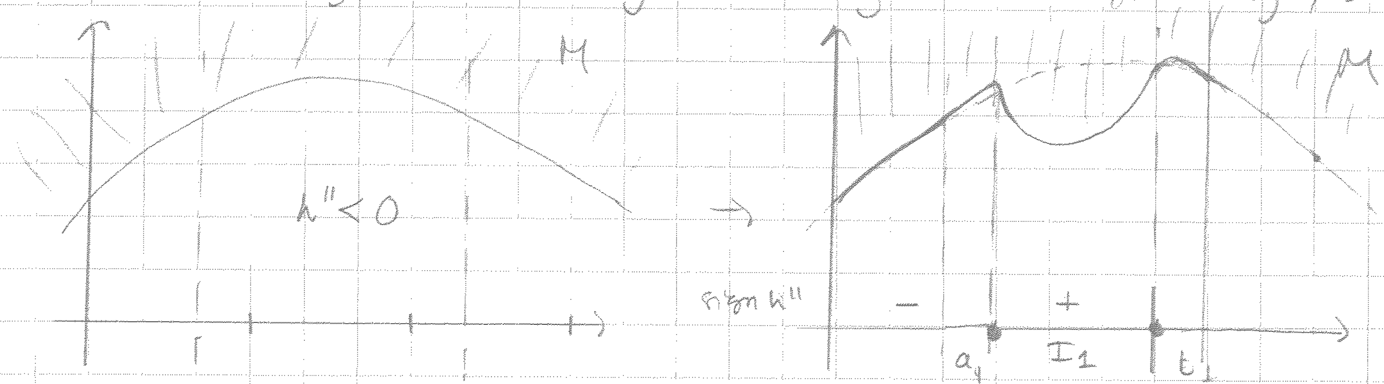
<u>Ex</u>: $n = 2$ Define boundary as the graph of a C^∞ function h obtained from its second derivative h''. Extract a sequence of open intervals $\{I_n\}$ from unit interval $[0,1]$.



$I_1 = [0,1] \setminus (\tilde{I}_{1,1} \cup \tilde{I}_{1,2})$

$I_2 = \tilde{I}_{1,1} \setminus (\tilde{I}_{2,1} \cup \tilde{I}_{2,2})$ etc...

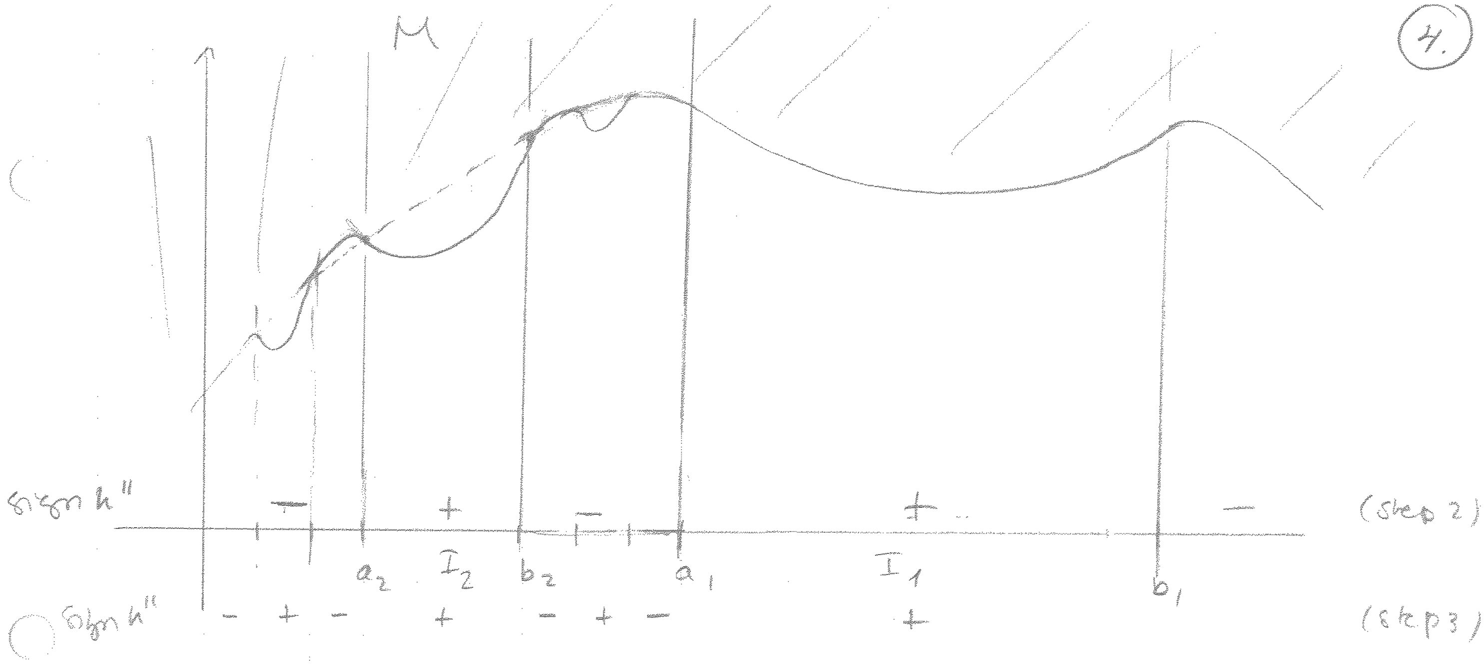
and modify h smoothly according to the following process



step 0. step 1.

1. choose $t \mapsto h''(t)$ smooth s.t. $h^{(k)}(a_1) = h^{(k)}(b_1) = 0$ $\forall\, k \geq 2$ and $h''|_{(a_1,b_1)} > 0$

and convex hull of $h|_{[0,1]\setminus(\tilde{I}_{22} \cup I_1 \cup \tilde{I}_{23})}$ remains unchanged



n. continue accordingly in a neighbourhood of $I_n = [a_n, b_n]$

deform h smoothly s.t. $h^{(k)}(a_n) = h^{(k)}(b_n) = 0 \ \forall k$

and $h''|[a_n, b_n] > 0$ without changing the

convex hull h, outside a neighbourhood of I_n

In the limit $(x, h(x)) \in \partial M \ \forall x \in [0,1]$

and for $y > h(x)$ $(x,y) \in M$. The boundary

of convex hull gives a geodesic whose intermittent

points is the Cantor set $\bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} \tilde{I}_{k,j}$. □.

<u>Good news:</u> At intermittent points $\exists D_{\dot\gamma}\dot\gamma = 0$.

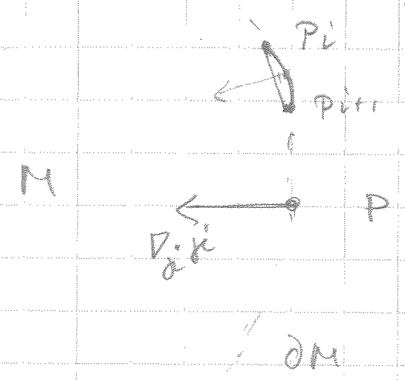
idea: If $\nexists \ddot\gamma$ enough to look at one sided $\ddot\gamma$.

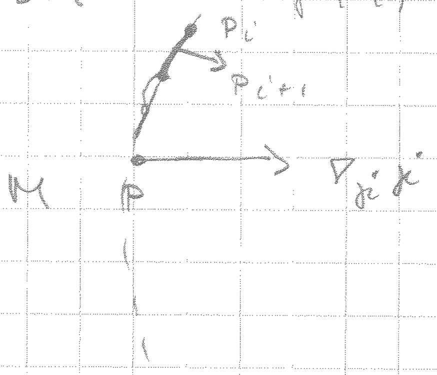
If $D_{\dot\gamma}\dot\gamma$ towards interior at intermittent p then

$\exists$ switch points $p_i \to p$ s.t. $\gamma(p_i, p_{i+1}) \subset \partial M$

for odd i and for large enough i one could shorten the geodesic by cutting across the interior.

$\nabla_{\dot\gamma}\dot\gamma \perp \partial M$ on $\gamma(t_i, t_{i+1})$ $\gamma(t_i) = p_i$ $\gamma(t_{i+1}) = p_{i+1}$





If $\nabla_{\dot\gamma}\dot\gamma$ towards exterior at p then the normal projection onto the boundary would shorten the curves which were not already on the boundary. □.

Especially the problem with intermittent point is not on the nonexistence of $\ddot\gamma$ at those points but at numerous nearby points.

Partial results to avoid intermittent pts:

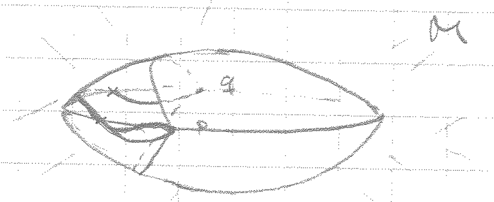
<u>Thm</u> [AB, -89] If M is obtained from $\mathbb{R}^n$ by removing an open set whose boundary is locally analytic i.e. locally of form $x_n = f(x_1, \dots, x_{n-1})$ with real analytic f then a geodesic can have only a finite number of switch pts. Hence no intermittent points.

No generalization of this known for nonflat M?

2° Local uniqueness of geodesics

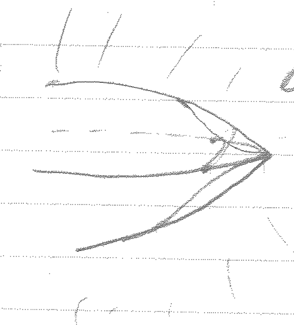
Given nearby points, when is geodesic between them unique? Hopeless task in general.

Example $M = \mathbb{R}^3 \setminus B$, where ∂B consists of two copies of a spherical cap glued together along a common circle (not great).



Euclidean metric. ∂M not smooth but take a look: $\forall\, p, q \in \partial M$ at most 2 minimal geodesics joining them but numerous (non-minimal) geodesics oscillating back and forth across the edge by fixing any number of points along the edge and joining them with minimal geodesics piecewise smoothly. The edge between p, q is a limit of geodesics but not a geodesic!

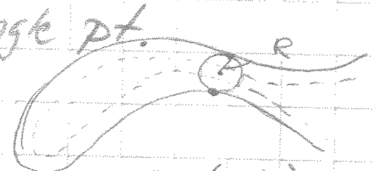
This example is not smooth but the boundary can be made C^1 without destroying the above phenomena,



but creating infinite normal curvature on the boundary.

To gain local uniquenes suppose M isometrically embedded to some $\mathbb{R}^n$ (means no restrictions for M : Nash -56)

Definition $r>0$ is a tubular radius for M in $\mathbb{R}^n$ if $\forall$ point at distance r or less from M is the center of a closed ball which meets M at a single pt.



Then a geodesic in M at every point has its radius of curvature in $\mathbb{R}^n$ extending beyond the tubular radius. One can show (ABBZ, -87)

Theorem If r is a tubular radius for M then two different geodesics in M starting from the same point must each travel more than πr before they can meet again.

Then one gains local uniqueness in the following sense:

Theorem Spse M has positive tubular radius. Then every point of M has a neighbourhood U s.t. $\forall\, p, q \in U$:

1) $\exists$ unique minimal geodesic joining p and q.

2) $\nexists$ other geodesic segment joining p and q and lying in U.

Note: in 1) no claim on the geodesic staying in U.
still open ??

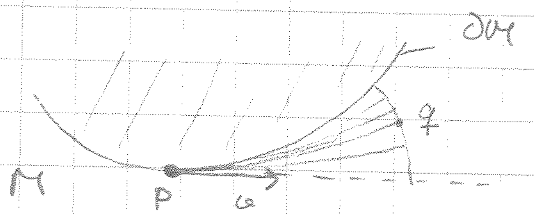
3° Cauchy uniqueness

If $\partial M = \emptyset$ then Cauchy-uniqueness:

Given any $p \in M$, $v \in T_pM$ $\exists$ unique (up to reparametrization) geodesic γ s.t. $\gamma(0) = p$ and $\gamma'(0) = v$.

Need not be true if $\partial M \neq \emptyset$

Ex:



even a continuous family of distinct geodesics close to p in the shadow

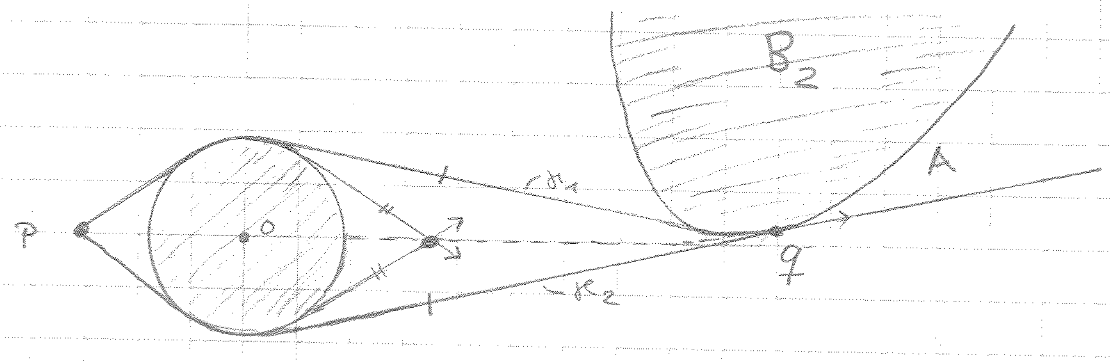
these are all involutes in the following sense:

Definition An involute of a geodesic β in M is another geodesic γ s.t. $\gamma(0) = \beta(0)$, $\gamma'(0) = \beta'(0)$ and $\ell(\beta) = \ell(\gamma)$ which consists of a segment in common with β following by a nontrivial segment of the interior.

Theorem (ABB1) $\forall$ boundary point of M has a neighbourhood in which: if two geodesic segments with the same initial point, initial tangent and length do not coincide then one of them has its right endpoint in the interior and is an involute of the other.

4° Primary minimizers ([ABB3], -93)

Example 1. $M = \mathbb{R}^2 \setminus (B(0,1) \cup B_2)$ with euclidean metric



B_2 unbounded obstacle as above

- each of the points on the dashed line have exactly two minimizing geodesics from p with different terminal velocities and no minimizing extensions

- minimizing geodesics γ_1 and γ_2 from p to q come together tangentially at equal length and extend beyond q (still minimizing) to reach all points of the "shadow" region A

Note: Wavefront from p (= metric circle $S(p,r) \subset M$) is separated into branches by the disk, colliding at the dashed line. Top branch of the wavefront is squeezed by B_2 and extinguished at q leaving only the bottom branch.

Idea: geodesic γ_2 is more "primary" than γ_1

Definition: If γ_1 and γ_2 are minimizing geodesics from p to q then γ_2 is preferred to γ_1 if they have the same terminal velocity vector at q and just before their common terminal segment (which may be trivial) γ_2 has a nontrivial segment in the interior of M but γ_1 does not.

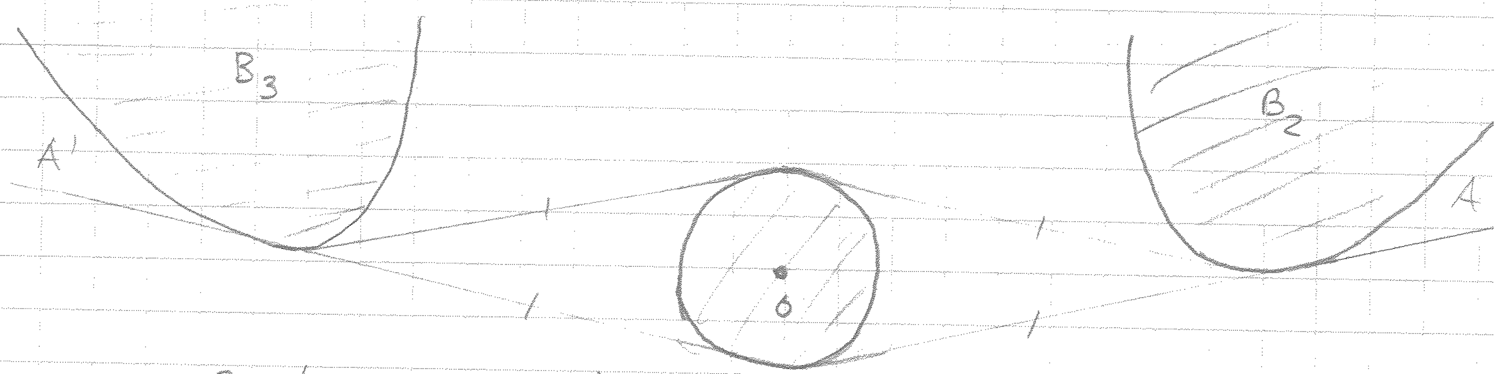
A <u>primary minimizer</u> from p to q is one that is preferred to all others having same velocity vector at q.

<u>Theorem</u> For each minimizer from p to q there is exactly one primary minimizer from p to q having the same terminal velocity vector.

How big is the set of points p that have nonprimary geodesics emanating from it?

Can be an open set!

<u>Example 2.</u> Reflect Ex 1 about the vertical axis!

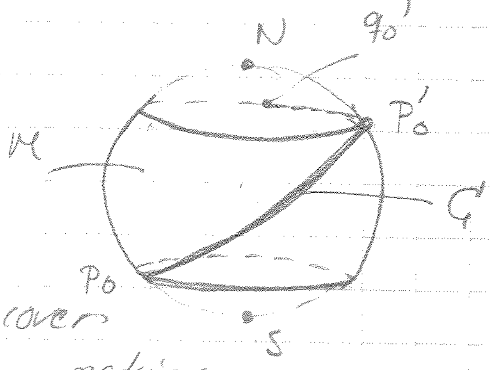


$M = \mathbb{R}^2 \setminus (B(0,1) \cup B_2 \cup B_3)$

"Shadow" A' is a set of points containing nonprimary minimizers

Bifurcating geodesics starting from p need not be isolated: 11.

Example 3: A continuous family of bifurcating minimizers starting from p.

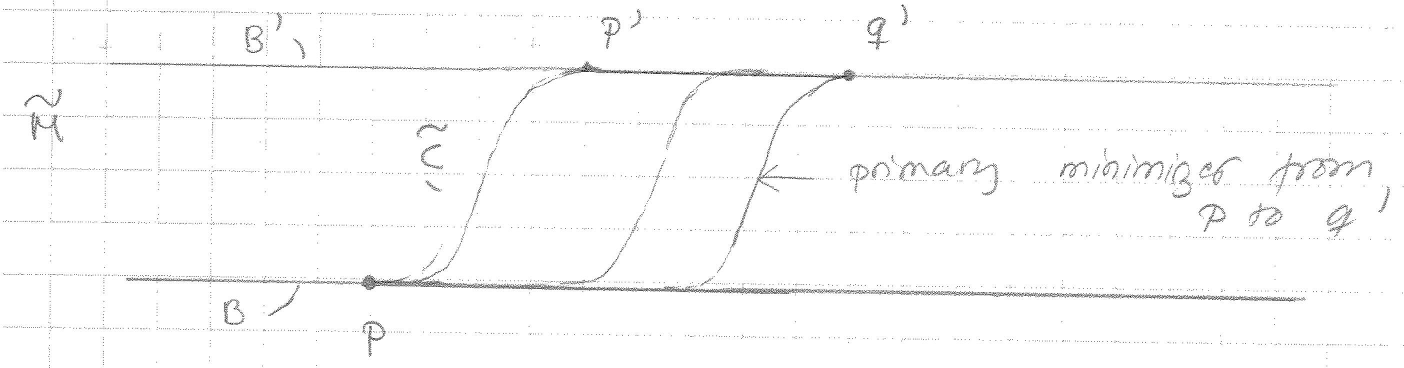


Let $M = S^2 \setminus (B(N,\varepsilon) \cup B(S,\varepsilon))$ with induced euclidean metric from $\mathbb{R}^3$.

Let $\tilde{M} \approx I \times \mathbb{R}$ be its universal cover with induced metric $\pi : \tilde{M} \to M$ making covering projection an isometry. Denote $\partial\tilde{M} = B \cup B'$.

Choose great semicircle C joining two antipodal points $p_0, p_0' \in \partial M$.

Consider an arc $\tilde{C} \subset \tilde{M}$ s.t. $\pi(\tilde{C}) = C$ joining points $p \in \pi^{-1}(p_0) \subset B$, $p' \in \pi^{-1}(p_0') \subset B'$



extend $\tilde{C}$ to q' by nontrivial segment $[p', q'] \subset B'$

$\pi(\tilde{C} \cup [p', q'])$ gives a minimizing geodesic γ.

between $q_0' = \pi(q')$ and p_0 : $|\gamma| = C \cup [p_0' \cup q_0']$.

γ belongs to a 1-parameter variation of minimizing geodesics from p_0' to q_0' all which have the same terminal velocity.

Is there uniform analysis to work in cases like ex 2 & 3 ?

5° Cut locus

Earlier: classical case $\partial M = \emptyset$:

$\mathrm{Cut}(p) := \{q \in M \,/\, q$ is the cut point of p along some geod. $\gamma\}$,

where cut point of p along some γ is $q = \gamma(t_0)$

s.t. $t_0 = \sup\{t > 0 \,/\, d(p, \gamma(t)) = t\} < \infty$.

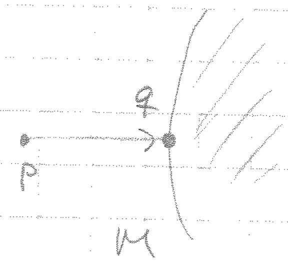
Characterization (Warner):

$\mathrm{Cut}(p) = \{q \in M \,/\, \exists$ two minimizing geodesics from p to $q\}$

Fails completely in the case $\partial M \neq \emptyset$:

$\not\supset$: ex above

$\not\subset$: geodesic hit the boundary:



Definition Cut locus of a manifold with boundary

$\mathrm{Cut}_\partial(p) := \{q \in M \,/\, \exists$ two primary minimizers from p to $q\}$

Then $\mathrm{Cut}_\partial(p) = \mathrm{Cut}(p)$ if $\partial M = \emptyset$ (all minimizers are primary).

In addition:

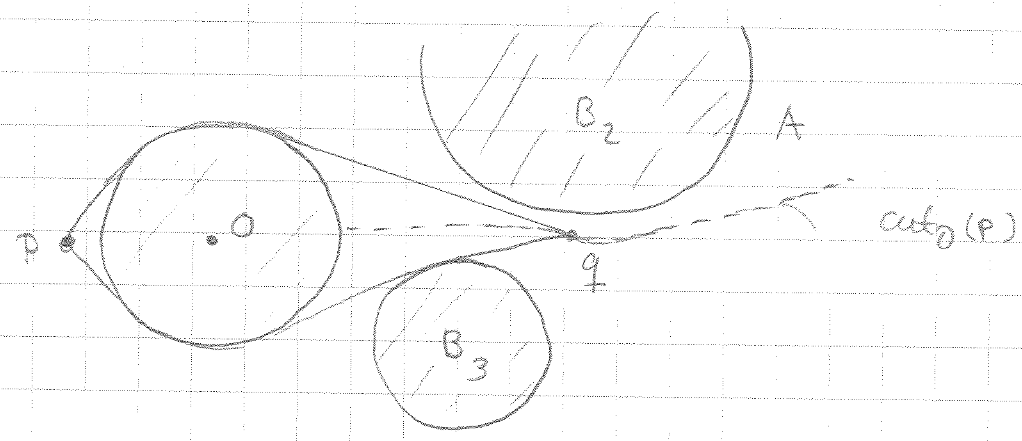
$\mathrm{Cut}_\partial(p) = \{q \,/\, q$ has two minimizers from p with different terminal velocities$\}$

Note: 1° can be points $q \in \mathrm{Cut}_\partial(p)$ that lie on extendible minimizers (ex 1)

2° such points can also lie in the interior of M (ex below)

If $\partial M = \emptyset$ can show: $M \setminus cut(p)$ is contractible (13)
In case $\partial M \neq \emptyset$ $M \setminus cut_0(p)$ need not be -"-

Example Starting from example 1° rotate the picture about a vertical axis through p to obtain a solid torus obstacle and unbounded obstacle. Remove further a 3-ball B_3 so that vertical cross-section through the center of B_3 is the following:



- unique minimizers to points in A
- $\nexists$ cut points along extendible minimizers
- $\exists$ noncontractible loop encircling the cut points that extend beyond and below the unbounded obstacle in all sections intersecting B_3.

× contractibility for $M \setminus$ gluing locus
× Counterparts for

exponential map $\to$ endpoint map

Jacobi fields $\to$ Jacobi equations

References:

[AA] R. Alexander, S. Alexander: Geodesics in riemannian manifolds with boundary, Indiana University Math. J. Vol 30 (1981)

[ABB1] S. Alexander, D. Berg, R.L. Bishop: Cauchy uniqueness in the riemannian obstacle problem. in Lect. Notes in Math 1209

[ABB2] —"— : The riemannian obstacle problem, Illinois J. Math. 31 (1987)

[ABB3] —"— : Cut loci, minimizers and wavefronts in riemannian manifolds with boundary. Michigan Math. J. 40 (1993)

[ABB4] —"— : Geometric curvature bounds in riemannian manifolds with boundary. Trans. Amer. Math. Soc. 339 (1993)

[AB] F. Albrecht, D. Berg: Geodesics in euclidean space with analytic obstacle, Proc. Amer. Math. Soc. 113 (1991)

[R] W. Rinow: Die Innere Geometrie der Metrischen Räume, Springer