Inverse Problems Seminar ①

Mikko Salo, 29.1.2007, Exactum B120

Boundary rigidity – review

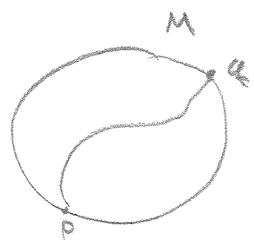
HY + TKK joint seminar

Mon 14–16, alternates between Exactum B120 and TKK U358

Today, we review topics from fall.

Let (M,g) be a smooth Riemannian manifold with boundary (assume (M,g) is embedded in a compact manifold (S,g) without boundary).

If $p,q \in \partial M$, define geodesic distance



$$d_g(p,q) = \inf_{\substack{\gamma : [a,b] \to M \\ \gamma(a)=p,\ \gamma(b)=q}} L(\gamma)$$

$$L(\gamma) = \int_a^b |\dot\gamma(t)|_g \, dt = \int_a^b \sqrt{g_{ij}(\gamma(t))\,\dot\gamma^i(t)\,\dot\gamma^j(t)}\, dt$$

The boundary distance function of (M,g) is

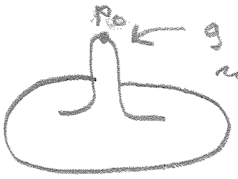
$$d_g : \partial M \times \partial M \to \mathbb{R}.$$

If $\psi : M \to M$ is a diffeo with $\psi|_{\partial M} = id$, then $d_{\psi^* g} = d_g$.

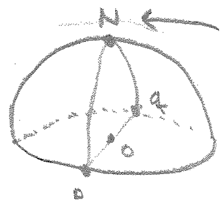
> Boundary rigidity problem: Let (M,g_i) be a Riemannian mfld with boundary, $i = 1,2$. If $d_{g_1} = d_{g_2}$, does this imply $g_2 = \psi^* g_1$ for some diffeo $\psi : M \to M$ with $\psi|_{\partial M} = id$?

In geophysics, this is known as the inverse kinematic problem: if one knows the travel times of earthquakes between points on the surface, can one determine the sound speed in the Earth?

The boundary rigidity problem does not have an affirmative answer without restrictions on g. Obstructions:



g very large
can change metric
near p_0



can change metric
near N

<u>Definition</u> We say that (M,g) is <u>simple</u> if ∂M is strictly convex, and there is an open set $\hat{M} \subseteq S$ with $M \subseteq \hat{M}$ such that $\exp_p : \exp_p^{-1}(\hat{M}) \to \hat{M}$ is diffeo $\forall\, p \in \hat{M}$.

Intuitively, this means that any two points in M can be joined by a unique geodesic.

Michel (1981) conjectured that boundary rigidity holds for simple manifolds.

Mukhometov (1981) proved this for conformal manifolds: if (M,g) is simple and $d_g = d_{\alpha g}$ where $\alpha > 0$, then $\alpha \equiv 1$. Juha-Matti gave a proof of this in the fall.

Also, Simopekka showed that if (M, g_i) are simple mflds and $d_{g_1} = d_{g_2}$, then $\exists$ diffeo $\Psi : M \to M$ with $\Psi|_{\partial M} = id$ such that $\partial^\alpha g_1|_{\partial M} = \partial^\alpha (\Psi^* g_2)|_{\partial M}$ in any local coordinates.

In the spring, we will discuss the following result.

<u>Thm</u> (Pestov–Uhlmann 2005) Let (M, g_i), $i = 1, 2$, be 2D simple compact manifolds with boundary ∂M. Assume

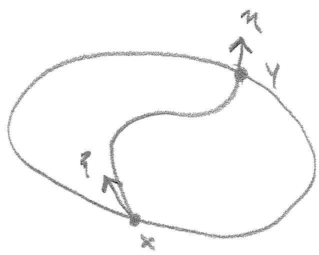
$$d_{g_1}(p,q) = d_{g_2}(p,q) \qquad \forall\, p,q \in \partial M.$$

Then $g_2 = \Psi^* g_1$ for some diffeo $\Psi : M \to M$ with $\Psi|_{\partial M} = id$.

To prove this, need scattering relation

$$\alpha_g : (x, \xi) \mapsto (y, \eta)$$

$$x, y \in \partial M, \quad |\xi|_g = |\eta|_g = 1$$

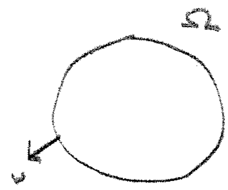


Juha-Matti showed that for simple (or SGM) metrics, α_g is determined by d_g.

<u>Conjecture</u> (lens rigidity) If (M,g) nontrapping, do d_g and α_g together determine g up to diffeomorphism?

We now consider the inverse conductivity problem arising in EIT.

$\gamma = (\gamma^{ij})_{i,j=1}^{n}$ conductivity in Ω



$$\begin{cases} \nabla\cdot(\gamma\nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

DN map

$\Lambda_\gamma : f \mapsto +\gamma^{ij}\frac{\partial u}{\partial x_i}\nu_j|_{\partial\Omega}$, $\Lambda_{\psi^*\gamma} = \Lambda_\gamma$ if $\psi : \Omega \to \Omega$ diffeo, $\psi|_{\partial\Omega} = id$

Inverse problem: find (γ^{ij}) from Λ_γ (up to ψ).

Geometrically, replace γ with metric g and $\nabla\cdot\gamma\nabla$ with

$$\Delta_g u = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij}\frac{\partial}{\partial x_j}u\right)$$

$$\begin{cases} \Delta_g u = 0 & \text{in } M \\ u = f & \text{on } \partial M \end{cases}$$

$$\Lambda_g : f \mapsto \langle \nu, \nabla u|_{\partial M}\rangle$$

When $n \geq 3$, $\gamma^{ij} = \sqrt{\det g}\, g^{ij}$ and the inverse problems for conductivity and metric are equivalent.

For $n = 2$, conformal invariance:

$$\Delta_{\beta g} = \frac{1}{\beta}\Delta_g \quad \text{where } \beta > 0$$

Therefore $\Lambda_{\beta(\psi^* g)} = \Lambda_g$.

Thm (Lassas–Uhlmann 2001) If (M, g_i) connected, $\Lambda_{g_1} = \Lambda_{g_2}$ and $n = 2$, then $\exists$ diffeo $\psi : M \to M$ with $\psi|_{\partial\Omega} = id$, and $\exists$ $\beta \in C^\infty(M)$ with $\beta \neq 0$, $\beta|_{\partial\Omega} = 1$, such that $g_2 = \beta\psi^* g_1$.

Thm (Lassas–Taylor–Uhlmann 2003) If $n \geq 3$, $\Lambda_{g_1} = \Lambda_{g_2}$ and g_1, g_2 are real-analytic up in $\bar{M}$, then $g_2 = \psi^* g_1$ for diffeo $\psi : M \to M$, $\psi|_{\partial M} = i$

Comparison of boundary rigidity and EIT

d_g / α_g	Λ_g
function of $2n-2$ variables	kernel is function of $2n-2$ variables
$d_{\psi^* g} = d_g$	$\Lambda_{\psi^* g} = \Lambda_g$
singularities determine $\partial^\alpha g|_{\partial M}$	singularities determine $\partial^\alpha g|_{\partial M}$
$d_g(x,y) = \inf_{\gamma(a)=x,\ \gamma(b)=y} \int_a^b \sqrt{g_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)}\,dt$	$Q_g(f) = \inf_{u|_{\partial\Omega}=f} \int_\Omega \gamma^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx$
graph of α_g Lagrangian mfld (finite dim.)	graph of Λ_g Lagrangian mfld (infinite dim.)
obtained from solutions of Hamilton ODE	obtained from solution of PDEs

Surprising result is

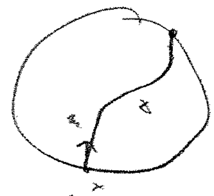
Thm (Pestov-Uhlmann 2005) (M, g_i) compact simple 2D mflds.
Then $\alpha_{g_1} = \alpha_{g_2} \Rightarrow \Lambda_{g_1} = \Lambda_{g_2}$.

Given this, proof that $d_{g_1} = d_{g_2} \Rightarrow g_2 = \psi^* g_1$ for simple 2D mflds goes as follows:

$$d_{g_1} = d_{g_2}$$
$$\overset{\text{simple}}{\Longrightarrow} \quad \alpha_{g_1} = \alpha_{g_2}$$
$$\overset{\text{Pestov-Uhlmann}}{\Longrightarrow} \quad \Lambda_{g_1} = \Lambda_{g_2}$$
$$\overset{\text{Lassas-Uhlmann}}{\Longrightarrow} \quad g_2 = \beta(\psi^* g_1)$$
$$\overset{\text{Mukhometov}}{\Longrightarrow} \quad g_2 = \psi^* g_1.$$

Geodesic X-ray transform

$$I_g f(\gamma) = \int_\gamma f = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,dt$$



Linearization of boundary rigidity problem about a simple metric results in injectivity questions for $I_g : C^\infty(M) \to C^\infty(\partial_+\Omega(M))$.
Need that I^* onto.

Open problems:

1) Nonsmooth metrics ($g \in C^{1,1}$)?
2) Reconstruction of g from d_g
3) $n = 2$, g nontrapping, does α_g determine g?
4) g nontrapping, is I_g injective?
5) Is there a connection between Λ_g and d_g?
6) $n \geq 3$, connection between d_g and Λ_g? (Gromov's result for Λ_g)