We consider now a class of asymptotic configurations that in the high-velocity limit have negligible interaction with the obstacle K. This will allow us to separate the scattering effect of the magnetic potential from that of the obstacle.

Given $\hat{v} \in S^1$ we denote

$$\Omega_{\hat{v}} = \{x \in \Omega : x + \tau \hat{v} \in \Omega, \forall \tau \in \mathbb{R}\}.$$

Take $\varphi_0 \in \mathcal{H}_2(\Omega)$ with support $\varphi_0 \in \Omega_{\hat{v}}$.

Recall that

$$e^{-im\bar{v}\cdot x} e^{-itH_0} e^{im\bar{v}\cdot x} \varphi_0 = e^{-imv^2t/2} e^{-i\bar{p}\cdot\bar{v}t} e^{-itH_0} \varphi_0$$

and that in the limit when $\bar{v} \to \infty$ with $\hat{v}$ fixed this can be replaced (modulo the unimportant phase factor $e^{-imv^2t/2}$) by

$$e^{-i\bar{p}\cdot\bar{v}t} \varphi_0 = \varphi_0(x - \bar{v}t),$$

Note that,

$$\text{support } e^{-i\bar{p}\cdot\bar{v}t} \varphi_0 \subset \Omega \quad \forall t \in \mathbb{R}.$$

To a good approximation this asymptotic configuration has negligible interaction with K $\forall t \in \mathbb{R}$ and it will only feel the scattering effects of the magnetic field.

we give a rigorous proof of this fact in the following lemma.

We denote

$$L_{A,\overset{n}{v}}(t) := \int_0^t \overset{n}{v} \cdot A(x+\tau \overset{n}{v})\, d\tau$$

<u>Lemma 5.5</u>

Let Ω_0 be a compact subset of $\Omega_{\overset{n}{v}}$, $\overset{n}{v} \in \mathbb{R}^2 \setminus \{0\}$.
Then, $\forall$ α_K and all $A \in \mathcal{A}_K(\alpha_K, \beta_R)$ there is a constant C such that

$$\left\| \left(e^{im\overline{v}\cdot x} W_{\pm}(A) e^{im\overline{v}\cdot x} - e^{-i L_{A,\overset{n}{v}}(\pm\infty)} \right) \varphi \right\| \leq \frac{C}{v} \|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}$$

for all $\varphi \in \mathcal{H}_2(\mathbb{R}^2)$ with support in Ω_0.

<u>Proof</u>: We give the proof for $W_+(A)$. The case of $W_-(A)$ follows in a similar way.

We first prove that

$$\left\| \left(e^{im\overline{v}\cdot x} W_+(A^{(\beta,\alpha)}) e^{im\overline{v}\cdot x} - e^{-i\int_0^\infty \overset{n}{v}\cdot A^{(\beta,\alpha)}(x+\tau \overset{n}{v})\, d\tau} \right) \varphi \right\| \leq$$

$$\leq \frac{C}{v} \|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}.$$

As support $\varphi_0 \subset \Omega_0$, we have that,

$$\text{distance}(K, \{x : x = y + \tau \overset{n}{v},\ y \in \text{support}\,\varphi,\ \tau \in \mathbb{R}\}) > 0.$$

Hence, there is $\chi \in C^\infty(\mathbb{R}^2)$, $\chi \equiv 0$ in a neighborhood of K and $\chi \equiv 1$ for $x \in \{x : x = y + \tau \overset{n}{v},\ y \in \text{support}\,\varphi,\ \tau \in \mathbb{R}\} \cup \{x : |x| \geq M\}$, for some large enough M.

We denote $\check{A}(x) = \chi(x) A(x) \in C^1(\mathbb{R}^2)$.

Let us designate,

$H_1 := \frac{1}{v} e^{-im\bar{v}\cdot x} H_0 e^{im\bar{v}\cdot x}$, $H_2 := \frac{1}{v} e^{-im\bar{v}\cdot x} H(A(P,Q)) e^{im\bar{v}\cdot x}$.

Then,

$$e^{-im\bar{v}\cdot x} W_+(A(P,Q)) e^{im\bar{v}\cdot x} = s\text{-}\lim_{t\to\infty} e^{itvH_2} \chi(p) e^{-itvH_1}$$

$$= s\text{-}\lim_{t\to\infty} e^{itH_2} \chi(p) e^{-itH_1} \qquad (5.4)$$

Note that by scaling time $t \to tv$ we gained one power of $\frac{1}{v}$ and we obtained H_1 and H_2 in the right-hand side of (5.4).

Moreover as $\tilde{A}(P,Q) = A(P,Q)$ on the support of φ,

$$e^{-im\bar{v}\cdot x} W_+(A(P,Q)) e^{im\bar{v}\cdot x} - e^{-iL^{(\infty)}_{\tilde{A}(P,Q),\hat{v}}}) \varphi =$$

$$= s\text{-}\lim_{t\to\infty} [e^{itH_2} \chi(p) e^{-itH_1} - e^{-iL^{(t)}_{\tilde{A}(P,Q),\hat{v}}}] \varphi.$$

Let $g \in C_0^\infty(\mathbb{R}^2)$ satisfy $g(p) = 1$, $|p| \le 1$, $g(p) = 0$, $|p| \ge 2$. Whence, as $\varphi \in \mathcal{H}_2(\mathbb{R}^2)$,

$$\| [g(\frac{p}{v^\rho}) - 1] \varphi \|_{L^2(\mathbb{R}^2)} \le \frac{C}{v^{2\rho}} \|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)},$$

$\rho \ge 0$. Hence, taking $\rho \ge 1/2$ we see that it is enough to prove that

$$\| \lim_{t\to\infty} [e^{itH_2} \chi(x) e^{-itH_1} - \chi(x) e^{-iL^{(t)}_{\tilde{A}(P,Q),\hat{v}}}] \tilde{\varphi} \| \le \frac{C}{v} \|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)} \qquad (5.5)$$

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By Duhamel's formula

$$\left(e^{itH_2}\chi(x)\,e^{-itH_1} - \chi(x)\,e^{-iL^{(t)}_{\tilde{A}^{(P,Q)},\hat{v}}}\right)\tilde{\varphi} =$$

$$= e^{itH_2}\left(\chi(x) - e^{-itH_2}\chi(x)\,e^{-iL^{(t)}_{\tilde{A}^{(P,Q)},\hat{v}}}\,e^{itH_1}\right)e^{-itH_1}\tilde{\varphi}$$

$$= e^{itH_2}\int_0^t d\tau\, e^{-i\tau H_2}\, i\Big[H_2\chi(x)\,e^{-iL^{(\tau)}_{\tilde{A}^{(P,Q)}}} - e^{-iL^{(\tau)}_{\tilde{A}^{(P,Q)}}}\chi(x)$$

$$\left(H_1 - \hat{v}\cdot\tilde{A}^{(P,Q)}(x+\tau\hat{v})\right)\Big]\,e^{i\tau H_1}\,e^{-itH_1}\tilde{\varphi}$$

$$= \int_0^t d\tau\, e^{i\tau H_2}\, i\Big[H_2\, e^{-iL^{(t-\tau)}_{\tilde{A}^{(P,Q)},\hat{v}}}\chi(x) - e^{-iL^{(t-\tau)}_{\tilde{A}^{(P,Q)},\hat{v}}}\chi(x)$$
$$\left(H_1 - \hat{v}\cdot\tilde{A}^{(P,Q)}(x+(t-\tau)\hat{v})\right)\Big]\,e^{-i\tau H_1}\tilde{\varphi}. \qquad (5.6)$$

We have the following identity

$$(\bar{p} - A(x))\,e^{-iL^{(t)}_{A,\hat{v}}} = e^{-iL^{(t)}_{A,\hat{v}}}\,(\bar{p} - b(x,t)), \qquad (5.7)$$

where

$$b(x,t) = A(x,t) + \int_0^t (\hat{v}\times B)(x+\hat{v}u)\,du. \qquad (5.8)$$

This follows from the identity:

$$\operatorname{grad}(\bar{a}\cdot A) = \bar{a}\times \operatorname{curl} A + A\times\operatorname{curl}\bar{a} +$$
$$+ (A\cdot\operatorname{grad})\,\bar{a} + (\bar{a}\cdot\operatorname{grad})A.$$

In our case $\bar{a} = \hat{v}$,

$B = \operatorname{curl} A = (\partial_1 A_2 - \partial_2 A_1)\,\bar{e}_3,$

$$\mathrm{grad}(\hat{v}\cdot A) = \hat{v}\times B + (\hat{v}\cdot \mathrm{grad})\,A .$$

We verify this identity.

component 1:

$$\hat{v}_1 \partial_1 A_1 + \hat{v}_2 \partial_1 A_2 = \hat{v}_2 (\partial_1 A_2 - \cancel{\partial_2 A_1}) + \hat{v}_1 \partial_1 A_1 + \cancel{\hat{v}_2 \partial_2 A_1} \quad \checkmark$$

$$\left(\begin{array}{l} (\hat{v}\times B)_1 = (\partial_1 A_2 - \partial_2 A_1)\,\hat{v}_2 \\ (\hat{v}\times B)_2 = -(\partial_1 A_2 - \partial_2 A_1)\,\hat{v}_1 \end{array}\right)$$

component 2

$$\hat{v}_1 \partial_2 A_1 + \hat{v}_2 \partial_2 A_2 = -\cancel{\partial_1 A_2 \hat{v}_1} + \partial_2 A_1 \hat{v}_1 + \cancel{\hat{v}_1 \partial_1 A_2} + \hat{v}_2 \partial_2 A_2$$

Then,

$$(\bar{p} - A(x))\, e^{-iL^{(t)}_{A,\hat{v}}}(x) = e^{-iL^{(t)}_{A,\hat{v}}}\,\bar{p} - e^{-iL^{(t)}_{A,\hat{v}}} \int_0^t \mathrm{grad}(\hat{v}\cdot A(x+\tau\hat{v})\,d\tau - e^{-iL^{(t)}_{A,\hat{v}}}\,A(x)$$

$$= e^{-iL^{(t)}_{A,\hat{v}}}\,\bar{p} - e^{-iL^{(t)}_{A,\hat{v}}} \int_0^t (\hat{v}\times B)(x+\tau\hat{v})\,d\tau$$

$$- e^{-iL^{(t)}_{A,\hat{v}}} \int_0^t \frac{d}{d\tau} A(x+\tau\hat{v})\,d\tau - A(x)\, e^{-iL^{(t)}_{A,\hat{v}}}$$

$$= e^{-iL^{(t)}_{A,\hat{v}}} (\bar{p} - b(x,t)) .$$

Using (5.7) we prove by a simple calculation that,

$$\left[H_2 e^{-iL^{(t-\tau)}_{\tilde{A}^{(P,Q)},\hat{v}}}\chi(x) - e^{-iL^{(t-\tau)}_{\tilde{A}^{(P,Q)},\hat{v}}}\chi(x)\left(H_1 - \hat{v}\cdot A^{(P,Q)}(x+\hat{v}(t-\tau))\right)\right]$$ (61)

$$= T_1 + T_2 + T_3, \qquad (5.9)$$

where

$$T_1 := \frac{-1}{2m v} e^{-iL^{(t-\tau)}_{\tilde{A}^{(P,Q)},\hat{v}}}\Big\{ \chi(x)\left[\bar{p}\cdot b(x,t-\tau) + b(x,t-\tau)\bar{p} - (b(x,t-\tau))^2\right] + (\Delta\chi(x)) - 2(\bar{p}\chi(x))\cdot\bar{p} + 2 b(x)\cdot(\bar{p}\chi(x))\Big\},$$

$$T_2 := \frac{1}{v} e^{-iL^{(t-\tau)}_{\tilde{A}^{(P,Q)},\hat{v}}}\Big\{ \frac{\chi(x)}{2m}\left[-(\bar{p}\cdot(\tilde{A}^{(P,Q)}(x) - \tilde{A}(x)) + |A^{(P,Q)}(x)|^2 - |\tilde{A}(x)|^2\right] - \frac{\chi(x)}{m}\left(A^{(P,Q)}(x) - \tilde{A}^{(P,Q)}(x)\right)\cdot \left(\tilde{A}^{(P,Q)}(x) + \bar{p} - b(x,t-\tau)\right) - \frac{1}{m}(\bar{p}\chi(x))\cdot \left(A^{(P,Q)}(x) - \tilde{A}^{(P,Q)}(x)\right)\Big\},$$

$$T_3 := e^{-iL^{(t-\tau)}_{\tilde{A}^{(P,Q)},\hat{v}}}\left[(\bar{p}\chi(x))\cdot\hat{v} - \chi(x)\left(A^{(P,Q)}(x) - \tilde{A}(x)\right)\cdot\hat{v}\right].$$

Here we compute $b(x,t)$ as in (5.8) replacing A by $\tilde{A}^{(P,Q)}$. The proof is given after the proof of Theorem 5.6.

Recall that

$$A^{(P,Q)}(x)\, F\left(|x-Q-\hat{v}t| \le \frac{|t|}{4}\right) = 0, \quad t\in\mathbb{R}.$$

Then for $0\le\tau\le t$ (recall that $e^{i\bar{p}\cdot\hat{v}\tau} f(x)\, e^{-i\bar{p}\cdot\hat{v}\tau} = f(x+\hat{v}\tau)$),

$$\left\|\tilde{A}^{(P,Q)}(x+\hat{v}(t-\tau))\, F\left(|x-Q-\hat{v}\tau| \le \frac{\tau}{4}\right)\right\|$$

$$= \left\| e^{i\bar{p}\cdot\hat{v}(t-\tau)}\tilde{A}^{(P,Q)}(x)\, F\left(|x-Q-\hat{v}t| \le \frac{\tau}{4}\right) e^{-i\bar{p}\cdot\hat{v}(t-\tau)}\right\|$$

$$\le \left\|\tilde{A}^{(P,Q)}\, F\left(|x-Q-\hat{v}t| \le \frac{t}{4}\right)\right\| = 0.$$

We prove in the same way that

$$b(x,t-\tau)\, F(|x-Q-\hat{v}\tau| \leq \frac{\tau}{4}) = 0, \quad 0 \leq \tau \leq t.$$

We estimate one of the terms in $T_1 e^{-i\tau H_1}\tilde{\varphi}$.
The others follow in a similar way.
For $0 \leq \tau \leq t$

$$\| \chi(x)\, b(x,t-\tau)\, \bar{p}\, e^{-i\tau H_1}\tilde{\varphi} \|$$

$$= \| \chi(x)\, b(x,t-\tau)\, F(|x-Q-\hat{v}\tau| \leq \frac{\tau}{4})\, e^{-im\bar{v}\cdot x}\, e^{-i\frac{\tau}{v}H_0}\, e^{im\bar{v}\cdot x}$$

$$g\left(\frac{\bar{p}}{v\rho}\right) e^{-im\bar{v}\cdot x}\, e^{im\bar{v}\cdot x}(\bar{p}\varphi) \| \leq C \| F(|x-Q-\hat{v}\tau| \leq \frac{\tau}{4})$$

$$e^{-i\frac{\tau}{v}H_0}\, g\left(\frac{\bar{p}-m\bar{v}}{v\rho}\right) F(|x-Q| \leq \frac{\tau}{8}) \| + \| F(|x-Q| > \frac{\tau}{8})(\bar{p}\varphi) \|$$

$$\leq \frac{C_l}{|1+\tau|^l}, \quad l = 0,1,2,\dots, \quad \text{where we}$$

used Corollary 2.2, taking $\frac{1}{2} \leq \rho < 1$, and the fact that φ has compact support.

In this way we prove that

$$\| T_1 e^{-i\tau H_1}\tilde{\varphi} \| \leq \frac{C_l}{v}(1+\tau)^{-l} \|\varphi\|_{\mathcal{H}_1(\mathbb{R}^2)} \qquad (5.10)$$

$$\| T_2 e^{-i\tau H_2}\tilde{\varphi} \| \leq \frac{C_l}{v}(1+\tau)^{-l} \|\varphi\|_{\mathcal{H}_1(\mathbb{R}^2)}, \quad l = 1,2,3,\dots \qquad (5.11)$$

The term T_3 requires a different proof since we do not have the factor $\frac{1}{v}$.
Denote

$$a(x) := |(\bar{p}\chi(x))| + |\chi(x)|\, |A^{(P,Q)}(x) - \tilde{A}^{(P,Q)}(x)|.$$

Then,

$$\|T_3 e^{-i\tau H_1}\tilde{\varphi}\| \le \|a(x)\, e^{-i\tau H_1}\tilde{\varphi}\|.$$

We prove as above that

$$\|a(x)\, e^{-i\tau H_1}\tilde{\varphi}\| \le C_l\,(1+|\tau|)^{-l}\,\|\tilde{\varphi}\|,\; l=1,2,3,\dots \qquad (5.12)$$

Moreover, by (2.1), (2.3) and as $a(x+\tau\hat{v})\varphi(x)=0$

$$a(x)\, e^{-i\tau H_1}\tilde{\varphi} = a(x)\, e^{-i\tau H_1}(\tilde{\varphi}-\varphi) + a(x)\, e^{-i\tau H_1}\varphi$$

$$= a(x)\, e^{-i\tau H_1}(\tilde{\varphi}-\varphi) + e^{-i(\hat{p}\cdot\hat{v}\tau + mv\tau/2)}\, a(x+\hat{v}\tau)\,(e^{-iH_0\frac{\tau}{v}} - I)\,\varphi.$$

But then

$$\|a(x)\, e^{-i\tau H_1}\tilde{\varphi}\| \le C\,\frac{(1+|\tau|)}{v}\,\|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}, \qquad (5.13)$$

where we used that

$$\|(e^{-iH_0\frac{\tau}{v}} - I)\varphi\| \le C\,\frac{|\tau|}{v}\,\|\hat{p}^2\varphi\|,$$

$$\|\tilde{\varphi}-\varphi\| \le \frac{C}{v}\,\|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}.$$

Interpolating (5.12) and (5.13) we obtain that

$$\|a(x)\, e^{-i\tau H_1}\tilde{\varphi}\| \le C_{\delta,l}\,\frac{1}{v^{\delta}}\,(1+|\tau|)^{-l}\,\|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)},\; l=1,2,\dots,$$

$$0 \le \delta < 1. \qquad (5.14)$$

We designate

$$I(v) := \int_{-\infty}^{+\infty} \gamma(v,\tau)\, d\tau, \text{ where}$$

$$\gamma(v,\tau) := \left[\|a(x)\, e^{-iH_1\tau}\tilde{\varphi}\|^2 + \varepsilon v^{-4}(1+|\tau|)^{-4}\right]^{\frac{1}{2}},$$

$\varepsilon > 0.$

By (5.14)

$I(\bar{v}) < \infty$ and $\lim_{v\to\infty} I(\bar{v}) = 0$

Moreover, as

$$\|a(x)\, e^{-iH_1\tau}\tilde{\varphi}\|^2 = \|a(x+\hat{v}\tau)\, e^{-iH_0\frac{\tau}{v}}\tilde{\varphi}\|^2,$$

we have that

$$\frac{\partial}{\partial v}\gamma(\bar{v},\tau) \leq C\left[\frac{|\tau|}{v^2}\|a(x+\hat{v}\tau)\, e^{-iH_0\frac{\tau}{v}} H_0\tilde{\varphi}\| + v^{-3}(1+|\tau|)^{-2}\right].$$

We prove as above, using Corollary 2.2 with $Q=0$ (note that $a(x)$ has compact support) that

$$\|a(x+\tau\hat{v})\, e^{-iH_0\frac{\tau}{v}} H_0\tilde{\varphi}\| \leq C_e(1+|\tau|)^{-e}\|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)},$$

and we obtain that

$$\left|\frac{\partial}{\partial v}\gamma(\bar{v},\tau)\right| \leq \frac{C}{v^2}(1+|\tau|)^{-2}\|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}.$$

This implies that

$$\left|\frac{\partial}{\partial v}I(\bar{v})\right| \leq \frac{C}{v^2}\|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}, \text{ and that,}$$

$$I(\bar{v}) = -\int_v^\infty \frac{\partial}{\partial s} I(s\hat{v})\, ds \leq \frac{C}{v}\|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}.$$

Hence

$$\int_{-\infty}^{\infty}\|\Gamma_3\, e^{-i\tau H_1}\tilde{\varphi}\|\, d\tau \leq I(\bar{v}) \leq \frac{C}{v}\|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}. \qquad (5.15)$$

By (5.6), (5.9), (5.10), (5.11) and (5.15),

$$\left\| \lim_{t\to\infty} \left[e^{itH_2} \chi(x) e^{-itH_1} - \chi(x)\, e^{-iL^{(t)}_{\tilde{A}(P,Q),C}} \right] \tilde{\varphi} \right\|$$

$$\leq \frac{C}{v} \|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)},$$

what proves the lemma for $A^{(P,Q)}$.

Let us now prove it for $A \in \mathcal{A}_K(\alpha_K, B_Q)$ using the gauge transformation formula in Lemma 5.4. We have that (see Lemma 5.3) $A = A^{(P,Q)} + \nabla\lambda$ and $\left|\lambda_\infty\left(\hat{v} + \frac{p}{mv}\right) - \lambda_\infty(\hat{v})\right| \leq C\left|\frac{p}{mv}\right|$, for $|p| \leq \frac{mv}{2}$.

Then as λ_∞ is homogeneous of degree zero,

$$\left\| \left(e^{-imv\cdot x} W_+(A)\, e^{imv\cdot x} - e^{-iL^{(\infty)}_{A,v}} \right) \varphi \right\|$$

$$= \left\| e^{i\lambda(x)} \left(e^{-imv\cdot x} W_+(A^{(P,Q)}) e^{imv\cdot x}\, e^{-i\lambda_\infty\left(\hat{v}+\frac{p}{mv}\right)} - e^{-i\lambda_\infty(\hat{v})}\, e^{-iL^{(\infty)}_{A^{(P,Q)}}} \right) \varphi \right\| \leq$$

$$\leq \left\| \left(e^{-imv\cdot x} W_+(A^{(P,Q)}) e^{imv\cdot x} - e^{-iL^{(\infty)}_{A^{(P,Q)}}} \right) e^{-i\lambda_\infty(\hat{v})} \varphi_0 \right\| + \left\| \left(e^{-i\lambda_\infty\left(\hat{v}+\frac{p}{mv}\right)} - e^{-i\lambda_\infty(\hat{v})} \right) \varphi \right\|$$

$$\leq \frac{C}{v} \|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}. \qquad \square$$

We now prove our main result on this section

<u>Theorem 5.6</u>

Let Ω_0 be a compact subset of $\Omega_{\hat{v}}$, $\hat{v} \in \mathbb{R}^2 \setminus \{0\}$. Then $\forall\, a_k$ and all $A \in \mathcal{A}(a_k, \beta_R)$ there is a constant C such that

$$\left\| \left(e^{-im\bar{v}\cdot x} S(A)\, e^{im\bar{v}\cdot x} - e^{i\int_{-\infty}^{+\infty} \hat{v}\cdot A(x+\tau\hat{v})\,d\tau} \right) \varphi \right\|$$

$$\leq \frac{C}{v} \|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)}, \qquad (5.16)$$

for all $\varphi \in \mathcal{H}_2(\mathbb{R}^2)$ with support in Ω_0

Proof: We denote $W_{\pm,\hat{v}} = e^{-im\bar{v}\cdot x} W_{\pm}(A)\, e^{im\bar{v}\cdot x}$

Then

$$\left\| \left(e^{-im\bar{v}\cdot x} S(A)\, e^{im\bar{v}\cdot x} - e^{i\int_{-\infty}^{+\infty} \hat{v}\cdot A(x+\tau\hat{v})\,d\tau} \right) \varphi \right\|$$

$$= \left\| W^*_{+,\hat{v}} W_{-,\hat{v}} \varphi - W^*_{+,\hat{v}} W_{+,\hat{v}}\, e^{i(L_{A,\hat{v}}(\infty) - L_{A,\hat{v}}(-\infty))} \varphi \right\|$$

$$\leq \left\| \left(W_{-,\hat{v}} - e^{-iL_{A,\hat{v}}(-\infty)} \right) \varphi - \left(W_{+,\hat{v}} - e^{-iL_{A,\hat{v}}(\infty)} \right) e^{i(L_{A,\hat{v}}(\infty) - L_{A,\hat{v}}(-\infty))} \varphi \right\| \leq \frac{C}{v} \|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)},$$

where we used that

$$\| e^{\pm i L_{A,\hat{v}}(\pm\infty)} \varphi \|_{\mathcal{H}_2(\mathbb{R}^2)} \leq C \|\varphi\|_{\mathcal{H}_2(\mathbb{R}^2)},$$

what is proven using (5.7, 5.8) to compute $\bar{p}^2 e^{\pm i L_{A,\vartheta}^{n}(t\omega)} \varphi_-$

$\square$

Proof of Equation (5.9)

For simplicity we denote $L_{\tilde{A}(\beta,\alpha\vartheta)}(t)$ by $L(t)$.
We have that,

$$\frac{1}{2m\vartheta}(\bar{p}-A)^2 e^{iL(t-\tau)}\chi(x) = \frac{1}{2m}(\bar{p}-\tilde{A}+(A-\tilde{A}))((\bar{p}-\tilde{A})-(A-\tilde{A}))$$

$$e^{-iL(t-\tau)}\chi(x) = d_1+d_2 \qquad (5.17)$$

$$d_1 = \frac{1}{2m\vartheta}(\bar{p}-\tilde{A})^2 e^{iL(t-\tau)}\chi(x), \qquad (5.18)$$

$$d_2 = \frac{-1}{2m\vartheta}\left[(\bar{p}-\tilde{A})(A-\tilde{A})+(A-\tilde{A})(\bar{p}-\tilde{A})+(A-\tilde{A})^2\right]e^{-iL(t-\tau)}\chi(x) \qquad (5.19)$$

Using (5.7) we compute,

$$d_1 = \frac{1}{2m\vartheta}(\bar{p}-\tilde{A})\, e^{iL(t-\tau)}(\bar{p}-b(x,t-\tau))\chi(x)$$

$$= \frac{1}{2m\vartheta}e^{-iL(t-\tau)}(\bar{p}-b(x,t-\tau))^2\chi(x) =$$

$$= \frac{1}{2m\vartheta}e^{iL(t-\tau)}(\bar{p}^2-\bar{p}\cdot b-b\cdot\bar{p}+b^2)\chi(x)$$

$$= \frac{1}{2m\vartheta}e^{iL(t-\tau)}\Big\{\chi(x)\bar{p}^2-(\Delta\chi)(x)+2(\bar{p}\chi)\cdot\bar{p}-\chi(x)\left[\bar{p}\cdot b+b\cdot\bar{p}-b^2\right]$$
$$-2b(p\chi)\Big\}. \qquad (5.20)$$

$$d_2 = \frac{-1}{2m\vartheta}e^{-iL(t-\tau)}\Big\{\chi(x)\left(2(A-\tilde{A})(\bar{p}-b(x,t-\tau))+(\bar{p}(A-\tilde{A}))+(A-\tilde{A})^2\right)$$

$$+2(A-\tilde{A})(\bar{p}-\chi)\Big\} = -\frac{e^{-iL(t-\tau)}}{2mv}\Big\{\chi(x)\Big((p\cdot(A-\tilde{A})) + (|\tilde{A}|^2-|A|^2)$$

$$+2(A-\tilde{A})\cdot(\tilde{A}+\bar{p}-p)\Big) + 2(\bar{p}\cdot\chi)(A-\tilde{A})\Big\} \qquad (5.21)$$

Then, by (5.17 – 5.21)

$$H_2 e^{-iL(t-\tau)}\chi(x) = \frac{1}{v} e^{-imv\cdot x}\frac{(\bar{p}-A)^2}{2mv} e^{imv\cdot x}$$

$$= T_1 + T_2 + \frac{\chi}{2mv} e^{-iL(t-\tau)}\Big(2(\bar{p}\cdot\chi)m\vec{v} - 2b\cdot m\vec{v} - 2(A-\tilde{A})m\vec{v}\Big) + e^{-iL(t-\tau)}\chi H_1$$

$$= T_1 + T_2 + T_3 - \chi\,\tilde{A}(x+\hat{v}(t-\tau))\cdot\hat{v} + e^{-iL(t-\tau)}\chi H_1,$$

and it follows that

$$H_2 e^{-iL(t-\tau)}\chi(x) - e^{-iL(t-\tau)}\chi(x)\left[H_1 - \hat{v}\cdot A(x+\hat{v}(t-\tau))\right]$$

$$= T_1 + T_2 + T_3,$$

what proves equation (5.9).

□

Reconstruction of the Magnetic Flux in K and B_R

Recall that the Coulomb potential for the regular magnetic field, B_R, was defined as,

$$A_R(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} B_R(x-y)\begin{bmatrix}-y_2\\ y_1\end{bmatrix}\frac{1}{|y|^2}\,dy, \qquad (5.22)$$

where we extended B_R to $\mathbb{R}^2$ as a C^1-function and that we proved that it satisfies;

$$A_R(x) = \frac{1}{2\pi}\frac{1}{|x|^2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}\int_{\mathbb{R}^2} B_R(y)\,dy + O\left(\frac{1}{|x|^2}\right),\ |x| \to \infty \qquad (5.23)$$

The Coulomb potential in the unshielded case $K = \{0\}$ was defined as,

$$A_{c,\{0\}} := A_S + A_R, \quad \text{where}$$

$$A_S := \frac{\alpha_{\{0\}}}{|x|^2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, \quad \alpha_{\{0\}} = \frac{B_S}{2\pi}, \quad \nabla \times A_S = B_S\,\delta(x). \qquad (5.24)$$

We have that

$$\nabla \times A_{c,\alpha} = B_S\,\delta(x) + B_R, \quad \operatorname{div} A_{c,\alpha} = 0.$$

In the shielded case the Coulomb potential was defined as

$$A_{c,K} := \frac{(\alpha_K - \alpha_R)}{|x|^2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + A_R(x), \qquad (5.25)$$

where

$$\alpha_R := \frac{1}{2\pi}\int_{\partial K} A_R.$$

We have that

$$\int_{\partial K} A_{c,K} = \alpha_K, \quad \text{as required.}$$

Equation (5.24) is a particular case of (5.25) if we take $\alpha_R = 0$ in the case $K = \{0\}$.

Let us denote

$$\alpha_{\mathbb{R}^2} := \frac{1}{2\pi}\int_{\mathbb{R}^2} B_\Omega(y)\,dy. \qquad (5.26)$$

Then, by (5.23)

$$A_{c,K} = \frac{(\alpha_K - \alpha_\Omega + \alpha_{\mathbb{R}^2})}{|x|^2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + O\left(\frac{1}{|x|^2}\right),\ |x| \to \infty. \qquad (5.27)$$

Note that, in the shielded case,

$$\alpha_{\mathbb{R}^2} - \alpha_\Omega = \frac{1}{2\pi}\int_{\mathbb{R}^2} B_\Omega(y)\,dy - \frac{1}{2\pi}\int_{\partial K} A_\Omega = \frac{1}{2\pi}\left[\int_{\mathbb{R}^2} B_\Omega(y) - \int_K B_\Omega(y)\,dy\right]$$

$$= \frac{1}{2\pi}\int_\Omega B_\Omega(y)\,dy := \alpha_\Omega.$$

Whence,

$$A_{c,K} = \frac{(\alpha_K + \alpha_\Omega)}{|x|^2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + O\left(\frac{1}{|x|^2}\right),\ |x| \to \infty. \qquad (5.28)$$

This shows that the leading term as $|x| \to \infty$ is independent of how we extended B_Ω to K. $(\alpha_K + \alpha_\Omega)$ is the total magnetic flux, normalized by 2π.

Integrating in the complex plane we prove that

$$(\alpha_K + \alpha_\Omega)\int_{-\infty}^{+\infty} \frac{1}{|x + \sigma\hat{v}|^2}\,\hat{v}\cdot(-x_2 - \sigma\hat{v}_2,\ x_1 + \sigma\hat{v}_1)\,d\sigma = \pm(\alpha_K + \alpha_\Omega)\pi, \qquad (5.29)$$

for $x \in \Omega^{\pm}_{\hat{v}} := \{x \in \Omega_{\hat{v}} : \pm x\cdot(\hat{v}_2, -\hat{v}_1) > 0\}.$

For $A \in \mathcal{A}_K(\alpha_K, B_R)$ denote

$$a(A, \hat{v}) := \int_{-\infty}^{+\infty} \hat{v} \cdot A(x + \tau \hat{v})\, d\tau$$

By (5.28), (5.29) for $x \in \Omega^{\pm}_{\hat{v}}$,

$$a(x, A_{c,K}, \hat{v}) = \pm(\alpha_K + \alpha_\Omega)\pi + O\left(\frac{1}{|x_\perp|}\right), \quad |x_\perp| \to \infty, \tag{5.30}$$

where $x_\perp = x - (x \cdot \hat{v})\hat{v}$.

Using the gauge transformation formula (see Lemma 5.3) with $A^{(2)} = A_{c,K}$, we obtain that

$$a(x, A, \hat{v}) = \pm(\alpha_K + \alpha_\Omega) + O\left(\frac{1}{|x_\perp|}\right) + \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}), \tag{5.31}$$

for $x \in \Omega^{\pm}_{\hat{v}}$.

By Theorem 5.6 from the high-velocity limit of $S(A)$ we uniquely reconstruct

$e^{i a(x, A, \hat{v})}$, $x \in \Omega_{\hat{v}}$. But by (5.31)

$$\lim_{\substack{|x_\perp| \to \infty \\ x \in \Omega^{\pm}_{\hat{v}}}} e^{i a(x, A, \hat{v})} = e^{i(\pm(\alpha_K + \alpha_\Omega)\pi + \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}))}$$

Then if $S(A^{(1)}) = S(A^{(2)})$, $A^{(1)}, A^{(2)} \in \mathcal{A}_K(\alpha_K, B_R)$, and denoting $\alpha_j := \alpha_{K,j} + \alpha_{\Omega,j}$, where $\alpha_{K,j}, \alpha_{\Omega,j}$ are, respectively, the magnetic fluxes for A_1 and A_2, we have that,

$$\alpha_1 \pi + \lambda_{1,\infty}(\hat{v}) - \lambda_{1,\infty}(-\hat{v}) = \alpha_2 \pi + \lambda_{2,\infty}(\hat{v}) - \lambda_{2,\infty}(-\hat{v}) + 2\pi k(\hat{v}), \qquad (5.32)$$

where the function $k(\hat{v})$ takes integer values. But as $\lambda_{j,\infty}(\hat{v})$ are continuous functions for $\hat{v}$ in the unit circle the function k is a constant independent of $\hat{v}$.

Then, substracting from both sides of (5.32) the same equation for $-\hat{v}$ we obtain that

$$2[\lambda_{1,\infty}(\hat{v}) - \lambda_{1,\infty}(-\hat{v})] = 2[\lambda_{2,\infty}(\hat{v}) - \lambda_{2,\infty}(-\hat{v})],$$

and then

$$\alpha_1 \pi = \alpha_2 \pi + 2\pi k, \qquad (5.33)$$

what proves that

$$\alpha_1 = \alpha_2 \text{ modulo } 2. \qquad (5.34)$$

Let us use the notation,

$$I(x,y,\overset{n}{v}) := \int_{-\infty}^{+\infty} \left(\overset{n}{v}\cdot A(x+\tau\overset{n}{v}) - \overset{n}{v}\cdot A(y+\tau\overset{n}{v}) \right) d\tau,$$

$$R_{\pm}(x,y,\overset{n}{v}) = e^{iI(x,y,\overset{n}{v})}, \quad x,y \in \Omega_{\overset{n}{v}}.$$

As $I(x,y,\overset{n}{v})$ is gauge invariant we can compute it with $A_{c,u}$. Then, by (5.22, 5.23), (5.25) (see also (5.29))

$I,(x,y,\overset{n}{v}), R(x,y,\overset{n}{v}) \in C^1(\Omega^{\pm}_{\overset{n}{v}}, \Omega^{\pm}_{\overset{n}{v}})$.

Then, by Theorem 5.6 from the high-velocity limit of $S(A)$ we reconstruct

$I(x,y,\overset{n}{v}) + 2\pi h(\overset{n}{v})$, where $h(\overset{n}{v})$ takes integer values and it is constant for $x,y \in \Omega^{\pm}_{\overset{n}{v}}$ because I is continuous.

Suppose that $\overset{n}{v} = (0,1)$. Using the gauge $A^{(F,Q)}$ given in page 43, taking the axis of the cone along the x_1 coordinate, we obtain that

$$\frac{\partial}{\partial x_1} I(x,y,\overset{n}{v}) = \frac{\partial}{\partial x_1} \int_{-\infty}^{+\infty} \overset{n}{v}\cdot A^{(F,Q)}(x+\tau\overset{n}{v})\, d\tau =$$

$$= \int_{-\infty}^{+\infty} \frac{\partial}{\partial x_1} A_2^{(F,Q)}(x+\tau\overset{n}{v})\, d\tau = \int_{-\infty}^{+\infty} B_{\Omega}(x+\tau\overset{n}{v})\, d\tau, \quad (5.35)$$

where we used that by our choice of gauge

$$-\int_{-\infty}^{+\infty} \frac{\partial}{\partial x_2} A_1^{(F,Q)}(x+\tau\overset{n}{v})\, d\tau = -\int_{-\infty}^{+\infty} \frac{\partial}{\partial x_2} A_1^{(F,Q)}(x_1,x_2)\, dx_2 = 0.$$

Remark 5.8.

Note that we only need to know the high-velocity limit of $S(A)$. See Theorem 5.6. Moreover if $B_\Omega \equiv 0$ we determine α_K modulo 2 without requiring that K is convex.

Remark 5.9

We can reconstruct α, and then also α_K, in a gauge invariant way as follows.

By (5.31)

$$\lim_{\substack{|v|\to\infty \\ x\in\Omega^+_{\hat{v}},\, y\in\Omega^-_{\hat{v}}}} R(x, y, \hat{v}) = e^{2\pi i\alpha}. \tag{5.36}$$

Then, from the high-velocity limit of $S(A)$ we reconstruct $2\pi\alpha$ modulo 2π, i.e., we reconstruct α modulo 1. Since B_Ω, and in consequence α_{Ω}, was reconstructed in a gauge invariant way, with this method

We write (5.35) in coordinate independent form as follows

$$\tilde{\hat{v}}\cdot\nabla_x f(x,y,\hat{v}) = \int B_\Omega(x+\sigma\hat{v})\,d\sigma, \text{ where } \quad (5.36)$$
$$\tilde{\hat{v}} = (\hat{v}_2, \hat{v}_1).$$

Then, by Theorem 5.6 from the high-velocity limit of $S(A)$ we uniquely reconstruct all the integrals (5.36) for $x\in\Omega_{\hat{v}}$, and if K is convex, by the support theorem for the Radon transform we uniquely reconstruct B_Ω in Ω.

Once B_Ω is known we compute α_Ω and by (5.34) $\alpha_{K,1} = \alpha_{K,2}$ modulo 2.

Hence, we have proven the following theorem.

<u>Theorem 5.7</u>. Assume that K is convex. Then, for any $A\in\mathcal{A}_K(\alpha_K, B_\Omega)$ the scattering operator $S(A)$ uniquely determines B_Ω and it determines α_K modulo 2.

we reconstruct α_K modulo 1 in a gauge invariant way.

Note that as if $S(A^{(1)}) = S(A^{(2)})$ we have that

$$\lambda_{1,\infty}(\hat{v}) - \lambda_{1,\infty}(-\hat{v}) = \lambda_{2,\infty}(\hat{v}) - \lambda_{2,\infty}(-\hat{v}),$$

$$\lim_{|x_\perp| \to \infty} e^{i a(x, A, \hat{v})} = e^{i(\pm(\alpha_K + \alpha_\varepsilon) + \lambda_\infty(\hat{v}) - \lambda_\infty(-\hat{v}))}$$

has a restricted gauge invariance, namely it is gauge invariant for all the potentials that have the same high-velocity limit for the scattering operator, i.e., given $A \in \mathcal{A}_K(\alpha_K, B_R)$ it is the same for all $\tilde{A} \in \mathcal{A}_K(\alpha_K, B_R)$ such that $S(\tilde{A})$ has the same high-velocity limit as $S(A)$.