Reconstruction of the Potential

We identify the vector $z=(z_1,z_2)\in\mathbb{R}^2$ with the vector $(z_1,z_2,0,0,\dots 0)\in\mathbb{R}^n$, or any other two-dimensional plane.

Then, for any φ,ψ with $\hat\varphi,\hat\psi\in C_0^\infty(\mathbb{R}^n)$ we define by

$$\varphi(z)=e^{-i\bar p\cdot z}\varphi,\quad \psi(z)=e^{-i\bar p\cdot z}\psi$$ the

states translated by z.

Define

$$f(z)=(V\varphi(z),\psi(z))=(Vg(\bar p)\varphi(z),\psi(z)),$$

$g\in C_0^\infty$ is such that $g(\bar p)\varphi=\varphi$.

Then $f\in C^\infty$, f is bounded and,

$$|f(z)|\le \|Vg(\bar p)\varphi(z)\|\,\|\psi\|\le\|\psi\|\left[\|Vg(\bar p)F(|x|\geq\tfrac{|z|}{2})\|\right.$$

$$\left.+\|Vg(\bar p)\|\,\|F(|x|<|z|/2)\,e^{-i\bar p\cdot z}\varphi\|\right]\in L^2(\mathbb{R}^2,dz).$$

The Radon transform of $f(z)$ for $\hat v$ in the z-plane is given by

$$\check f(\hat v,z)=\int_{-\infty}^{+\infty} f(z+\tau\hat v)\,d\tau=$$

$$=\int_{-\infty}^{+\infty}(V(x+\tau\hat v)\varphi(z),\psi(z))\,d\tau=$$

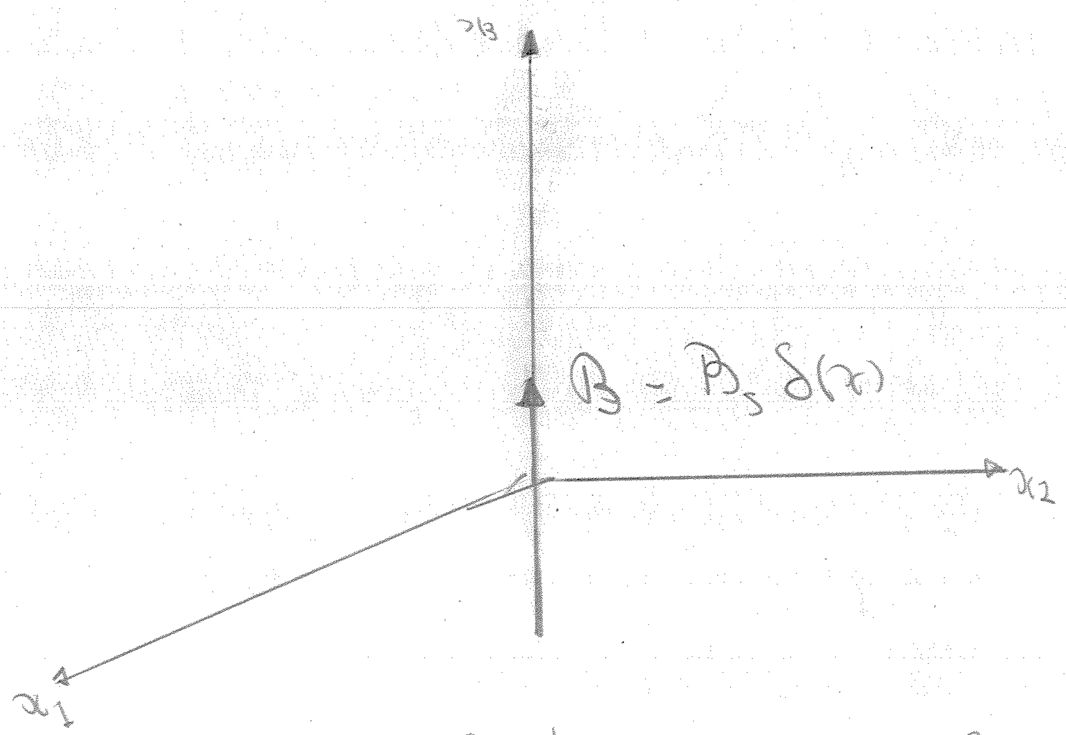
$$=\lim_{v\to\infty}(i(S-I)\varphi_{\bar v},\psi_{\bar v}).$$

From the Radon transforms $\check{f}(\hat{v}, \tau)$ we uniquely reconstruct $f(z)$, and in particular

$f(0) = (V\varphi, \psi)$.

Since $\{\varphi : \hat{\varphi} \in C_0^\infty\}$ is dense in L^2 we reconstruct $V(x)$ a.e.w., or if we think of V as an operator we reconstruct the scalar products of V in a basis.

5. The Aharonov-Bohm Effect



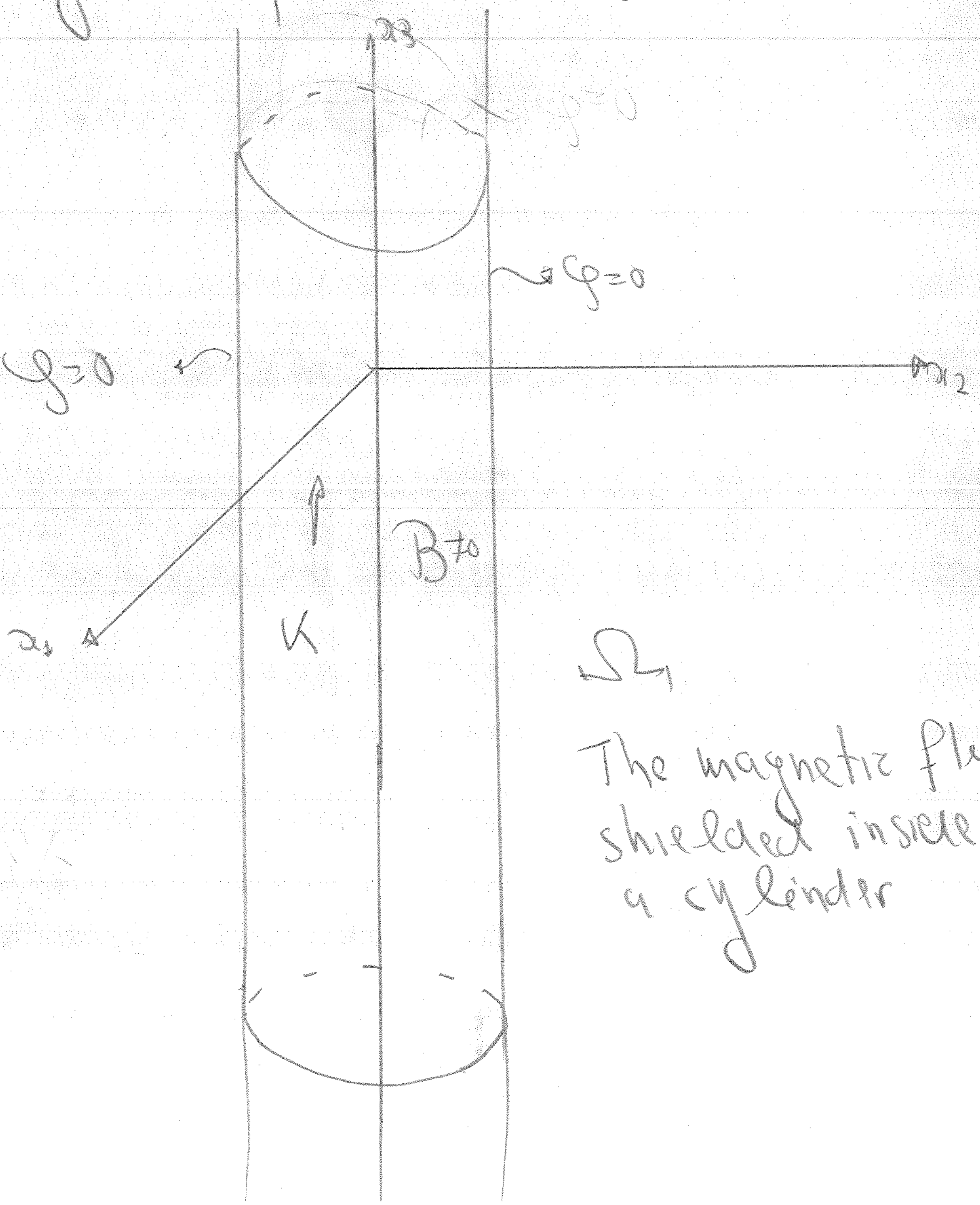
Unshielded infinitely long solenoid that is infinitely thin

Schrödinger equation in $\mathbb{R}^2$

$i\frac{\partial}{\partial t}\varphi = \frac{1}{2m}(\bar{p}-A)^2\varphi, \quad B=\nabla\times A \quad , \bar{p}=-i\nabla$

$\varphi(0)=0.$

Note that the electron is never at $x=0$ (i.e. the probability is zero), $B=0$ for $x\neq 0$ but still the electron feels the magnetic flux at $x=0$.



Ω_1

The magnetic flux shielded inside a cylinder

Now the magnetic flux is shielded by an infinite cylinder

By translation invariance along x_3 we reduce the problem to $\mathbb{R}^2$.

$$i \frac{\partial}{\partial t} \varphi = \frac{1}{2m} (\bar{p} - A)^2 \varphi, \quad \text{in } \Omega = \mathbb{R}^2 \setminus K.$$

K a compact set that is the cross-section of the cylinder.

$$\varphi \Big|_{\partial\Omega} = 0.$$

Even though $B = 0$ in Ω the electron feels the magnetic flux shielded inside K.

In classical mechanics the force on the electron is proportional $q\, \dot{x}(t) \times B(x(t))$. Then, if $B = 0$ the force is zero and the electron moves in a straight line. The Aharonov-Bohm effect is a genuinely quantum effect

Note that if the magnetic flux is

nontrivial the magnetic potentials have to be long-range, since if

$|A(x)| \leq C(1+|x|)^{-\delta}$, $\delta > 1$, by Stokes theorem

$$\Phi = \lim_{R\to\infty} \int_{|x|\leq R} B\,dx = \lim_{R\to\infty} \int_{|x|=R} A = 0.$$

This is an example of long-range effects in quantum mechanics. Even if the magnetic potential is very small far away we can not assume that it is zero, or that it is short-range, unless the flux is zero. So no matter how small the flux is, the magnetic potentials are long-range. However, since the magnetic field will have compact support the wave operators will exist without modifying the free dynamics, what is physically reasonable.

We assume that there is also a regular magnetic field for $x \neq 0$ in the case of the unshielded solenoid, and for $x \in \Omega$ in the shielded case. This allows us to take into account the imperfections of the solenoid and of the cylinder.

We define appropriate classes of magnetic potentials.

The Unshielded Case

We assume that

$$B = B_s \delta(x) + B_R, \qquad B_R \in C_0^1(\mathbb{R}^2).$$

We define

$$A_s := \frac{\alpha_{\{0\}}}{|x|^2} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, \quad \alpha_{\{0\}} = \frac{B_s}{2\pi}, \text{ is the flux}$$

normalized by 2π.

$$\nabla \times A_s = B_s \delta(x), \quad \operatorname{div} A_s = 0.$$

The Coulomb potential for B_R is given by

$$A_R^{(c)} := \frac{1}{2\pi} \int B_R(x-y) \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix} \frac{1}{|y|^2} \, dy.$$

$A_R \in C^1$, $\nabla \times A_R = B_R$, $\operatorname{div} B_R = 0$

The magnetic potential in the Coulomb gauge is given by

$A_{c,f,d,} := A_s + A_R$.

The Shielded case

$K \subset \mathbb{R}^2$, $0 \in \mathring{K}$, K is compact with boundary ∂K a simple, closed, C^1-curve.

In $\Omega := \mathbb{R}^2 \setminus K$ we have a regular magnetic field $B_R \in C_0^1(\overline{\Omega})$.

We extend B_R to a C^1-function on $\mathbb{R}^2$.

The Coulomb potential for B_R is defined as in the unshielded case.

We wish to fix the magnetic flux inside K. According to Stokes theorem this amounts to fixing the circulation of the magnetic potential on ∂K.

So, we wish to construct $A \in C^1(\overline{\Omega})$ with $\nabla \times A = B_R$ and such that for $\alpha_K \in \mathbb{R}$:

$$\alpha_K = \frac{1}{2\pi} \int_{\partial K} A(x).$$

Denote

$$\alpha_R := \frac{1}{2\pi}\int_{\partial K} A_R.$$

Then, the Coulomb potential,

$$A_{c,K} := \frac{(\alpha_K - \alpha_R)}{|x|^2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + A_R(x)$$

satisfies

$$\nabla \times A_{c,K} = B_R, \quad \frac{1}{2\pi}\int_{\partial K} A_{c,K} = \alpha_K.$$

We introduce now general classes of potentials that are natural for our problem.

<u>Definition 5.1</u> (Unshielded Case)

We denote by $\mathcal{A}_{\{0\}}(\alpha_{\{0\}}, B_R)$ the set of all real-valued $A \in C^1(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}^2) \cap L^1_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ with $\nabla \times A = 2\pi\alpha_{\{0\}}\delta(x) + B_R$ in $\mathcal{D}'$ such that $A(x) = O(\frac{1}{|x|})$, $|x| \to \infty$, and

$$a(r) := \sup_{|x| \geq r} |A(x)\hat{x}| \in L^1([0,\infty)).$$

$\square$

Clearly $A_{c,\{0\}} \in \mathcal{A}(\alpha_{\{0\}}, B_R)$.

Definition 5.2 (Shielded Case)

We denote by $\mathcal{A}_K(\alpha_K, B_\Omega)$ the set of all real-valued $A \in C^1(\bar{\Omega}, \mathbb{R}^2)$ with $\nabla \times A = B_\Omega$ and

$$\alpha_K = \frac{1}{2\pi}\int_{\partial K} A.$$

Moreover, we assume that $A(x) = O(\frac{1}{|x|})$, $|x| \to \infty$, and that

$$a(r) := \sup_{x \in \Omega, |x| \geq r} |A(x) \cdot \hat{x}| \in L^1([0,\infty)).$$

$\square$

Clearly $A_{c,K} \in \mathcal{A}_K(\alpha_K, B_\Omega)$.

We now construct special magnetic potentials with support contained in a cone with small opening angle.

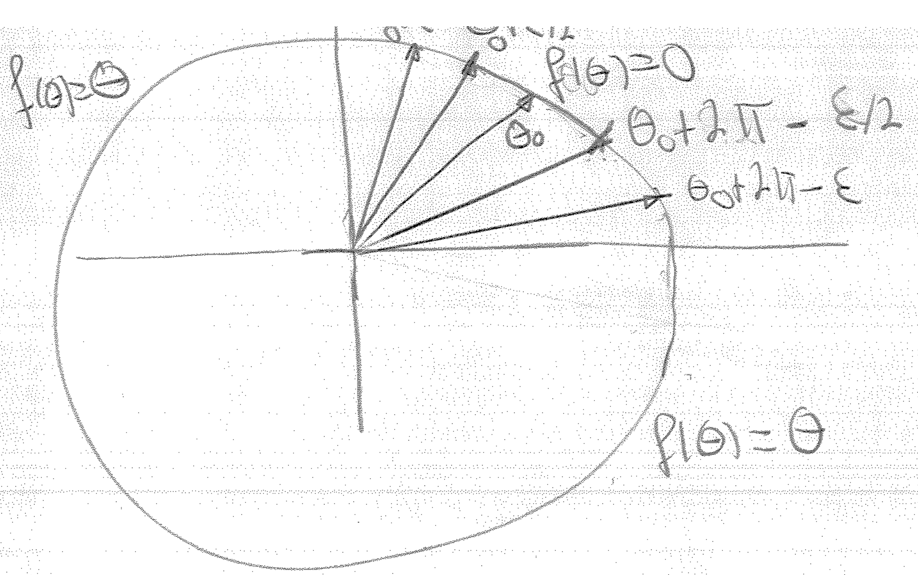
5.12.07

Let us take polar coordinates (r, θ), $r > 0$, $\theta_0 \leq \theta < \theta_0 + 2\pi$, for some $0 \leq \theta_0 < 2\pi$.

Let $f(\theta) \in C^1(\mathbb{S}^1)$ satisfy $f(\theta) = 0$, for $\theta_0 \leq \theta \leq \theta_0 + \varepsilon/2$ and for $\theta_0 + 2\pi > \theta \geq \theta_0 + 2\pi - \varepsilon/2$.

and Moreover, $f(\theta) = \theta$, for

$$\theta_0 + \varepsilon \leq \theta \leq \theta_0 + 2\pi - \varepsilon$$



We define

$$A^{(f)} := \frac{(\alpha_L - \alpha_R)}{|x|^2} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} (1 - f'(\theta)) \;,\; \alpha_L, \alpha_R \in \mathbb{R}.$$

$A^{(f)}$ has support in a cone with vertex zero, axis, $(\cos\theta_0, \sin\theta_0)$, and opening angle ε.

For any $Q = (q_1, q_2) \in \mathbb{R}^2$, let $A^{(Q)}$ be the following shifted transverse potential

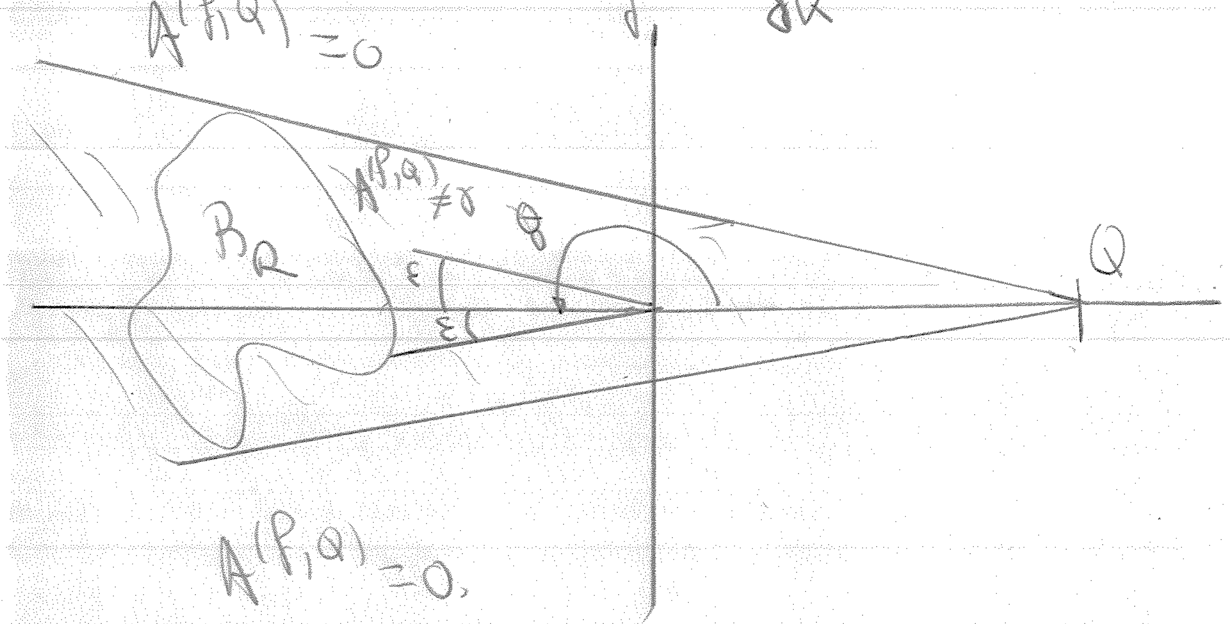
$$A^{(Q)} := \begin{bmatrix} q_2 - x_2 \\ x_1 - q_1 \end{bmatrix} \int_0^1 B_R(x\mu + (1-\mu)Q)\, d\mu$$

If we take Q far enough from the support of B_R, $A^{(Q)}$ will have support in a cone with vertex Q and opening angle as small as we wish.

We define

$$A^{(P,Q)} := A^{(P)} + A^{(Q)} \in \mathcal{A}_K(\alpha_K, B_R)$$

where if $K = \{0\}$ we take $\alpha_R = 0$, and otherwise, $\alpha_R = \frac{1}{2\pi}\int_{\partial K} A^{(Q)}$.



Let us analyse the behaviour of A_R for large $|x|$. Recall that

$$A_R(x) = \frac{1}{2\pi}\int B_R(x-y)\begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}\frac{1}{|y|^2}\,dy$$

$$= \frac{1}{2\pi}\int \begin{bmatrix} -(x-y)_2 \\ (x-y)_1 \end{bmatrix}\frac{1}{|x-y|^2} B_R(y)\,dy.$$

We have that

$$\begin{bmatrix} -(x-y)_2 \\ (x-y)_1 \end{bmatrix}\frac{1}{|x-y|^2} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}\frac{1}{|x-y|^2} + \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix}\frac{1}{|x-y|^2}$$

Suppose that support $B_R \subset B_{M/2}(0)$, the closed ball of center zero and radius $M/2$

For $|y| \le M/2$

$$\left| \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix} \frac{1}{|x-y|^2} \right| \le \frac{C}{|x|^2}, \quad |x| \ge M.$$

$$\left| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right| \le \frac{C}{|x|^3}, \quad |x| \ge M.$$

Then,

$$A_\Omega(x) = \frac{1}{2\pi}\left[\int B_\Omega(y)\,dy\right] \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \frac{1}{|x|^2} + O\!\left(\frac{1}{|x|^2}\right), \ |x| \to \infty.$$

In particular, if the flux is zero i.e. if

$\int B_\Omega(y)\,dy = 0$, then $A_\Omega(x) = O\!\left(\frac{1}{|x|^2}\right)$, $|x| \to \infty$

Gauge Transformations

Lemma 5.3

Suppose that $A^{(1)}, A^{(2)} \in \mathcal{A}_K(\alpha_K, B_\Omega)$. Then, there is a real-valued function λ, such that if $K = \{0\}$ $\lambda \in C^2(\mathbb{R}^2 \setminus \{0\}) \cap L^\infty(\mathbb{R}^2)$ and in the case of Definition 5.2 $\lambda \in C^2(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$, and moreover, $A^{(1)} = A^{(2)} + \nabla\lambda$. Furthermore, for $x \neq 0$ $\lambda_\infty(x) = \lim_{r\to\infty} \lambda(rx)$ exists and it is a continuous function on $\mathbb{R}^2 \setminus \{0\}$ that is homogeneous of degree zero, $\lambda_\infty(rx) = \lambda_\infty(x)$.

Moreover,

$|\lambda_\infty(x+y) - \lambda_\infty(x)| \le C|y|$ $\forall x$ with $|x| = 1$, and $\forall y$, $|y| \le \frac{1}{2}$.

Proof: Denote $A = A^{(1)} - A^{(2)}$. Then

$\nabla \times A = 0$. Hence, for any simple closed C^1-curve in $\Omega := \mathbb{R}^2 \setminus K$

$$\int_C A = 0,$$

by Stoke's theorem. Recall that in the case of Definition 5.2 $\int_{\partial K} A = 0$.

Take any $x_0 \in \Omega$ and define

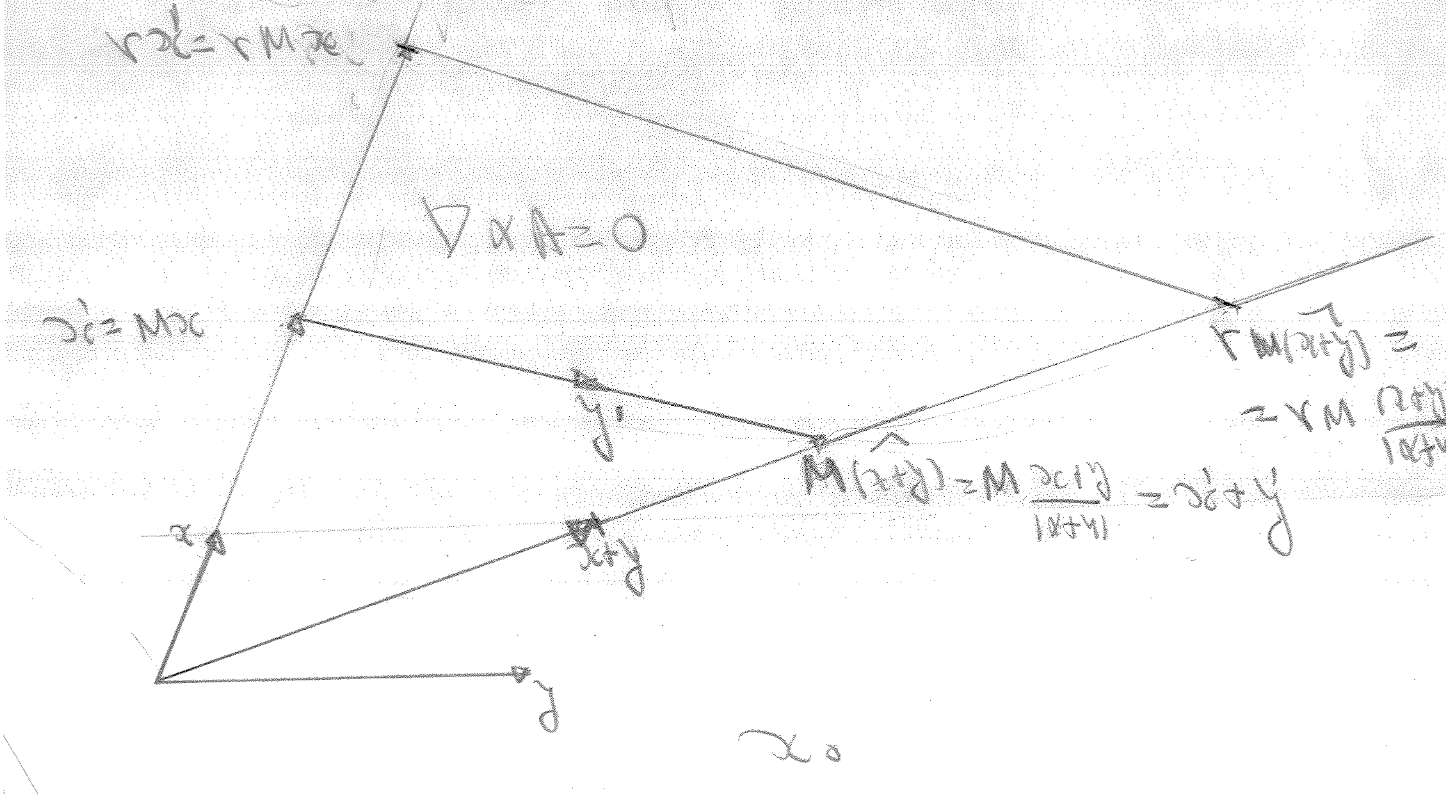
$\lambda(x) = \int_{C_{x_0}^{x}} A$, where $C_{x_0}^{x}$ is any simple C^1-curve in Ω that goes from x_0 to x. Clearly,

$A = \nabla\lambda$, i.e. $A^{(1)} = A^{(2)} + \nabla\lambda$. The existence and the continuity of $\lambda_\infty(x)$ follows from the condition

$a(r) := \sup_{x \in \Omega, |x| \geq r} |A(x) \cdot \hat{x}| \in L^1([0,\infty))$.

The continuity follows from

$$\left| \int_1^\infty \hat{x} \cdot A(r x)\, dr \right| \leq \frac{\varepsilon}{|x|}, \; |x| \geq r_0 .$$



$x' = Mx, \quad y' = M\frac{x+y}{|x+y|} - x'.$

Then, $\lambda_\infty(x) = \lambda_\infty(x')$ $\lambda_\infty(x+y) = \lambda_\infty(M\widehat{(x+y)}) = \lambda_\infty(x'+y')$

We have that

$$\lambda_\infty(x'+y') - \lambda_\infty(x') = \cancel{\int_{x_0}^{x'} A} + \int_{x'}^{x'+y'} A + \int_{x'+y'}^{\infty} A - \cancel{\int_{x_0}^{x'} A} - \int_{x'}^{\infty} A$$

$$= \int_{x'}^{x'+y'} A + \lim_{r\to\infty}\left[\int_{x'+y'}^{r} A - \int_{x'}^{r} A\right]$$

$$= \lim_{r\to\infty} \int_{rx'}^{r(x'+y')} A.$$

In the last equality we used Stoke's theorem and $\nabla \times A = 0$. Finally,

$$\lim_{r\to\infty}\left|\int_{rx'}^{r(x'+y')} A\right| \leq \lim_{r\to\infty} C\,\frac{1}{rM}\left|rM\frac{x+y}{|x+y|} - rMx\right|$$

$$= C\left|\frac{x+y}{|x+y|} - x\right|.$$

Recall that $|x| = 1$, $|y| \leq 1/2$, and then $|x+y| \geq \frac{1}{2}$.

We have that,

$$\left|\frac{x+y}{|x+y|} - x\right| \leq \frac{|y|}{|x+y|} + \left|\frac{x}{|x+y|} - x\right| \leq 2|y| + 2\left|x - x(|x+y|)\right|$$

$$= 2|y| + 2|x|\,|1 - |x+y|| = 2|y| + ||x+y| - 1|\,2$$

$$= 2|y| + ||x+y| - |x||\,2 \leq 2|y| + |x+y-x|\,2 = 4|y|.$$

Finally

$$\lim_{r \to \infty} \left| \int_{rx}^{r(x+y)} A \right| \leq C|y|,$$

and we have proved that

$$|\lambda_\infty(x+y) - \lambda_\infty(x)| \leq C|y|, \quad |x| = 1, \quad |y| \leq \frac{1}{2}.$$

$\square$

The Hamiltonian

The formal Hamiltonian is the operator

$$h_A := \frac{(\bar{p}-A)^2}{2m} \quad \text{with domain } C_0^2(\Omega); \quad \Omega := \mathbb{R}^2 \setminus K.$$

The associated quadratic form is given by

$$q_A(\varphi, \psi) := ((\bar{p}-A)\varphi, (p-A)\psi), \quad D(q_A) = C_0^1(\Omega) \times C_0^1(\Omega),$$

$A \in \mathcal{A}_K(\alpha_K, B\Omega)$

Clearly $q_A \geq 0$.

Let us prove that it is closable. Let $\varphi_n \in C_0^1(\Omega)$ $\varphi_n \to 0$ in $L^2(\Omega)$ and also

$$q_A(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \xrightarrow[n,m \to \infty]{} 0.$$

Whence,

$$\|(\bar{p}-A)(\varphi_n - \varphi_m)\| \xrightarrow[n,m \to \infty]{} 0.$$

It follows that $\exists\ \phi \in L^2(\Omega)$ such that $\lim_{n \to \infty} (\bar{p}-A)\varphi_n = \phi$ in $L^2(\Omega)$. But $\forall \psi \in C_0^\infty(\Omega)$

$$\lim_{n \to \infty} ((\bar{p}-A)\varphi_n, \psi) = \lim_{n \to \infty} (\varphi_n, (p-A)\psi) = (\varphi, (p-A)\psi)$$

Then, we have that

$$(\phi, \psi) = (\varphi, (p-A)\psi) = 0, \quad \text{since } \varphi \to 0,$$

and we have proven that $q_A \geq 0$

Let $\bar{q}_A$ be the closure of q_A.

The associated positive operator, H_A, is the Hamiltonian.

Suppose that $A^{(1)}, A^{(2)} \in A_K(a_{K_1}, B_\Omega)$ and that $A^{(2)} = A^{(1)} + \nabla\lambda$.

Take $\varphi \in D(H_{A^{(2)}})$.

Let us prove that $e^{-i\lambda}\varphi \in D(H_{A^{(1)}})$.

$\exists\, \varphi_n \in C_0^1(\Omega)$: $\varphi_n \to \varphi$ in L^2 and $\forall \psi \in C_0^1(\Omega)$

$$(H_{A^{(2)}}\varphi, \psi) = \bar{q}_{A^{(2)}}(\varphi, \psi) = \lim_{n\to\infty} q_{A^{(2)}}(\varphi_n, \psi)$$

$$= \lim_{n\to\infty}\left((\bar{p} - A^{(2)})\varphi_n, (\bar{p} - A^{(2)})\psi\right) =$$

$$= \lim_{n\to\infty}\left((\bar{p} - A^{(1)})e^{-i\lambda}\varphi_n, (\bar{p} - A^{(1)})e^{-i\lambda}\psi\right)$$

$$= \left((\bar{p} - A^{(1)})e^{-i\lambda}\varphi, (\bar{p} - A^{(1)})e^{-i\lambda}\psi\right).$$

Then

$$(H_{A^{(2)}}\varphi, \psi) = \left((\bar{p} - A^{(1)})e^{-i\lambda}\varphi, (\bar{p} - A)^{(1)}e^{-i\lambda}\psi\right).$$

<u>To be more precise:</u>

We know that $\varphi_n \to \varphi$ in $L^2(\Omega)$

$e^{-i\lambda}\varphi_n \in C_0^1(\Omega)$,

$e^{-i\lambda}\varphi_n \longrightarrow e^{-i\lambda}\varphi$ in $L^2(\Omega)$

$$q_{A^{(1)}}\left((e^{-i\lambda}\varphi_n - e^{-i\lambda}\varphi_m), (e^{-i\lambda}\varphi_n - e^{-i\lambda}\varphi_m)\right)$$

$$= q_{A^{(2)}}(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \xrightarrow[n,m\to\infty]{} 0 .$$

Then, $e^{-i\lambda}\varphi \in D(\bar{q}_{A^{(1)}})$ and

$$q_{A^{(1)}}(e^{-i\lambda}\varphi, e^{-i\lambda}\psi) = \lim_{n\to\infty} q_{A^{(1)}}(e^{-i\lambda}\varphi_n, e^{-i\lambda}\psi)$$

$$= \lim_{n\to\infty} q_{A^{(2)}}(\varphi_n, \psi) = (H_{A^{(2)}}\varphi, \psi)$$

Then, we have, $e^{-i\lambda}\varphi \in D(\bar{q}_{A^{(1)}})$ and

$$\bar{q}_{A^{(1)}}(e^{-i\lambda}\varphi, e^{-i\lambda}\psi) = (H_{A^{(2)}}\varphi, \psi).$$

But as $e^{-i\lambda}C_0^1(\Omega) = C_0^1(\Omega)$ it follows that

$$\bar{q}_{A^{(1)}}(e^{-i\lambda}\varphi, \psi) = (H_{A^{(2)}}\varphi, e^{i\lambda}\psi) \quad \forall \psi \in C_0^1(\Omega),$$

and this extends by continuity to all $\psi \in D(\bar{q}_{A^{(1)}})$. This implies that $e^{-i\lambda}\varphi \in D(H_{A^{(1)}})$ and also that

$$H_{A^{(1)}} e^{-i\lambda}\varphi = e^{-i\lambda} H_{A^{(2)}}\varphi \text{, or equivalently}$$

that

$$H_{A^{(2)}} = e^{i\lambda} H_{A^{(1)}} e^{-i\lambda} .$$

We have proven that the Hamiltonians corresponding to different gauges are unitarily equivalent. In physical terms this means that they describe the same physics.

The Wave Operators

Let J be the identification operator from $L^2(\mathbb{R}^2)$ onto $L^2(\Omega)$ given by multiplication by the characteristic function of Ω, χ_Ω, i.e.

$(J\varphi)(x) = \chi_\Omega(x)\,\varphi(x)$.

In the case $K = \{0\}$, $J = I$.

The wave operators are defined as follows,

$$W_\pm(A) := s\text{-}\lim_{t \to \pm\infty} e^{itH_A} J e^{-itH_0},$$

provided that the strong limits exist.

Lemma 5.4

The wave operators $W_\pm(A)$ exist and are isometric for all $A \in \mathcal{A}_K(\alpha_K, B_R)$.

Moreover, if $A^{(1)}, A^{(2)} \in \mathcal{A}_K(\alpha_K, B_R)$ with $A^{(2)} = A^{(1)} + \nabla\lambda$, then,

$$W_\pm(A^{(2)}) = e^{i\lambda(x)} W_\pm(A^{(1)})\, e^{-i\lambda_\infty(\pm\hat{p})}$$

Proof: Let $\chi \in C^\infty(\mathbb{R}^2)$ satisfy $\chi(x) = 0$ for x in a neighborhood of K and $\chi(x) = 1$ for $|x| \geq M$ with M large enough. Then, as $(1-\chi(x))(H_0+1)^{-1}$ is compact, (51)

$$W_\pm(A)\varphi = s\text{-}\lim_{t\to\pm\infty} e^{itH_A}\chi\, e^{-itH_0}\varphi \qquad (5.1)$$

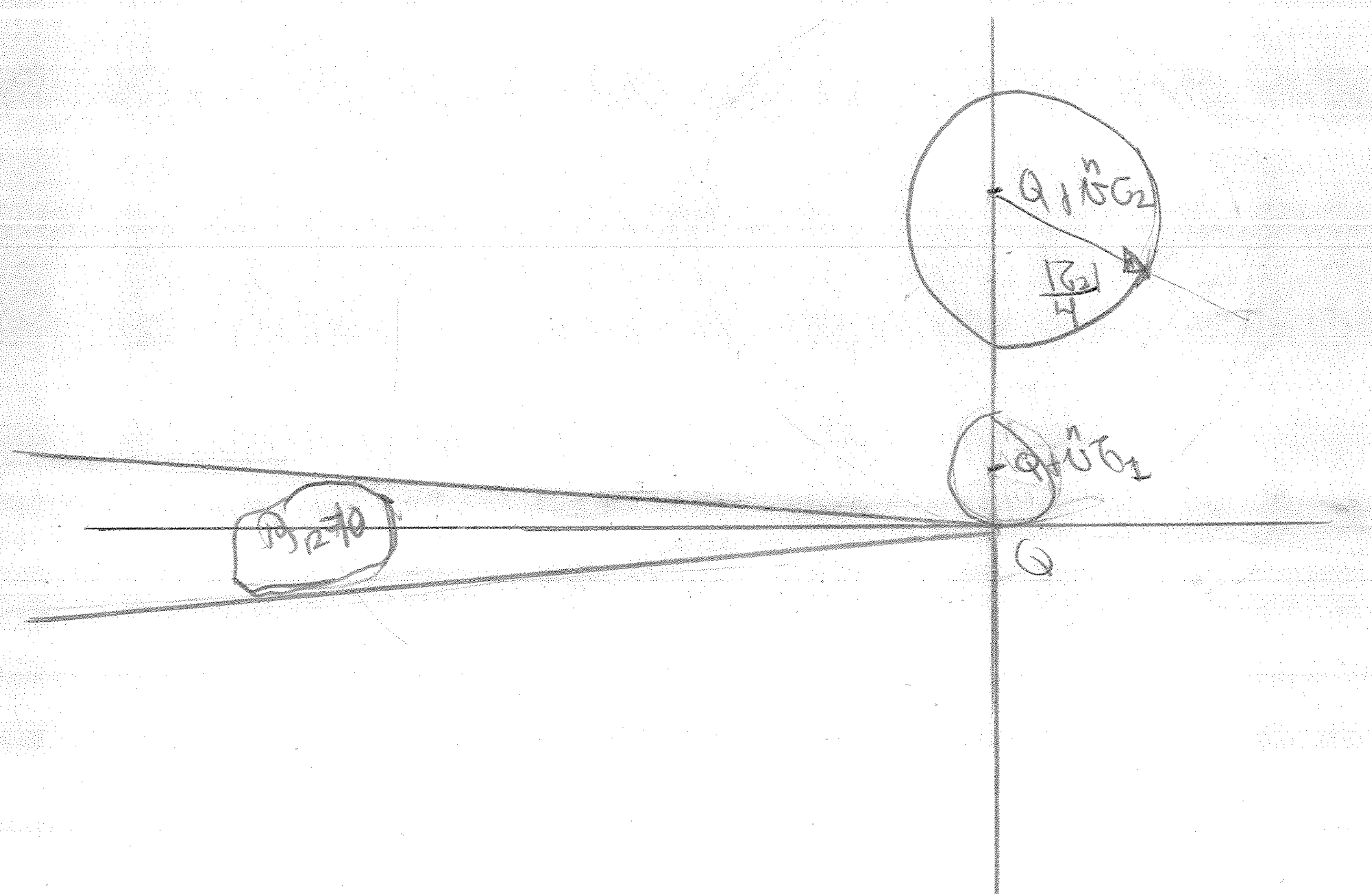
$\forall \varphi \in D(H_0)$, and by density $\forall \varphi \in L^2(\mathbb{R}^2)$.—

We give the proofs for $W_+(A)$. The case of $W_-(A)$ follows in the same way. It is enough to prove the existence of the strong limit (5.1) for all $\varphi_{\vec{v}} = e^{im\vec{v}\cdot x}\varphi_0$, $\hat{\varphi}_0 \in C_0^\infty(B_{m\eta}(0))$,

with $v > 8\eta$ since this is a dense set.

Let us take a $A^{(\beta,Q)}$ with

$$A^{(\beta,Q)}(x)\, F\left(|x - Q - \hat{v}\sigma| \le \frac{|\sigma|}{4}\right) = 0, \quad \sigma \in \mathbb{R}$$



By Duhamel's formula

$$W_+(A^{(f,Q)})\varphi_{\bar v} = \chi(x)\varphi_{\bar v} + i\int_0^\infty dt\, e^{itH_{A^{(f,Q)}}} V(x,\bar p)\cdot e^{-itH_0}\varphi_{\bar v},$$

where

$$V(x,\bar p) = \chi(x)\left[-\frac{A^{(f,Q)}\cdot \bar p}{m} + \frac{(-\bar p\cdot A^{(f,Q)}) + (A^{(f,Q)})^2}{2m}\right]$$

$$-\frac{1}{2m}\left[(\Delta\chi) + 2i(\nabla\chi)\cdot \bar p + 2A^{(f,Q)}\cdot(\bar p\chi)\right]$$

Take $g \in C_0^\infty(B_{m\eta})$ with $g \equiv 1$ on the support of $\hat\varphi_0$. Hence,

$$\|\chi A^{(f,Q)} e^{-itH_0}(\bar p - m\bar v)\varphi_{\bar v}\| \le C\big[\|\chi A^{(f,Q)} F(|x-Q-\bar v t| > |\bar v t|/4)\, e^{-itH_0} g(\bar p - m\bar v)(\bar p - m\bar v)$$

$$F(|x-Q| \le |\bar v t|/8)\|\,\|\varphi_0\| + \|F(|x-Q| > |\bar v t|/8)\varphi_0\|\big].$$

As φ_0 has rapid decay

$$\|F(|x-Q| > |\bar v t|/8)\varphi_0\| \le C_l(1+|\bar v t|)^{-l},\ l = 1,2,3,\ldots$$

By our propagation estimate, Corollary 2.2 with $f = 0$,

$$\|\chi A^{(f,Q)} F(|x-Q-\bar v t| > |\bar v t|/4)\, e^{-itH_0} g(\bar p - m\bar v)$$

$$(\bar p - m\bar v) F(|x-Q| \le |\bar v t|/8)\| \le C_l(1+|\bar v t|)^{-l},$$

$l = 1, 2, 3, \ldots$

We estimate the other terms in a similar way to obtain that

$$\| V(x, \bar{p})\, e^{-itH_0} \varphi_{\bar{v}} \| \leq C_\ell\, v\, (1+|\bar{v}t|)^{-\ell}, \quad \ell = 1,2,3.$$

The v comes from the term

$$\left\| \left[\nabla\chi \cdot \frac{A^{(f,Q)}}{m} \pm \frac{i(\Delta\chi)}{m} \right] \cdot m \bar{v}\, e^{-itH_0} \varphi_{\bar{v}} \right\|.$$

This implies that

$W_+(A^{(f,Q)}) \varphi_{\bar{v}}$ exists.

Let $A \in \mathcal{A}_{V}(q_{V_1}, B_R)$ and let λ satisfy

$$A = A^{(f,Q)} + \nabla \lambda.$$

We have that,

$$W_+(A)\varphi_{\bar{v}} = e^{i\lambda(x)} \operatorname*{s-lim}_{t \to +\infty} e^{itH_{A^{(f,Q)}}} e^{-i\lambda(x)} J e^{-itH_0} \varphi_{\bar{v}}$$

$$= e^{i\lambda(x)} \operatorname*{s-lim}_{t \to +\infty} e^{itH_{A^{(f,Q)}}} J e^{-i\lambda_\infty(x)} e^{-itH_0} \varphi_{\bar{v}}. \qquad (5.2)$$

In the last step we used the compactness of $(e^{-i\lambda(x)} - e^{-i\lambda_\infty(x)})(H_0+1)^{-1}$, that follows from Rellich's local compactness theorem, as for $|x| \geq M$

$$|\lambda_\infty(x) - \lambda(x)| \leq \int_1^\infty |\hat{x} \cdot (A(rx) - A^{(f,Q)}(rx))|\, dr$$

$$\leq \frac{1}{|x|} \int_{|x|}^\infty \sup_{|y| \geq r} |\hat{y} \cdot (A(y) - A^{(f,Q)}(y))|\, dr = o\left(\frac{1}{|x|}\right).$$

Let us now prove that

$$e^{itH_0}\, x\, e^{-itH_0} = x + \frac{\bar{p}\, t}{m}. \qquad (5.3)$$

Let $\mathcal{S}$ be Schwartz's space.

As $e^{itH_0}\mathcal{S}=\mathcal{S}$ and x is essentially self-adjoint in $\mathcal{S}$ it is enough to prove (5.3) in $\mathcal{S}$.
For $\varphi\in\mathcal{S}$ we have that,

$$e^{itH_0}x\,e^{-itH_0}\varphi=\frac{1}{(2\pi)}\int e^{ipx}e^{it\frac{p^2}{2m}}(-i\nabla_p\, e^{-it\frac{p^2}{2m}}\hat{\varphi}(p))\,dp$$

$$=\frac{1}{(2\pi)}\int e^{ipx}\left[\frac{p}{m}t+i\nabla_p\right]\hat{\varphi}(p)\,dp=$$

$$=(x+\frac{\bar{p}}{m}t)\,\varphi(x).$$

By functional calculus and as λ_∞ is homogeneous,

$$e^{itH_0}\lambda_\infty(x)\,e^{-itH_0}=\lambda_\infty(x+\frac{\bar{p}t}{m})=\lambda_\infty(\frac{mx}{t}+\bar{p}),\ t>0.$$

But for $\varphi\in\mathcal{S}$

$$e^{-i\frac{mx^2}{2t}}\,\bar{p}\,e^{i\frac{mx^2}{2t}}\varphi=(\frac{mx}{t}+\bar{p})\varphi \text{ for } \varphi\in\mathcal{S},$$ and

as $\bar{p}$ is essentially self-adjoint in $\mathcal{S}$ it holds $\forall\varphi\in D(\bar{p})$. Whence,

$$\underset{t\to\infty}{s\text{-}\lim}\, e^{itH_0}\lambda_\infty(x)\,e^{-itH_0}=\underset{t\to\infty}{s\text{-}\lim}\, e^{-i\frac{mx^2}{2t}}\lambda_\infty(\bar{p})\,e^{i\frac{mx^2}{2t}}=\lambda_\infty(\bar{p}).$$

By (5.2)

$$W_+(A)\varphi_{\bar{v}}=e^{i\lambda(x)}\,\underset{t\to\infty}{s\text{-}\lim}\, e^{itH_{A(P,Q)}}\,J\,e^{-i\lambda_\infty(x)}$$

$$e^{-itH_0}\varphi_{\bar{v}}=e^{i\lambda(x)}\,\underset{t\to\infty}{s\text{-}\lim}\left[e^{itH_{A(P,Q)}}\,J\,e^{-itH_0}\right]$$

$$\left[e^{itH_0}\,e^{-i\lambda_\infty(x)}\,e^{-itH_0}\right]\varphi_{\bar{v}}=$$

$$=e^{i\lambda(x)}\,W_+(A(P,Q))\,e^{-i\lambda_\infty(\bar{p})}\,\varphi_{\bar{v}}.$$

We have proven two things:

1) $W_+(A)$ exists, and

2) the gauge transformation formula holds

$$W_+(A) = e^{i\lambda(x)} W_+(A^{(f,Q)})\, e^{-i\lambda_\infty(\hat{p})}.$$

The isometry of $W_+(A)$ is immediate because e^{itH_A} and e^{itH_0} are unitary, and $(1-\chi_\Omega)(H_0+1)^{-1}$ is compact. For $\varphi \in D(H_0)$

$$\lim_{t\to\infty} \| e^{itH_A} J e^{-itH_0}\varphi \| = \lim_{t\to\infty} \| J e^{-itH_0}\varphi\|$$

$$= \lim_{t\to\infty} \| \chi_\Omega (H_0+1)^{-1} e^{-itH_0}(H_0+1)\varphi\|$$

$$= \lim_{t\to\infty} \| e^{-itH_0}\varphi\| = \|\varphi\|. \text{ Hence}$$

$$\|W_+(A)\varphi\|_{L^2(\Omega)} = \|\varphi\|_{L^2(\mathbb{R}^2)}, \quad \varphi \in D(H_0), \text{ and this}$$

extends to $\varphi \in L^2(\mathbb{R}^2)$ by continuity.

$\square$