The Wave Operators

The wave operators are defined as follows,

$$W_\pm = s\text{-}\lim e^{itH} e^{-itH_0},$$

provided that the strong limits exist.

Let $\varphi_0 \in L^2$ have momentum support in $B_{m\eta}$ and momentum wave function $\hat{\varphi}_0 \in C_0^\infty(B_{m\eta})$

The boosted state

$$\varphi_{\bar{v}} := e^{im\bar{v}\cdot x}\varphi_0 \iff \hat{\varphi}_{\bar{v}}(p) = \hat{\varphi}_0(p - m\bar{v}) \qquad (4.1)$$

has compact velocity support contained in $B_\eta(\bar{v})$, the ball of center $\bar{v}$ and radius η.

Moreover,

$$\|(1+x^2)^{3/2}\varphi_{\bar{v}}\| \leq C, \text{ uniformly in } \bar{v}$$

We will use the high-velocity limit in an arbitrary fixed direction $\hat{v} = \frac{\bar{v}}{|\bar{v}|}$

<u>Theorem 4.2</u>. Suppose that $V \in \mathcal{V}$. Then, the wave operators exist and, furthermore

$$\|(W_\pm - I)\, e^{-itH_0}\varphi_{\bar{v}}\| = O\left(\frac{1}{\bar{v}}\right), \qquad (4.2)$$

uniformly for $t \in \mathbb{R}$

Proof: by Duhamel's formula,

$$(W_+ - I)\, e^{-itH_0}\varphi_{\bar{v}} = i\int_0^\infty dt'\, e^{it'H}\, V\, e^{-it'H_0}\varphi_{\bar{v}},$$

provided that the integral in the right-hand side exists.

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Take $f \in C_0^\infty(B_{mn})$ such that $f(\bar{p}-m\bar{v})\varphi_{\bar{v}} = \varphi_{\bar{v}}$. Then,

$$\|(W_+ - I)e^{-itH_0}\varphi_{\bar{v}}\| \leq C\int_{-\infty}^{+\infty} \|V e^{-it''H_0} f(\bar{p}-m\bar{v})(1+x^2)^{-3/2}\| \leq \frac{C}{v}\int_{-\infty}^{+\infty} d(vt'')\, h(|vt''|).$$

Taking $t=0$ this proves the existence of a wave operator for W_+ in a dense set, and then the existence on L^2 follows by continuity. The results for W_- is proven similarly. □

As the $W_\pm$ are partially isometric

$$W_\pm^* W_\pm = I.$$

Recall that the scattering operator is defined as

$$S = W_+^* W_-$$

Then,

$$S - I = (W_+ - W_-)^* W_-$$

By Duhamel's formula

$$i(S-I)\varphi_{\bar{v}} = \int_{-\infty}^{+\infty} e^{itH_0} V e^{-itH} W_- \varphi_{\bar{v}}$$

By the intertwining relations

$e^{-itH} W_- = W_- e^{-itH_0}$. Hence,

$$i(S-I)\varphi_{\bar v} = \int_{-\infty}^{+\infty} e^{itH_0}\, V W_-\, e^{-itH_0}\varphi_{\bar v}$$

$$v\left(i(S-I)\varphi_{\bar v}, \psi_{\bar v}\right) = \int_{-\infty}^{+\infty} \ell_{\bar v}(\tau)\,d\tau + R(\bar v). \quad (4.3)$$

<u>The leading</u> - single scattering - term is

$$\ell_{\bar v}(\tau) = \left(V(x)\, e^{-iH_0 \frac{\tau}{v}}\varphi_{\bar v},\ e^{-iH_0 \tau/v}\psi_{\bar v}\right), \quad (4.4)$$

$\tau = vt$ is the distance at time t.

We have re-scaled from time integral to classical distance integral.

The remainder - multiple scattering - term is

$$R(\bar v) = v\int_{-\infty}^{+\infty} dt\left((W_- - I)\, e^{-itH_0}\varphi_{\bar v}, V e^{-itH_0}\psi_{\bar v}\right) \quad (4.5)$$

<u>Theorem 4.3</u> (Reconstruction formula)

Suppose that $V \in \mathcal{V}$ and that for some $0 \le \delta \le 1$

$$(1+R)^{\delta}\, \| V(x)\, g(\bar p)\, F(|x| > R)\| \in L^1(0,\infty),\ g \in C_0^\infty$$

Then, for all $\varphi_{\bar v}, \psi_{\bar v}$ as in (4.1)

$$v\, i\left((S-I)\varphi_{\bar v}, \psi_{\bar v}\right) = \left(\int_{-\infty}^{+\infty} d\tau\, V(x+\tau\hat v)\, \varphi_0, \psi_0\right) + \begin{cases} o(v^{-\delta}), & 0 \le \delta < 1, \\ O(\frac{1}{v}), & \delta = 1 \end{cases} \quad (4.6)$$

Proof: By Lemma 4.1 and Theorem 4.2

$$|R(\bar v)| \le C\int_{-\infty}^{+\infty} \| V e^{-itH_0}\varphi_{\bar v}\|\, dt \le \frac{C}{v}\int_{-\infty}^{+\infty} h(|\tau|)\,d\tau.$$

Recall that

$$e^{i\bar{p}\cdot\hat{v}\tau} f(x)\, e^{-i\bar{p}\cdot\hat{v}\tau} = f(x+\tau\hat{v})$$

$$e^{-im\bar{v}\cdot x}\, e^{-i\frac{\tau}{v}H_0}\, e^{im\bar{v}\cdot x} = e^{-imv\tau/2}\, e^{-i\bar{p}\cdot\hat{v}\tau}\, e^{-i\frac{\tau}{v}H_0}.$$

Remember that $\tau = vt$ is the classical distance traveled at time t in the direction $\hat{v} = \frac{\bar{v}}{|\bar{v}|}$.

$e^{-i\bar{p}\cdot\hat{v}\tau}$ generates the translation of the wave packet along the classical trajectory.

$e^{-i\frac{\tau}{v}H_0}$ produces the <u>spreading of the wave packet</u>. It is of order $\frac{1}{v}$ with respect to the translation along the classical trajectory given by $e^{-i\bar{p}\cdot\hat{v}\tau}$. In the high-velocity limit it can be neglected. Scattering is over and the particle is outside of the interaction region <u>before the wave packet spreading is relevant.</u>

<u>This means that for high-velocity we can replace the free evolution</u>

$e^{-imv\cdot x}\, e^{-i\frac{\tau}{v}H_0}\, e^{imv\cdot x}$ by the translation along the classical trajectory $e^{-ip\cdot \hat{v}\tau}$.

Note that for this analysis it is important that we have re-scaled and we have taken as parameter the distance $\tau = vt$.

In pointwise sense in τ

$$\lim_{v\to\infty} \ell_v(\tau) = (V(x)\, e^{-ip\cdot\hat{v}\tau}\varphi_0,\ e^{-ip\cdot\hat{v}\tau}\varphi_0)$$

$$= (V(x+\tau\hat{v})\,\varphi_0, \varphi_0).$$

This shows that in the high-velocity limit, as we can replace the free evolution by the translation along the classical trajectory, scattering simplifies and our leading term is the mean value of the potential along the classical trajectory.

Note that

$$V(x+\tau\hat{v}) = e^{ip\cdot\hat{v}\tau}\, V(x)\, e^{-ip\cdot\hat{v}\tau}.$$

is the potential at distance τ in the

Heisenberg representation under the evolution $e^{-i\bar{p}\cdot\hat{v}\,\tau}$.

Lemma 4.1 gives us the bound

$$|l_{\bar{v}}(\tau)| \leq \|\psi_0\| \, \| V(x)\, e^{-iH_0 \frac{\tau}{v}} \varphi_{\bar{v}}\|$$

$$\leq C \| V(x)\, e^{-iH_0\frac{\tau}{v}} f(\bar{p}-m\bar{v})(1+x^2)^{-3/2}\|$$

$$\leq C h(|\tau|).$$

Then, (4.6) with $\rho = 0$ follows from the dominated convergence theorem.

Furthermore,

$$l_{\bar{v}}(\tau) - (V(x+\tau\hat{v})\varphi_0, \psi_0) = l^{(1)}_{\bar{v}}(\tau) + l^{(2)}_{\bar{v}}(\tau)$$

$$l^{(1)}_{\bar{v}}(\tau) = (V(x+\tau\hat{v})\, e^{-iH_0\frac{\tau}{v}}\varphi_0, (e^{-iH_0\frac{\tau}{v}} - I)\psi_0)$$

$$l^{(2)}_{\bar{v}}(\tau) = ((e^{-iH_0\frac{\tau}{v}} - I)\varphi_0, V(x+\tau\hat{v})\psi_0)$$

Since $\hat{\psi}_0$ has compact support

$$\|(e^{-iH_0\frac{\tau}{v}} - I)\psi_0\| \leq \|H_0\psi_0\| \frac{|\tau|}{v}.$$

Furthermore,

$\|(e^{-iH_0\frac{\tau}{v}} - I)\psi_0\| \leq 2\|\psi_0\|$. Hence,

$$\|(e^{-iH_0\frac{\tau}{v}} - I)\psi_0\| \leq C\left(\frac{|\tau|}{v}\right)^{\rho}, \quad 0 \leq \rho \leq 1$$

It follows from Lemma 4.1 that

$$|e^{(1)}_v(\sigma)| \le \frac{C}{v^\rho} |\sigma|^\rho \, h(|\sigma|)$$

Furthermore, as for $0 \le \rho < 1$

$$\lim_{v \to \infty} v^\rho e^{(1)}_v(\sigma) = 0,$$

$$\int_{-\infty}^{+\infty} e^{(1)}_v(\sigma)\, d\sigma = \begin{cases} o(v^{-\rho}), & 0 \le \rho < 1, \\ O(v^{-1}), & \rho = 1 \end{cases}$$

Moreover, denoting $\tilde{x} = x \cdot \overset{n}{v}$

$$\int_{-\infty}^{+\infty} (1+|\sigma|)^\rho \, \|V(x+\sigma \overset{n}{v}) \varphi_0\| \, d\sigma \le$$

$$\|\varphi_0\| \int_{-\infty}^{+\infty} (1+|\sigma|)^\rho \Big[\|V(x+\sigma \overset{n}{v}) g(\bar{p}) F(|\tilde{x}+\sigma| \ge \tfrac{|\sigma|}{2})\| \, \|\varphi_0\|$$

$$+ \|V(x+\sigma \overset{n}{v}) g(\bar{p})\| \, \|F(|\tilde{x}+\sigma| < \tfrac{|\sigma|}{2}) \varphi_0\| \Big] d\sigma < \infty,$$

by our assumption on V and as,

$$(1+|\sigma|)^\rho \, \|F(|\tilde{x}| > |\sigma|/2) \varphi_0\| \in L^1(0,\infty).$$

We prove as above that

$$\int_{-\infty}^{+\infty} e^{(2)}_v(\sigma)\, d\sigma = \begin{cases} o(v^{-\rho}), & 0 \le \rho < 1, \\ O(v^{-1}), & \rho = 1 \end{cases}$$

$\square$