point of stationary phase.

$\dot{x} = \frac{p}{m} t'$ $\quad |p| \le m\eta$ on support of f

Then, if $\left|\frac{x}{t}\right| \geq \eta$

$$|\tilde{f}_{t'}(x)| \le C_m (1 + |x| + |t|)^{-m}$$

$$|\tilde{f}_t(x)| \le C_m (1 + |x| v^{\delta} + |t| v^{2\delta})^{-m} \; v^{n\delta}$$

$$\left|\frac{x v^{\delta}}{t v^{2\delta}}\right| \geq \eta \Rightarrow |x| \geq \eta |t| v^{\delta}.$$

Finally

$$\int_{|x| \geq r + \eta v^{\delta} |t|} |\tilde{f}_t(x)| \, dx \le C_m v^{n\delta} \int_{|x| \geq r + \eta v^{\delta} |t|} (1 + |x| v^{\delta} + |t| v^{2\delta})^{-m} dx$$

and estimating the integral we obtain the result.

<u>Corollary</u>

For any $Q \in \mathbb{R}^2$, $f \in C_0^{\infty}(B_{m\eta})$, $0 \le \delta < 1$, $l = 1, 2, 3 \ldots$ there is a C_l such that

$$\left\| F(|x - Q - \bar{v}t| > |\bar{v}t|/4) \, e^{-iH_0 t} f\left(\frac{p - m\bar{v}}{v^{\delta}}\right) F(|x - Q| \le \frac{|\bar{v}t|}{8}) \right\| \le C_l (1 + |\bar{v}t|)^{-l}.$$

For $v > (8\eta)^{\frac{1}{1-\delta}}$ with $v = |\bar{v}|$.

Proof

$\operatorname{dist}(\tilde{M}, M + v t) - \eta v^{\delta} |t| \geq \frac{|v t|}{8} - \eta v^{\delta} |t| \geq 0$

$x \in \tilde{M}$, $y \in M$

$|x - y - vt| = |x - a - vt + (a - y)|$

$\geq \frac{|vt|}{4} - \frac{|vt|}{8} = \frac{|vt|}{8}$

$\frac{|vt|}{8} \geq \eta v^{\delta} |t| \Rightarrow v^{1-\delta} \geq 8\eta$

$v \geq (8\eta)^{\frac{1}{1-\delta}}$

Denote $I = \| \cdots \|$. Then,

$I \leq C_{\ell}\left(1 + r v^{\delta} + \eta v^{2\delta} |t|\right)^{-\ell}$

$r v^{\delta} + \eta v^{2\delta} |t| = v^{\delta}(r + \eta v^{\delta} |t|) =$

$= v^{\delta} \operatorname{dist}(\tilde{M}, M + vt) \geq v^{\delta} \frac{|vt|}{8}.$

Finally

$I \leq C_{\ell}\left(1 + |vt|\right)^{-\ell}.$

$\square$

• The Hamiltonian

The potential V satisfies.

Enss condition: Definition of V

V is H_0-small, i.e. it is H_0-bounded with relative bound zero (or just < 1)

$\|V\varphi\| \leq \varepsilon \|H_0 \varphi\| + K_{\varepsilon} \|\varphi\|$, $\forall \varepsilon > 0$, and

$$\int_0^\infty dR \, \| V(x) \, (H_0+I)^{-1} \, F(|x| \geq R) \| < \infty.$$

This is equivalent to the more intuitive condition

$$\int_0^\infty dR \, \| V(x) \, F(|x| \geq R) \, (H_0+I)^{-1} \| < \infty.$$

Let us prove this

Denote

$h(R, z) := \| V \, (H_0 - z)^{-1} \, F(|x| \geq R) \|$, $z \in \mathbb{C} \setminus [0, \infty)$

Take j: $1 - j \in C_0^\infty$, $j(x) = 1$, $|x| \geq 1$, $j(x) = 0$, $|x| \leq 1/2$, $0 \leq j \leq 1$

$j_R(x) = j(x/R)$, $j_R(x) = 1$, $|x| \geq R$,

$j_R(x) = 0$, $|x| \leq \frac{R}{2}$.

$h_1(R, z) := \| V \, R_0(z) \, j_R \|$

$h_2(R, z) := \| V \, j_R \, R_0(z) \|$

Then (certain abuse of notation)

$h(R, z) = \| F(|x| \geq R) \, R_0(\bar{z}) \, V \| \leq \| j_R \, R_0(\bar{z}) \, V \|$

$= \| V \, R_0(z) \, j_R \| = h_1(R, z).$

$h_1(R, z) = \| j_R \, R_0(\bar{z}) \, V \| \leq \| F(|x| \geq R/2) \, R_0(\bar{z}) \, V \|$

$= h(R/2, z)$

Then,

$$h(R,z) \le h_1(R,z) \le h\left(\frac{R}{2},z\right) \quad \text{and}$$

$$h(R,z) \in L^1(0,\infty) \iff h_1(R,z) \in L^1(0,\infty).$$

Moreover

$$|h_1(R,z) - h_2(R,z)| \le \| V R_0(z) J_R - V J_R R_0(z) \|$$

$$= \| V R_0(z) [J_R, H_0] R_0(z) \|$$

$$[J_R, H_0] = \frac{1}{2m}\left[\frac{1}{R^2}(\Delta J)_R + \frac{2}{R}(\nabla J)_R \cdot \nabla\right]$$

Then

$$|h_1(R,z) - h_2(R,z)| \le \frac{1}{2m}\frac{1}{R^2}\| V R_0(z) (\Delta J)_R \| \frac{1}{|\operatorname{Im} z|}$$

$$+ \frac{1}{m}\frac{1}{R}\sum_{i=1}^{n} \| V R_0(z) (\partial_i J)_R \| \| \partial_i R_0(z) \|$$

$$\le \frac{C_z}{R^2} h_1\left(\frac{R}{2},z\right) + \frac{C_z}{R} h_1\left(\frac{R}{2},z\right) \le \frac{C_z}{R} h_1\left(\frac{R}{2},z\right)$$

It follows that

$$h_1(R,z) \in L^1(0,\infty) \implies h_2(R,z) \in L^1(0,\infty).$$

Furthermore,

$$|h_1(R,z) - h_2(R,z)| \le \| V R_0(z) J_{R/2} [J_R, H_0] R_0(z) \|$$

~~Define~~

$$\le \| V [R_0(z), J_{R/2}] [J_R, H_0] R_0(z) \|$$

$$+ \| V J_{R/2} R_0(z) [J_R, H_0] R_0(z) \| \le$$

$\leq \| V R_0(\beta) [J_{R/2}, H_0] R_0(\beta) [J_R, H_0] R_0(\beta) \|$

$+ \frac{C_\beta}{R} h_2(R/2) \leq \frac{C_\beta}{R^2} + \frac{C_\beta}{R} h_2(R/2)$

Then, $h_2(R,\beta) \in L^1 \Rightarrow h_1(R,\beta) \in L^1$.

So, we have proved that

$h(R,\beta) \in L^1(0,\infty) \Leftrightarrow h_1(R,\beta) \in L^1(0,\infty) \Leftrightarrow$

$h_2(R,\beta) \in L^1(0,\infty)$.

Define $h_3(R,\beta) = \| F(|x|>R) V(x) R(\beta) \|$

We have that

$h_2(R,\beta) \leq h_3(R/2,\beta)$, and also

$h_3(R,\beta) \leq h_2(R,\beta)$

Hence

$h_2(R,\beta) \in L^1(0,\infty) \Leftrightarrow h_3(R,\beta) \in L^1(0,\infty)$.

In consequence

$h(R,\beta) \in L^1(0,\infty) \Leftrightarrow h_3(R,\beta) \in L^1(0,\infty)$.

Finally

$|h_3(R,\beta_2) - h_3(R,\beta_1)|$

$\leq \| V(x) F(|x|>R) (R(\beta_2) - R(\beta_1)) \|$

$= |\beta_2 - \beta_1| \| V(x) F(|x|>R) R(\beta_2) R(\beta_1) \|$

(23)

$\le C_3\, h(R, \mathfrak{z}_i)$, $i = 1, 2$.

This implies that

$h_{\mathfrak{z}}(R, \mathfrak{z}_1) \in L^1(0, \infty) \Leftrightarrow h_{\mathfrak{z}}(R, \mathfrak{z}_2) \in L^1(0, \infty)$

$\forall\, \mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C} \setminus [0, \infty)$, and the same is true for $h(R, \mathfrak{z})$.

By Kato-Rellich theorem

$H = H_0 + V$ is self-adjoint on

$\mathcal{D}(H_0) = \mathcal{H}_2$

<u>Lemma</u>

Suppose that $V \in \mathcal{V}$ and that for some $0 \le \rho \le 1$ and all $g \in C_0^\infty(\mathbb{R}^n)$

$$(1+R)^\rho \| V(x) g(\bar{p})\, F(|x| > R) \| \in L^1(0, \infty)$$

Then, for any $f \in C_0^\infty(B_{m\eta})$ there is a function h with $(1+\sigma)^\rho h(\sigma) \in L^1(0, \infty)$ such that for every $\bar{v} \in \mathbb{R}^n$ satisfying $v \ge 4\eta$, $v = |\bar{v}|$, we have

$$\| V(x)\, e^{-itH_0} f(\bar{p} - m\bar{v})\, (1 + x^2)^{-3/2} \|$$

$$= \| V(x + t\bar{v})\, e^{-itH_0} f(\bar{p})\, (1 + x^2)^{-3/2} \| \le h(|\bar{v} t|).$$

Proof: (24)

$$I = \| V(x)\, e^{-itH_0} f(\bar p - m\bar v)\,(1+x^2)^{-3/2} \|$$

$$= \| V(x)\, e^{-itH_0}\, e^{im\bar v x} f(\bar p)\, e^{-im\bar v \cdot x}\,(1+x^2)^{-3/2} \|$$

$e^{im\bar v\cdot x}\ e^{-im\bar v\cdot x}$

$$= \| e^{im\bar v x} V(x)\, e^{-i\bar p\cdot \bar v t}\, e^{-itH_0} f(\bar p)\,(1+x^2)^{-3/2} \|$$

$$= \| V(x+\bar v t)\, e^{-itH_0} f(\bar p)\,(1+x^2)^{-3/2} \|.$$

Take $g \in C_0^\infty(\mathbb{B}_{m})$ with $g \equiv 1$ on the support of f. Then

$$\| V(x)\, g(\bar p - m\bar v)\| = \| V(x)\, e^{im\bar v\cdot x} g(\bar p)\, e^{-im\bar v\cdot x}\|$$

$= \| V(x)\, g(\bar p)\|$ is independent of $\bar p$

Recall our Corollary:

$$\| F(|x-\bar v t| > |\bar v t|/4)\, e^{-itH_0} g(\bar p - m\bar v)\, F(|x| \le \tfrac{|\bar v t|}{8}) \|$$

$$\le C_\ell\, (1+|\bar v t|)^{-\ell}$$

$$I = \| V(x)\, g(\bar p - m\bar v)\big[F(|x-\bar v t| > \tfrac{|\bar v t|}{4}) + F(|x-\bar v t| \le \tfrac{|\bar v t|}{4}) \big)$$

$$e^{-itH_0} f(\bar p - m\bar v)\,\big(F(|x| \le \tfrac{|\bar v t|}{8}) + F(|x| > \tfrac{|\bar v t|}{8}) \big)$$

$$(1+x^2)^{-3/2} \| \le I_1 + I_2 + I_3$$

$$I_1 = \| V(x)\, g(\bar{p} - m\bar{v})\, F(|x - \bar{v}t| \geq \tfrac{|\bar{v}t|}{4})$$
$$e^{-itH_0} f(\bar{p} - m\bar{v})\, F(|x| \leq \tfrac{|\bar{v}t|}{8})\, (1+x^2)^{-3/2} \|$$

$$I_2 = \| V(x)\, g(\bar{p} - m\bar{v})\, F(|x - \bar{v}t| \leq \tfrac{|\bar{v}t|}{4})$$
$$e^{-itH_0} f(\bar{p} - m\bar{v})\, F(|x| \leq \tfrac{|\bar{v}t|}{8})\, (1+x^2)^{-3/2} \|$$

$$I_3 = \| V(x)\, g(\bar{p} - m\bar{v})\, e^{-itH_0} f(\bar{p} - m\bar{v})\, F(|x| \geq \tfrac{|\bar{v}t|}{8})$$
$$(1+x^2)^{-3/2} \|$$

I_1 The classically forbidden region

$$I_1 \leq \| V g(\bar{p} - m\bar{v}) \|\, C_\ell\, (1 + |\bar{v}t|)^{-\ell}, \quad \ell = 1, 2, 3$$

I_2 Classically allowed region.

Here the potential is small

$$I_2 \leq C_1 \| V g(\bar{p} - m\bar{v})\, F(|x - \bar{v}t| \leq \tfrac{|\bar{v}t|}{4}) \|$$

$$\leq \| V g(\bar{p} - m\bar{v})\, F(|x| \geq \tfrac{3}{4}|\bar{v}t|) \| =: h_1(|\bar{v}t|)$$

with $(1+\tau)^{\delta} h_1(\tau) \in L^1(0, \infty)$

I_3. The weight is small

$$I_3 \leq C\, (1 + |\bar{v}t|^2)^{-3/2}$$

$\square$