2 The Free Propagation

We denote by $\bar{p} = -i\nabla$ the operator of momentum

Then,

$H_0 = \frac{-\Delta}{2m} = \frac{\bar{p}^2}{2m}$ $\quad D(H_0) = H_2$.

We define operators $f(\bar{p})$, f a Borel function by functional calculus or as $f(\bar{p}) = \mathcal{F}^{*} f(p)\, \mathcal{F}$ where $\mathcal{F}$ denotes the Fourier transform

$$(\mathcal{F}\varphi)(p) = \frac{1}{(2\pi)^{n/2}} \int e^{-ip\cdot x} \varphi(x)\, dx.$$

Note that

$$e^{i\bar{p}\cdot\bar{v}t} f(x)\, e^{-i\bar{p}\cdot\bar{v}t} = f(x+\bar{v}t) \qquad (1)$$

$$e^{-im\bar{v}\cdot x} f(\bar{p})\, e^{im\bar{v}\cdot x} = f(\bar{p}+m\bar{v}) \qquad (2)$$

In particular

$$e^{-im\bar{v}\cdot x} e^{-itH_0} e^{im\bar{v}\cdot x} = e^{-imv^2/2}\, e^{-i\bar{p}\cdot\bar{v}t}\, e^{-itH_0} \qquad (3)$$

$v = |\bar{v}|$.

Proof of (2):

$$e^{-im\bar{v}\cdot x} f(\bar{p})\, e^{im\bar{v}\cdot x} \varphi(x) = e^{-im\bar{v}\cdot x} \mathcal{F}^{*} f(p)\, \mathcal{F} e^{im\bar{v}\cdot x}\varphi.$$

$$\mathcal{F} e^{im\bar{v}\cdot x}\varphi = \frac{1}{(2\pi)^{n/2}} \int e^{-ip\cdot x} e^{im\bar{v}\cdot x} \varphi(x)\, dx$$

$= \hat{\varphi}(p - m\bar{v}), \quad \hat{\varphi} = \mathcal{F}\varphi$

$e^{im\bar{v}\cdot x} f(\bar{p})\, e^{im\bar{v}\cdot x} \varphi(x) = \frac{1}{(2\pi)^{n/2}} e^{im\bar{v}\cdot x} \int e^{ipx} f(p)$

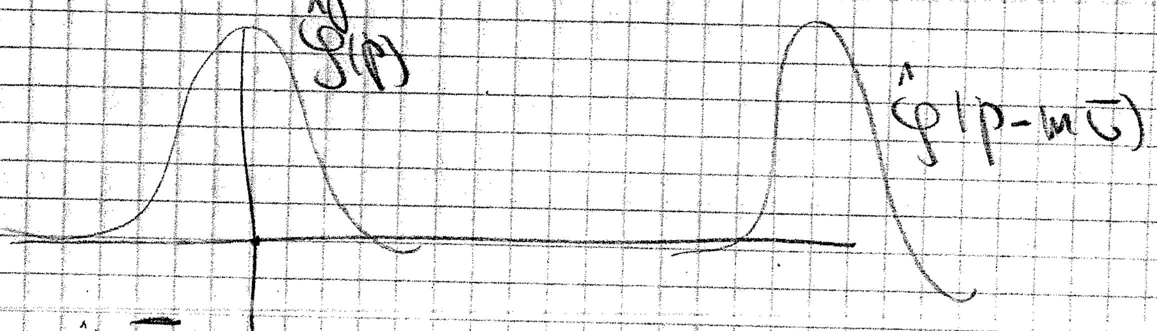
$\hat{\varphi}(p - m\bar{v})\, dp = \frac{1}{(2\pi)^{n/2}} \int e^{i(p - m\bar{v})x} f(p - m\bar{v} + m\bar{v})$

$\hat{\varphi}(p - m\bar{v})\, dp = \mathcal{F}^{-1} f(p + m\bar{v}) \mathcal{F}\varphi = f(\bar{p} + m\bar{v})\varphi$

Note that

$\mathcal{F}\{e^{im\bar{v}\cdot x}\varphi(x)\} = \hat{\varphi}(p - m\bar{v})$

$e^{im\bar{v}\cdot x}$ gives a "boost" $m\bar{v}$ to the state φ



$e^{im\bar{v}\cdot x}\varphi(x)$ is a state with velocity $\sim \bar{v}$

<u>Notation</u>

$B_{m\eta}(m\bar{v})$: open ball of center $m\bar{v}$ and radius $m\eta$.

<u>Lemma</u>

For any $f \in C_0^\infty(B_{m\eta})$, for some $\eta > 0$, and any $l = 1, 2, \ldots$ there is a constant C_l such

$$\left\| F(x \in \tilde{M}) \, e^{-itH_0} f\left(\frac{p - mv}{v^{\delta}}\right) F(x \in M) \right\|$$

$$\le C_\ell \left(1 + r v^{\delta} + \eta v^{2\delta} |t|\right)^{-\ell}$$

for every $\bar{v} \in \mathbb{R}^n$, $t \neq 0$, $v > 0$, $\delta \in \mathbb{R}$ and any measurable sets $M, \tilde{M} \subset \mathbb{R}^n$ such that

$r := \operatorname{dist}(\tilde{M}, M + \bar{v} t) - \eta v^{\delta} |t| \geq 0$.

Before we give the proof we try to understand the meaning of the Lemma.

For simplicity take $\delta = 0$.

$$\left\| F(x \in \tilde{M}) \, e^{-itH_0} f(\bar{p} - m\bar{v}) F(x \in M) \right\|$$

$$\le C_\ell \left(1 + r + \eta |t|\right)^{-\ell}$$

other side

$$\operatorname{dist}(\tilde{M}, M + \bar{v} t) = r + \eta |t|$$

f localizes on $|p - m\bar{v}| \le m\eta$

the velocity differs from $\bar{v}$ by less than η

Proof.

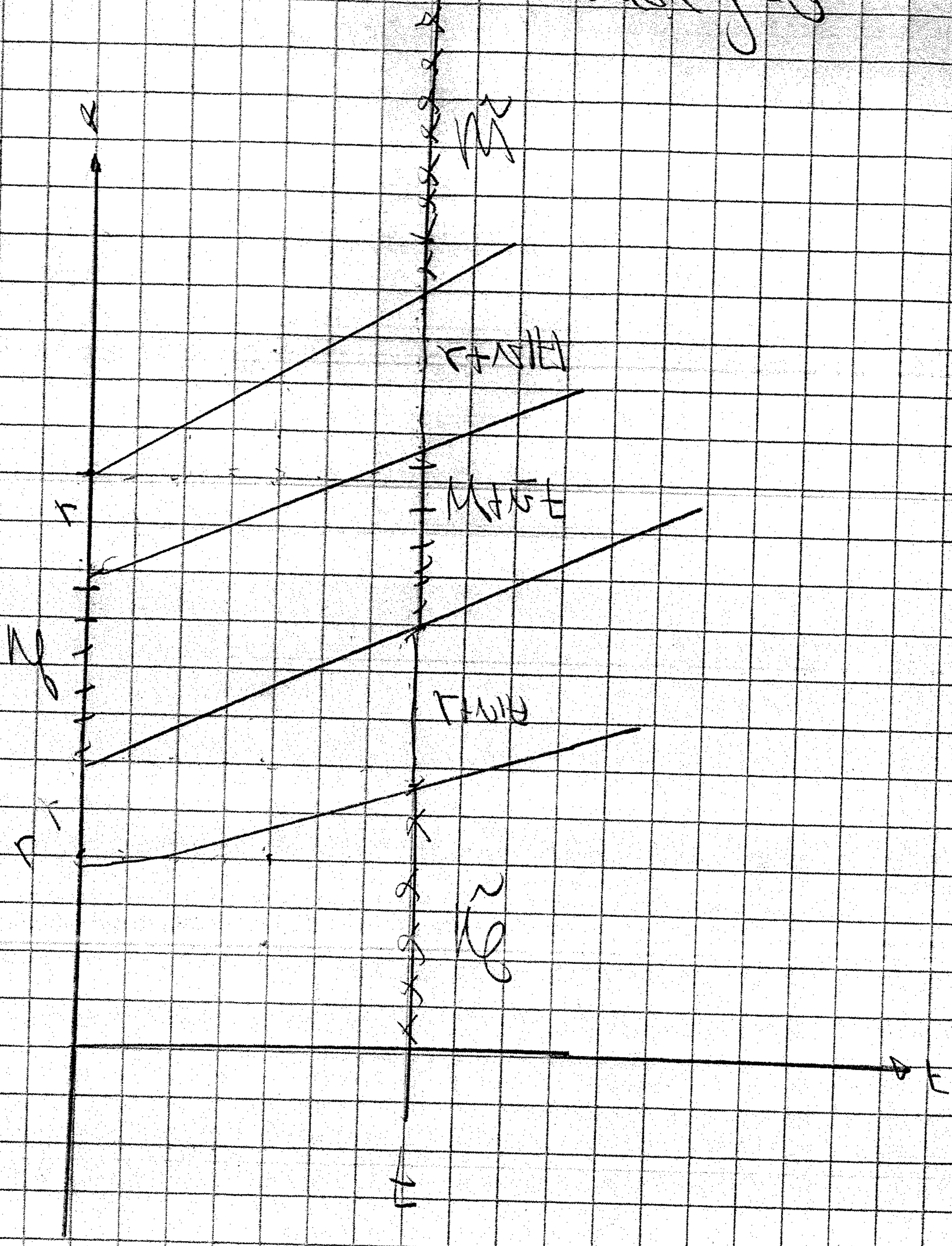
$$\left\| F(x \in \tilde{M}) \, e^{-itH_0} f\left(\frac{\bar{p} - m\bar{v}}{v^{\delta}}\right) F(x \in M) \right\| =$$

$$= \left\| F(x \in \tilde{M}) \, e^{-itH_0} e^{im\bar{v}\cdot x} f\left(\frac{\bar{p}}{v^{\delta}}\right) e^{-im\bar{v}\cdot x} F(x \in M) \right\|$$

$$= \left\| F(x \in \tilde{M}) \, e^{-im\bar{v}\cdot x} e^{-itH_0} e^{im\bar{v}\cdot x} f\left(\frac{\bar{p}}{v^{\delta}}\right) F(x \in M) \right\|$$

$$= \left\| F(x \in \tilde{M}) \, e^{-itH_0} f\left(\frac{\bar{p}}{v^{\delta}}\right) e^{-i\bar{p}\cdot\bar{v}t} F(x \in M) \, e^{i\bar{p}\cdot\bar{v}t} \right\|$$

Case $g=0$ (156)



$\mathrm{dist}(\tilde{M}, M+\tilde{u}t) = r + |\tilde{u}||t|, \quad r \geq 0$

$= \| F(x \in \tilde{M}) \, e^{-itH_0} f\left(\frac{p}{v\delta}\right) F(x - vt \in M) \|$

$= \| F(x \in \tilde{M}) \, e^{-itH_0} f\left(\frac{p}{v\delta}\right) F(x \in M + vt) \|$

Replace $M \to M + vt$, and we have to prove that

$\| F(x \in \tilde{M}) \, e^{-itH_0} f\left(\frac{p}{v\delta}\right) F(x \in M) \| \leq$

$$\leq C_\ell \left(1 + r v^\delta + \eta v^{2\delta} |t|\right)^{-\ell},$$

$r := \mathrm{dist}(\tilde{M}, M) - \eta v \delta |t| \geq 0.$

We have that

$I = \| F(x \in \tilde{M}) \, e^{-itH_0} f\left(\frac{p}{v\delta}\right) F(x \in M) \phi \|^2 =$

$= \Big(F(x \in \tilde{M}) \, e^{-itH_0} f\left(\frac{p}{v\delta}\right) F(x \in M) \phi ,$

$\quad F(x \in \tilde{M}) \, e^{-itH_0} f\left(\frac{p}{v\delta}\right) F(x \in M) \phi \Big)_{\mathcal{H}}$

$$e^{-itH_0} f\left(\frac{p}{v\delta}\right) F(x \in M) \phi = \frac{1}{(2\pi)^{n/2}} \int \check{f}_t(x - y) \, \phi(y) \, dy$$

$$\check{f}_t(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ipx} \, e^{-ip^2 t/2m} f\left(\frac{p}{v\delta}\right) dp.$$

$$I = \frac{1}{(2\pi)^n} \int_{\tilde{M}} dx \int_M dy \, \check{f}_t(x - y) \, \phi(y) \, dy \int_M dz \, \overline{\check{f}_t(x - z)} \, \overline{\phi(z)} \, dz.$$

But as $2|\Phi(y)||\Phi(z)| \le |\Phi(y)|^2 + |\Phi(z)|^2$. (17)

$$I \le \frac{1}{2(2\pi)^n} \int dx \int_{\tilde{M}} \left[|\tilde{f}_t(x-y)| \, |\tilde{f}_t(x-z)| \left[|\Phi(y)|^2 + |\Phi(z)|^2 \right] \right] dy\, dz$$

$x \in M$, $y \in \tilde{M}$ Max

$|x-y| \ge \mathrm{dist}(M, \tilde{M}) = \underline{r} + \eta v^{\delta} |t|$.

replace $x-y \to \tilde{x}$, $x-z \to \tilde{z}$, $y \to y$ and

$x-y \to \tilde{y}$, $x-z \to \tilde{x}$, $z \to z$ and we get

$$I \le \frac{1}{(2\pi)^n} \left[\int_{|x| \ge r + \eta v^{\delta} |t|} |\tilde{f}_t(x)|^2 dx \right]^2 \|\Phi\|^2.$$

We have to estimate $\tilde{f}_t(x)$ using (non) stationary phase

Hörmander, Reed-Simon III, Corollary to Theorem XI.14, page 38

$$\tilde{f}_t(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ip\cdot x} e^{-i\frac{p^2}{2m}t} f\left(\frac{p}{v^{\delta}}\right) dp$$

$$\tilde{f}_t(x) = v^{n\delta} \frac{1}{(2\pi)^{n/2}} \int e^{ip\cdot x v^{\delta}} e^{-i\frac{p^2}{2m} t v^{2\delta}} f(p)\, dp$$

$$\tilde{f}_{t'}(x') = \frac{v^{n\delta}}{(2\pi)^{n/2}} \int e^{ip\cdot x'} e^{-i\frac{p^2}{2m}t'} f(p)\, dp$$

$x' = x v^{\delta}$ $\quad t' = t v^{2\delta}$

point of stationary phase. (18)

$\dot{x} = \frac{p}{m} t' \qquad |p| \leq m\eta$ on support of f

Then, if $\left|\frac{x}{t}\right| \geq \eta$

$$|\tilde{f}_{t'}(x)| \leq C_m (1 + |x| + |t|)^{-m}$$

$$|\tilde{f}_t(x)| \leq C_m (1 + |x| v^{\delta} + |t| v^{2\delta})^{-m} \, v^{n\delta}$$

$$\left|\frac{x v^{\delta}}{t v^{2\delta}}\right| \geq \eta \Rightarrow |x| \geq \eta |t| v^{\delta}.$$

Finally

$$\int_{|x| \geq r + \eta v^{\delta} |t|} |\tilde{f}_t(x)| \, dx \leq C_m v^{n\delta} \int_{|x| \geq r + \eta v^{\delta} |t|} (1 + |x| v^{\delta} + |t| v^{2\delta})^{-m} dx$$

and estimating the integral we obtain the result.

<u>Corollary</u>

For any $Q \in \mathbb{R}^2$, $f \in C_0^\infty(B_{m\eta})$, $0 \leq \delta < 1$, $l = 1, 2, 3 \ldots$ there is a C_l such that

$$\left\| F(|x - Q - \bar{v}t| > |\bar{v}t|/4)\, e^{-itH_0} f\left(\frac{p - m\bar{v}}{v^{\delta}}\right) F(|x - Q| \leq \frac{|\bar{v}t|}{8}) \right\| \leq C_l (1 + |\bar{v}t|)^{-l}$$

for $v > (8\eta)^{\frac{1}{1-\delta}}$ with $v = |\bar{v}|$.