$\hat{S}(E)$, is "the scattering matrix", is a unitary operator on $L^2(S^{n-1})$.

Relation with the Stationary Theory

The weighted L^2 spaces

$$L^2_s := \{ f : (1+x^2)^{s/2} f(x) \in L^2 \}$$

$$\|f\|_{L^2_s} := \|(1+x^2)^{s/2} f\|_{L^2}, \quad s \in \mathbb{R}, \text{ and for}$$

any $a, s \in \mathbb{R}$

$$\mathcal{H}_{a,s} := \{ f : (1+x^2)^{s/2} f(x) \in \mathcal{H}_a \},$$

$$\|f\|_{\mathcal{H}_{a,s}} := \|(1+x^2)^{s/2} f(x)\|_{\mathcal{H}_a},$$

where $\mathcal{H}_a$ is the Sobolev space

$$\mathcal{H}_a := \{ f \in \mathcal{S}' : (1+p^2)^{a/2} \hat{f}(p) \in L^2 \}$$

$$\|f\|_{\mathcal{H}_a} := \|(1+p^2)^{a/2} \hat{f}\|_{L^2}.$$

The limiting absorption principle is the following statement. For $z \in \rho(H)$, the resolvent set of H, let $R(z) = (H-z)^{-1}$ be the resolvent. Then for every $E \in \mathbb{R}^+$

the following limits

$$R(E\pm i0) := \lim_{\varepsilon\to 0} R(E\pm i\varepsilon)$$

exist in the uniform operator topology in $B(L^2_s, H_{\alpha,-s})$, $s>1/2$, $|\alpha|\le 2$.

The functions

$$R_\pm(E) = \begin{cases} R(E), & \operatorname{Im} E \neq 0 \\ R(E\pm i0), & E\in \mathbb{R}^+ \end{cases}$$

defined for $E \in \mathbb{C}^\pm \cup \mathbb{R}^+$ with values in $B(L^2_s, H_{\alpha,-s})$ are analytic for $\operatorname{Im} E \neq 0$ and locally Hölder continuous for $E\in\mathbb{R}^+$ with exponent γ satisfying $\gamma<1$, $\gamma< s-\frac{1}{2}$.

Let us denote by $T_0(E)$ the following trace operator

$$(T_0(E)\varphi)(\omega) := 2^{-1/2} E^{(n-2)/4} \frac{1}{(2\pi)^{n/2}} \int e^{-i\sqrt{E}\, x\cdot\omega} \varphi(x)\,dx,$$

is bounded from L^2_s, $s>1/2$, into $L^2(S^{n-1})$ and is locally Hölder continuous with exponent $\gamma<1$, $\gamma<s-\frac{1}{2}$. Moreover

$$(\mathcal{F}_0\varphi)(E,\omega) = (T_0(E)\varphi)(\omega), \quad \varphi\in L^2_s.$$

The perturbed trace operators are defined as follows

$$(T_\pm(E)\varphi)(\omega) := T_0(E)\,(I - V R_\pm(E))\,\varphi, \quad E \in \mathbb{R}^+$$

They are bounded from L^2_s, $s > 1/2$ into $L^2(S^{n-1})$ and furthermore the operator valued functions $E \in \mathbb{R}^+ \to T_\pm(E)$ from $\mathbb{R}^+$ into $\mathcal{B}(L^2_s, L^2(S^{n-1}))$ are locally Hölder continuous with exponent $\gamma < 1$, $\gamma < s - \frac{1}{2}$.

The operators

$$(\mathcal{F}_\pm \varphi)(E, \omega) := (T_\pm(E)\varphi)(\omega),$$

extend to unitary operator from $\mathcal{H}_{ac}$ onto $\hat{\mathcal{H}}$ and they give a spectral representation for $H_{ac} := H P_{ac}(H)$:

$$\mathcal{F}_\pm H_{ac} \mathcal{F}_\pm^* = E.$$

Moreover,

$$W_\pm = \mathcal{F}_\pm^* \mathcal{F}_0.$$

$$\hat{S} = \mathcal{F}_+ \mathcal{F}_-^* = \oplus \int_{\mathbb{R}^+} S(E)\, dE.$$

Note that in this formulation time has

disappeared and the scattering problem is equivalent to the problem of finding two spectral representation for a self-adjoint operator, what is a classical problem in operator theory

Stationary Representation for the Scattering Matrix

$$S(E) = I - 2\pi i\, \mathcal{F}_0 V \left[I - R_+(E) V \right] \mathcal{F}_0^*$$

<u>Unperturbed</u> <u>Averaged</u> <u>Scattering</u> <u>Solutions</u>

$$\phi_{0,f}(x;E) := \int_{S^{n-1}} e^{i E^{1/2} x \cdot \omega} f(\omega)\, d\omega \quad , f \in L^2(S^{n-1})$$

$$\phi_{0,f} \in L^2_{-s} \, , s > 1/2$$

$$H_0 \phi_{0,f} = E \phi_{0,f}$$

<u>Perturbed</u> <u>Averaged</u> <u>Scattering</u> <u>Solutions</u>

$$\phi_{\pm,f}(x;E) := \left[I - R_\mp(E) V \right] \phi_{0,f} , \quad f \in L^2(S^{n-1})$$

$\phi_{\pm,f} \in L^2_{-s}$, and

$$H \phi_{\pm,f} = E\, \phi_{\pm,f}$$

Moreover, for $f, g \in L^2(S^{n-1})$ ⑨

$$(S(E) f, g) = (f, g) - i \frac{E^{(n-2)/2}}{2(2\pi)^{n-1}} (V\phi_{+,f}, \phi_{g,g})$$

Suppose that $\rho := 1+\varepsilon > (n+1)/2$. Then $V \in L^2_{s,1}$, $s > 1/2$ and we can define the the scattering solutions

$$\phi_\pm(x, \omega; E) := e^{i\sqrt{E}\, x\cdot\omega} - R_\pm(E)\left(V e^{i\sqrt{E}\, x\cdot\omega}\right)$$

$$H \phi_E = E \phi_E$$

In this case

$$\phi_{E,f}(x; E) := \int_{S^{n-1}} \phi_\pm(x, \omega; E) f(\omega)\, d\omega.$$

Moreover

$$(S(E) f, g) = (f, g) - \frac{i E^{(n-2)/2}}{2(2\pi)^{n-1}} \int_{S^{n-1}\times S^{n-1}} \left[\int_{\mathbb{R}^n} \phi_+(x, \omega; E) V e^{i\sqrt{E}\, x\cdot\nu}\right]. f(\omega)\, \overline{g}(\nu)\, d\omega\, d\nu.$$

Then

$$S(E) = I + T(E), \quad \text{where}$$

$$(T(E) g)(\nu) = \int E(\nu, \omega; E)\, g(\omega)\, d\omega,$$

with

$$E(\nu, \omega; E) = \frac{-i E^{(n-2)/2}}{2(2\pi)^{n-1}} \int \phi_+(x, \omega; E) V(x) e^{i\sqrt{E}\, x\cdot\nu}\, dx$$

$T(E)$ is a Hilbert-Schmidt operator

Moreover

$$\phi_+(x,\omega;E) \underset{|x|\to\infty}{\simeq} e^{i\sqrt{E}\,x\cdot\omega} + \frac{1}{|x|^{\frac{n-1}{2}}} e^{i\sqrt{E}|x|} f(\nu,\omega;E)$$

where $\nu = \frac{x}{|x|}$,

i.e. it behaves as a plane wave plus a out-going spherical wave that satisfies Sommerfeld's radiation condition

$f(\nu,\omega;E)$ is the scattering amplitude and

$$t(\nu,\omega;E) = e^{i\pi(n-3)/4} E^{(n-2)/4} (2\pi)^{-(n-1)/2} f(\nu,\omega;E)$$

We have yet another formulation of the scattering problem:

Find a solution of the time independent Schrödinger equation

$$H\phi_+(x,\omega;E) = E\,\phi_+(x,\omega;E)$$ that

as $|x|\to\infty$ is a plane wave plus a outgoing spherical wave that satisfies Sommerfeld's radiation condition

Note that

$$\Psi_+(x,\omega,t,E) = \varphi_+(x,\omega,E)\, e^{-i\sqrt{E}t}$$ is a

solution of the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi_+ = H\Psi_+,$$

but $\Psi_+ \notin L^2$, i.e. it is not a physical solution with finite total probability. Also the energy of Ψ_+ is infinite.

We construct solutions in L^2 as follows

$$\Psi(x,t) = \int_{S^{n-1}\times\mathbb{R}^+} \Psi_+(x,\omega,t;E)\, \hat{g}(\omega,E)\, d\omega\, dE$$

$$\hat{g} \in \hat{\mathcal{H}} := L^2(\mathbb{R}^+, L^2(S^{n-1})).$$

Note that $\Psi_+(x,\omega,t;E)$ are periodic in time, there is no propagation as time evolves. This is why this method is called

<u>stationary or time-independent</u>

R. Weder, Spectral and Scattering Theory for Perturbed Stratified Media. Applied Mathematical Sciences, vol 87, Springer, 1991

Two Hilbert Space Scattering Theory

$\mathcal{H}_i$, $i=0,1$, Hilbert spaces.

H_i self-adjoint operators on $\mathcal{H}_i$

J: operator from $\mathcal{H}_0$ into $\mathcal{H}_1$

$$W_{\pm} = s-\lim_{t \to \pm\infty} e^{itH_1} J e^{-itH_0} P_{ac}(H_0)$$

Example

$\mathcal{H}_0 = L^2(\mathbb{R}^n)$, $\mathcal{H}_1 = L^2(\Omega)$,

Ω a exterior domain in $\mathbb{R}^n$

$J: \mathcal{H}_0 \to \mathcal{H}_1$

$(J\varphi)(x) = \chi_{\Omega}(x)\varphi(x)$.

In this case J is bounded.

In classical wave propagation J is a bijection, in quantum field theory it is not even closable.