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Fall Semester 2007

Inverse Scattering: The Time-Dependent Approach

1 Introduction

Unperturbed equation.

$$i\partial_t \varphi = H_0\varphi, \quad H_0 = -\Delta$$

Perturbed equation.

$$i\partial_t \varphi = H\varphi, \quad H = H_0 + V(x).$$

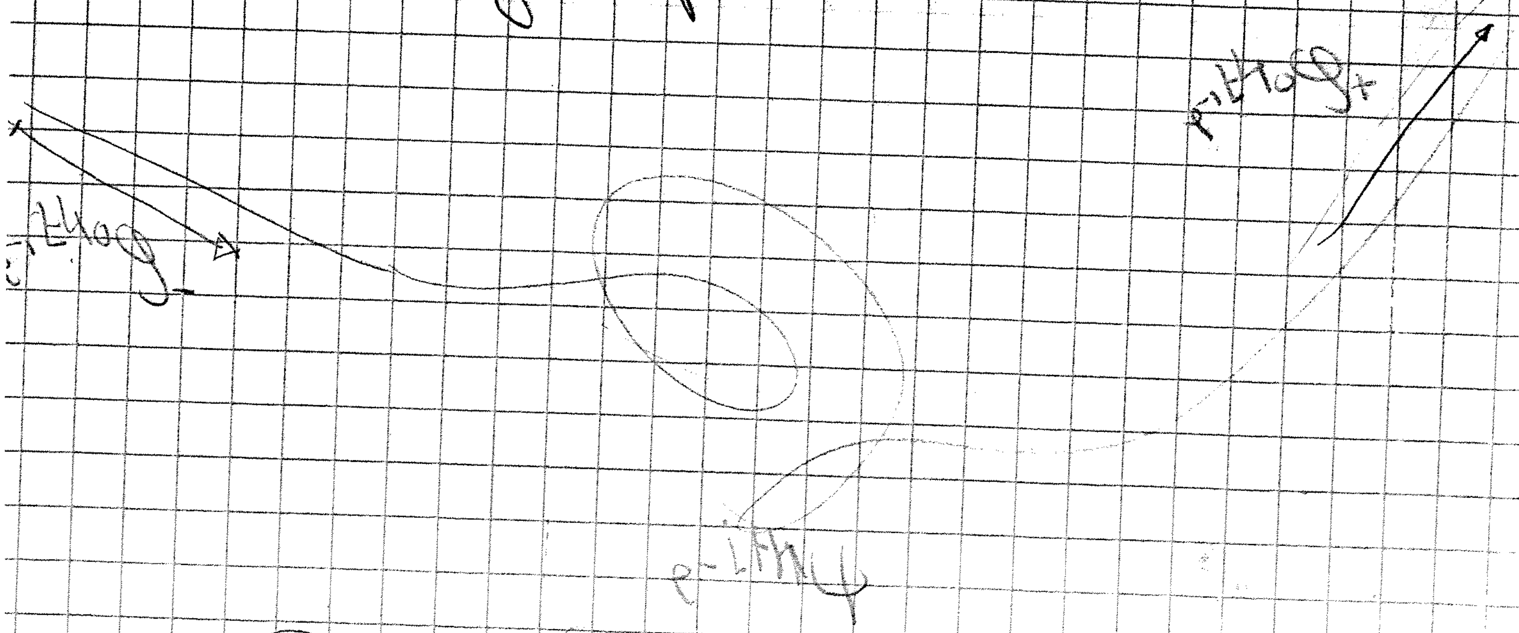
Suppose $|V(x)| \le C(1+|x|)^{-1-\epsilon}$. Later we consider more general potentials.

H_0 and H are self-adjoint operators in $L^2(\mathbb{R}^n)$, $D(H) = D(H_0) = H_2(\mathbb{R}^n)$ Sobolev space.

Free solutions $\varphi = e^{-itH_0}\varphi_0$

Interacting solution $\psi = e^{-itH}\psi_0$

A scattering experiment



$$S\varphi_- = \varphi_+$$

$W_\pm \varphi = s\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0}\varphi$

Denote

$\psi = W_- \varphi_-$. Then

$\lim_{t\to-\infty} \| e^{-itH}\psi - e^{-itH_0}\varphi_- \| = \lim_{t\to-\infty} \| \psi - e^{itH} e^{-itH_0}\varphi_- \|$

Then, the interacting solution is,

$e^{-itH} W_- \varphi_-$

Moreover, we look for φ_+ such that

$\lim_{t\to\infty} \| e^{-itH}\psi - e^{-itH_0}\varphi_+ \| = 0$, i.e.,

$\lim_{t\to\infty} \| \psi - e^{itH} e^{-itH_0}\varphi_+ \| = 0.$

$\psi = W_+ \varphi_+ \Rightarrow \varphi_+ = W_+^{-1}\psi = W_+^{*}\varphi,$

since $W_\pm$ turn out to be unitary

Finally

$\varphi_+ = S\varphi_- = W_+^{*} W_- \varphi_-$

$S = W_+^{*} W_-$ is a unitary operator.

Note that $e^{-itH_0}\varphi_\pm$, $e^{-itH}\psi$ are solutions in L^2, and since H_0, H are self-adjoint

$\| e^{-itH_0}\varphi_\pm \| = \| \varphi_\pm \|$, $\| e^{-itH}\psi \| = \|\psi\|$

(3)

In Q.M. $|\varphi(x)|^2$ is the probability density and

$\int |\varphi(x)|^2 dx < \infty$ means that the total probability is finite. For normalized states with $\|\varphi\| = 1$ the total probability is finite. The mean value of the energy of the state is

$$(H\varphi, \varphi) = \int (H\varphi)(x)\, \bar{\varphi}(x)\, dx.$$

This finite if $\varphi \in Q(H)$, the quadratic form domain, and in particular if $\varphi \in D(H)$, the operator domain.

The $W_\pm$ have the following properties.

1) They are isometric and asymptotically complete

$$\text{Range } W_\pm = \mathcal{H}_{p.p.}^{\perp} = \mathcal{H}_{ac}(H)$$

2) They have the intertwining property:

$$e^{-itH} W_\pm = W_\pm e^{-itH_0}:$$

$$e^{itH} W_\pm = e^{itH} \text{s-}\lim_{\tau \to \pm\infty} e^{-i\tau H} e^{i\tau H} e^{-i\tau H_0}$$

$$= \text{s-}\lim_{\tau \to \pm\infty} e^{i(\tau - t)H} e^{-i(\tau - t)H_0} e^{itH_0} = W_\pm e^{itH_0}$$

By Stone's theorem (4)

$W_\pm D(H_0) \subset D(H)$ and

$H W_\pm = W_\pm H_0$, $H_{ac} = W_\pm H_0 W_\pm^*$, $H_{ac} = H P_{ac}(H)$

Time-Dependent Definition of the Scattering Matrix

S^{n-1} unit sphere on $\mathbb{R}^n$, $n \geq 2$

$\mathbb{R}^+ = (0,\infty)$, $\mathcal{H} = L^2(\mathbb{R}^+, L^2(S^{n-1}))$.

Define (the Fourier transform in spherical coordinates)

$(\mathcal{F}_0 \varphi)(E,\omega) = 2^{-1/2} E^{\frac{(n-2)}{4}} \hat{\varphi}(\sqrt{E}\,\omega)$, $E \in \mathbb{R}_+$, $\omega \in S^{n-1}$

$$\hat{\varphi}(p) = \frac{1}{(2\pi)^{n/2}} \int e^{-ip\cdot x} \varphi(x)\, dx$$

Then

$\mathcal{F} H_0 \mathcal{F}^{-1} = E$,

the operator of multiplication by E.

By the intertwining property

$$S e^{-itH_0} = W_+^* W_- e^{-itH_0} = W_+^* e^{-itH} W_-$$
$$= e^{-itH_0} S$$

Then, S can be diagonalized by $\mathcal{F}$.

$$\hat{S} = \mathcal{F}_0 S \mathcal{F}_0^* = \int_{\mathbb{R}^+}^{\oplus} \hat{S}(E)\, dE, \text{ i.e.}$$

$(\hat{S}\varphi)(\omega,E) = \hat{S}(E)\,\hat{\varphi}(E)$, where

$\hat{S}(\lambda)$, is "the scattering matrix", is a unitary operator on $L^2(S^{n-1})$.

<u>Relation with the Stationary Theory</u>

The weighted L^2 spaces

$$L^2_s := \{ f : (1+x^2)^{s/2} f(x) \in L^2 \}$$

$$\|f\|_{L^2_s} := \|(1+x^2)^{s/2} f\|_{L^2}$$, $s \in \mathbb{R}$, and for any $a, s \in \mathbb{R}$

$$\mathcal{H}_{a,s} := \{ f : (1+x^2)^{s/2} f(x) \in \mathcal{H}_a \},$$

$$\|f\|_{\mathcal{H}_{a,s}} := \|(1+x^2)^{s/2} f(x)\|_{\mathcal{H}_a},$$

where $\mathcal{H}_a$ is the Sobolev space

$$\mathcal{H}_a := \{ f \in \mathcal{S}' : (1+p^2)^{a/2} \hat{f}(p) \in L^2 \}$$

$$\|f\|_{\mathcal{H}_a} := \|(1+p^2)^{a/2} \hat{f}\|_{L^2}.$$

The limiting absorption principle is the following statement. For $z \in \rho(H)$, the resolvent set of H, let $R(z) = (H-z)^{-1}$ be the resolvent. Then for every $E \in \mathbb{R}^+$