CONJUGATE POINTS, JACOBI FIELDS & CURVATURE

1° Riemannian curvature as a family of linear mappings through geodesic variation.

(M,g) complete riemannian manifold

spse: γ geodesic, $|\dot\gamma| = 1$ and H a geodesic variation, that is $H: (-\varepsilon, \varepsilon) \times [a,b] \to M$ smooth and

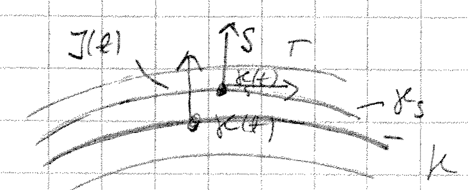
(1) $H(0,t) = \gamma(t) = \gamma_0(t) \quad \forall\, t \in [a,b]$

(2) $\forall\, s \in (-\varepsilon, \varepsilon) \quad t \mapsto \gamma_s(t) := H(s,t)$ is a geodesic

Denote $J(t) := \frac{\partial H}{\partial s}(s,t)\Big|_{s=0}$ variation field of H.

and $T = T^i \frac{\partial}{\partial x^i} := \frac{\partial H}{\partial t} = \dot\gamma_s$

$S = S^i \frac{\partial}{\partial x^i} := \frac{\partial H}{\partial s}$



$\forall\, s \ \gamma_s$ is a geodesic $\Leftrightarrow D_T T = 0 \quad \forall\, (s,t)$

$$\Leftrightarrow V^k := \frac{\partial T^k}{\partial t} + T^j T^i \Gamma^k_{ij} = 0 \quad \forall\, k \qquad (1.0.)$$

$D_T T = 0 \Rightarrow D_S D_T T = 0 \Rightarrow$

$$0 = \frac{\partial V^k}{\partial s} + V^i S^j \Gamma^k_{ij}$$

$$= \frac{\partial^2 T^k}{\partial s \partial t} + \frac{\partial T^j}{\partial s} T^i \Gamma^k_{ij} + T^j \frac{\partial T^i}{\partial s} \Gamma^k_{ij} + T^j T^i \frac{\partial \Gamma^k_{ij}}{\partial x^p} \underbrace{\frac{\partial H^p}{\partial s}}_{= S^p}$$

Riem. conn $\Rightarrow \Gamma_{ij} = \Gamma_{ji}$

$$= \frac{\partial^2 T^k}{\partial s \partial t} + 2 \frac{\partial T^j}{\partial s} T^i \Gamma^k_{ij} + T^j T^i S^p \frac{\partial \Gamma^k_{ij}}{\partial x^p} \qquad (1.1)$$

$$D_T D_T S = D_T \left\{ \left(\frac{\partial S^k}{\partial t} + S^i T^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k} \right\} =$$

$$\left\{ \frac{\partial}{\partial t} \left(\frac{\partial S^k}{\partial t} + S^i T^j \Gamma^k_{ij} \right) + \left(\frac{\partial S^l}{\partial t} + S^i T^j \Gamma^l_{ij} \right) T^m \Gamma^k_{lm} \right\} \frac{\partial}{\partial x^k}$$

(2)

$$= \left\{ \frac{\partial^2 S^k}{\partial t^2} + \frac{\partial S^i}{\partial t} T^j \Gamma_{ij}^k + S^i \frac{\partial T^j}{\partial t} \Gamma_{ij}^k + S^i T^j \frac{\partial \Gamma_{ij}^k}{\partial x^p} \underbrace{\frac{\partial H^p}{\partial t}}_{= T^p} \right.$$

$$\left. + \frac{\partial S^l}{\partial t} T^m \Gamma_{lm}^k + S^i T^j T^m \Gamma_{ij}^l \Gamma_{lm}^k \right\} \frac{\partial}{\partial x^k}$$

Note: $\frac{\partial T^i}{\partial s} = \frac{\partial^2 H^i}{\partial s \partial t} = \frac{\partial^2 H^i}{\partial t \partial s} = \frac{\partial S^i}{\partial t}$ (1.2)

$$\Rightarrow \frac{\partial^2 S^i}{\partial t^2} = \frac{\partial^2 T^i}{\partial t \partial s} \overset{(1.1)}{=} -2 \frac{\partial T^j}{\partial s} T^k \Gamma_{jk}^i - T^j T^k S^p \frac{\partial \Gamma_{jk}^i}{\partial x^p}$$

$$\overset{(1.2)}{=} -2 \frac{\partial S^j}{\partial t} T^k \Gamma_{jk}^i - T^j T^k S^p \frac{\partial \Gamma_{kj}^i}{\partial x^p}$$

$$\Rightarrow D_T D_T S = \left\{ S^i \underset{(3.0)}{\frac{\partial T^j}{\partial t}} \Gamma_{ij}^k + S^i T^j T^m \Gamma_{ij}^l \Gamma_{lm}^k + S^i T^j T^p \frac{\partial \Gamma_{ij}^k}{\partial x^p} \right.$$

$$-T^m T^s \Gamma_{ms}^j \qquad \left. - T^j T^i S^p \frac{\partial \Gamma_{ij}^k}{\partial x^p} \right\} \frac{\partial}{\partial x^k}$$

$$= -S^p \left\{ T^m T^s \Gamma_{ms}^j \Gamma_{pj}^k - T^j T^m \Gamma_{pj}^l \Gamma_{lm}^k - T^j T^i \frac{\partial \Gamma_{pj}^k}{\partial x^i} \right.$$

$$\left. + T^j T^i \frac{\partial \Gamma_{ij}^k}{\partial x^p} \right\} \frac{\partial}{\partial x^k}$$

$$=: -S^p R_p^k(T) \frac{\partial}{\partial x^k}$$

$\forall y \in T_xM$ we define a linear mapping

$$R_y = R_j^i(y) \frac{\partial}{\partial x^i} \otimes dx^j \Big|_x : T_xM \to T_xM$$

Now: $R_T(S) = R_j^i(T) S^j \frac{\partial}{\partial x^i}$ holds

$$\Rightarrow D_T D_T S - R_T(S) = 0 .$$

By restricting to geodesic $t \mapsto \gamma_0(t) = \gamma(t)$

$J(t) = \frac{\partial H}{\partial s}(s,t)\Big|_{s=0} = S(0,t)$ gain!

$$\boxed{D_{\dot\gamma} D_{\dot\gamma} J + R_{\dot\gamma}(J) = 0}$$ the Jacobi

equation of geodesic γ.

The family of transformations

$$R = \{ R_y : T_xM \to T_xM \mid y \in T_xM \setminus \{0\},\ x \in M \}$$

is the Riemannian curvature,

$$R^i_j(y) = y^k y^l \Gamma^s_{kl} \Gamma^i_{js} - y^k y^l \Gamma^m_{jk} \Gamma^i_{ml} - y^k y^l \frac{\partial \Gamma^i_{jk}}{\partial x^l} + y^k y^l \frac{\partial \Gamma^i_{kl}}{\partial x^j}$$

$$= y^k y^l \underbrace{\left\{ \Gamma^s_{kl} \Gamma^i_{js} - \Gamma^m_{jk} \Gamma^i_{ml} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \frac{\partial \Gamma^i_{kl}}{\partial x^j} \right\}}_{=:\ R_{k\ jl}^{\ i}(x)}$$

Connection to Riemann curvature tensor

$$R = R_{ijkl}(x)\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l, \quad R_{ijkl}(x) := g_{im} R_{j\ kl}^{\ m}$$

$$R(u,v)w := R_{k\ jl}^{\ i}(x)\, u^j v^l w^k \frac{\partial}{\partial x^i}. \quad (\Rightarrow R_y(u) = R^i_j(y) u^j = R_{k\ jl}^{\ i} y^k y^l \ldots = R(u,y)y.$$

Denote $\underline{P} \subset T_xM$ 2-plane $\underline{P} = sp\{u, v\}$

Sectional curvature of $\underline{P}$

$$K(\underline{P}) := \frac{\langle R_v(u), u \rangle}{|v|^2 |u|^2 - (\langle v, u \rangle)^2} \ \underset{\text{when } |v| = 1 = |u| \text{ and } \langle u, v \rangle = 0}{=} \ \langle R_v(u), u \rangle$$

can show: $K(P)$ independent of the choice of u and v.

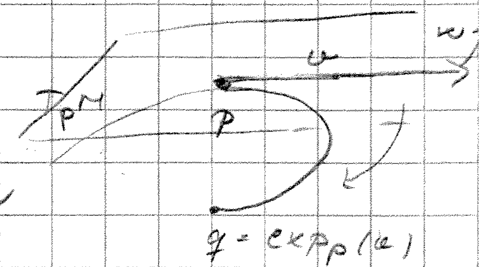
2° Conjugate points vs. Jacobi fields

(4.)

Spse: $p \in M$ and q conjugate to p along γ.

$\Rightarrow \exists\, v \in T_pM$ s.t. $d(\exp_p)_v(w) = 0$ for some $0 \neq w \in T_v(T_pM) \cong T_pM$, $\gamma(0) = p$, $\dot\gamma(0) \parallel v$

Set $H(s,t) := \exp_p t(v + sw)$.



Variation of $t \mapsto \exp_p(tv)$ through geodesics. Variation field for H is

$$V_t = \frac{d}{ds} H(s,t)\Big|_{s=0} = d(\exp_p)_{tv}(tW), \text{ where}$$

W parallel field of $w \in T_pM$: $D_{\dot\gamma} W = 0$, $W_p = w$

V_t is a Jacobi field that vanishes at $t=0$ and $t=1$

$V_0 = d(\exp_p)_0(0) = 0$

$V_1 = d(\exp_p)_v(w) = 0$

$V \neq 0$ since exp local diffeo at $0 \in T_pM$

Also converse holds:

Theorem: Spse $\gamma: [0,1] \to M$ geodesic, $\gamma(0) = p$. Then $q = \gamma(1)$ is conjugate to p along γ $\Leftrightarrow$ $\exists$ non-trivial Jacobi field V along γ s.t. $V_0 = 0 = V_1$.

"$\Leftarrow$": Idea: Study variation $H(s,t) = \exp_p t(\dot\gamma(0) + s D_{\dot\gamma(0)} V_0)$

$q = \exp_p(\dot\gamma(0))$ conjugate to p along γ

3° Second variation formula (à la Synge -26)

Spse: $\gamma : [a,b] \to M$ geodesic $|\dot\gamma| = 1$ and $H:$

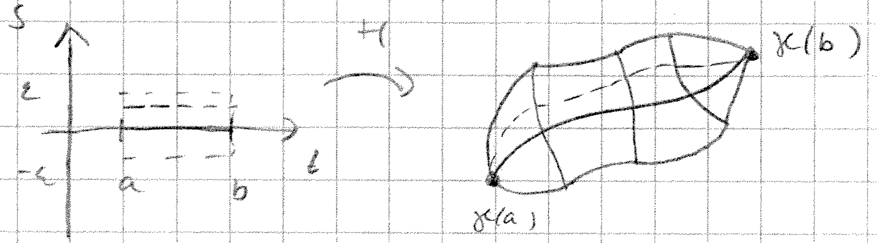
$]-\varepsilon,\varepsilon[\times [a,b] \to M$ (general) variation of γ ($\gamma(t) = H(0,t)$)

so that (1) $H|_{]-\varepsilon,\varepsilon[\times [a_{i-1},a_i]}$ C^∞ $\forall i$ for some $a=a_0 < a_1 < \dots < a_k = b$

(2) $\forall$ $s \in]-\varepsilon,\varepsilon[$ $H_s : [a,b] \to M$ $H_s(t) = H(s,t)$)

is piecewise regular (C^∞ & $\dot H_s \neq 0$)

(3) $H_s(a) = \gamma(a)$, $H_s(b) = \gamma(b)$ $\forall s \in]-\varepsilon,\varepsilon[$



Then

(3.1) $$\frac{d^2}{ds^2} \ell(H_s)\Big|_{s=0} = \int_a^b \left(|D_{\dot\gamma} V|^2 - \langle R(V,\dot\gamma)\dot\gamma, \dot\gamma \rangle \right) dt,$$

where V is the variation of H along γ

s.t. $V(t) = \frac{d}{ds} H(s,t)\Big|_{s=0} \perp \dot\gamma$. (For gen. V (3.1)

holds for $V^\perp = V - \langle V,\dot\gamma\rangle\dot\gamma$.) Denote $T = T^i \frac{\partial}{\partial x^i} := \frac{\partial H}{\partial t}$ and

$S = S^i \frac{\partial}{\partial x^i} = \frac{\partial H}{\partial s}$ as in geodesic variation on page 1.

Proof: (1) $\Rightarrow$

$$\frac{d}{ds} \ell(H_s|_{[a_{i-1},a_i]}) = \frac{d}{ds} \int_{a_{i-1}}^{a_i} \langle T,T\rangle^{1/2} dt = \int_{a_{i-1}}^{a_i} \frac{1}{2} \langle T,T\rangle^{-\frac{1}{2}} \underbrace{\frac{d}{ds}\langle T,T\rangle}_{= 2\langle D_S T, T\rangle} dt$$

$$D_S T \overset{(1.2)}{=} D_T S$$

$$= \int_{a_{i-1}}^{a_i} \frac{\langle D_T S, T\rangle}{|T|} dt.$$

(6.)

$$\Rightarrow \frac{d^2}{ds^2} \, l(H_s|_{[a_{i-1},a_i]}) = \int_{a_{i-1}}^{a_i} \frac{d}{ds}\left(\frac{\langle D_t S, T\rangle}{|T|} \right) dt$$

$$= \int_{a_{i-1}}^{a_i} \left\{ \frac{\langle D_s D_t S, T\rangle + \langle D_t S, \underbrace{D_s T}_{= D_t S}\rangle}{|T|} - \frac{\frac{d}{ds}\left(\langle T,T\rangle^{1/2}\right)\langle D_t S, T\rangle}{|T|^2} \right\} dt$$

$$\frac{d}{ds}\left(\langle T,T\rangle^{1/2}\right) = \frac{1}{2}\langle T,T\rangle^{-\frac{1}{2}} \, 2 \cdot \langle D_s T, T\rangle = \frac{\langle D_t S, T\rangle}{|T|}$$

$$= \int_{a_{i-1}}^{a_i} \left\{ \frac{\langle D_s D_t S, T\rangle + |D_t S|^2}{|T|} - \frac{\langle D_t S, T\rangle^2}{|T|^3} \right\} dt$$

<u>Need</u>: $D_s D_t S - D_t D_s S = -R_S(T)$!

$$D_t S = \left\{ \frac{\partial S^k}{\partial t} + S^i T^j \Gamma_{ij}^k \right\} \frac{\partial}{\partial x^k}$$

$$D_s D_t S = \left[\frac{\partial}{\partial s}\left\{ \frac{\partial S^k}{\partial t} + S^i T^j \Gamma_{ij}^k \right\} + \left\{ \frac{\partial S^i}{\partial t} + S^p T^j \Gamma_{pj}^i \right\} S^l \Gamma_{il}^k \right] \frac{\partial}{\partial x^k}$$

$$= \Bigg[\frac{\partial^2 S^k}{\partial s \partial t} + \frac{\partial S^i}{\partial s} T^j \Gamma_{ij}^k + S^i \underbrace{\frac{\partial T^j}{\partial s}}_{(1.2) = \frac{\partial S^j}{\partial t}} \Gamma_{ij}^k + S^i T^j \frac{\partial \Gamma_{ij}^k}{\partial x^p} \underbrace{\frac{\partial H^p}{\partial s}}_{= S^p} +$$

$$\frac{\partial S^i}{\partial t} S^l \Gamma_{il}^k + S^p T^j S^l \Gamma_{pj}^i \Gamma_{il}^k \Bigg] \frac{\partial}{\partial x^k}$$

$$D_s S = \left\{ \frac{\partial S^k}{\partial s} + S^i S^j \Gamma_{ij}^k \right\} \frac{\partial}{\partial x^k}$$

$$D_t D_s S = \left\{ \frac{\partial}{\partial t}\left(\frac{\partial S^k}{\partial s} + S^i S^j \Gamma_{ij}^k \right) + \left(\frac{\partial S^i}{\partial s} + S^p S^j \Gamma_{pj}^i \right) T^l \Gamma_{il}^k \right\} \frac{\partial}{\partial x^k}$$

$$= \Bigg[\frac{\partial^2 S^k}{\partial t \partial s} + 2 \frac{\partial S^i}{\partial t} S^j \Gamma_{ij}^k + S^i S^j \frac{\partial \Gamma_{ij}^k}{\partial x^p} \underbrace{\frac{\partial H^p}{\partial t}}_{= T^p} + \frac{\partial S^i}{\partial s} T^l \Gamma_{il}^k + S^p S^j T^l \Gamma_{pj}^i \Gamma_{il}^k \Bigg] \frac{\partial}{\partial x^k}$$

$$\Rightarrow D_s D_t S - D_t D_s S = \left\{ S^i S^p T^j \left(\frac{\partial \Gamma_{ij}^k}{\partial x^p} + \Gamma_{ij}^m \Gamma_{mp}^k - \frac{\partial \Gamma_{ip}^k}{\partial x^j} - \Gamma_{ip}^m \Gamma_{mj}^k \right) \right\} \frac{\partial}{\partial x^k}$$

$$= - R_j^k(S) T^j \frac{\partial}{\partial x^k} = -R_S(T) \, .$$

$$\Rightarrow \quad \frac{d^2}{ds^2}\, \ell\left(H_s|_{[a_{i-1},a_i]}\right) = \int_{a_{i-1}}^{a_i} \left\{ \frac{\langle D_T D_S S - R_S(T), T\rangle + |D_T S|^2}{|T|} - \frac{\langle D_T S, T\rangle^2}{|T|^3} \right\} dt \, .$$

Kun $s=0$ $\quad |T|=1 \quad \Rightarrow$

$$\frac{d^2}{ds^2}\, \ell\left(H_s|_{[a_{i-1},a_i]}\right)\Big|_{s=0} = \int_{a_{i-1}}^{a_i} \left\{ \langle D_T D_S S, T\rangle - \langle R_S(T), T\rangle + |D_T S|^2 - \langle D_T S, T\rangle^2 \right\} dt$$

$T(0,t) = \dot\gamma(t) \quad \Rightarrow \quad D_T T = D_{\dot\gamma}\dot\gamma = 0$ kun $s=0$ (γ geod.),

joten

a) $$\int_{a_{i-1}}^{a_i} \langle D_T D_S S, T\rangle \, dt \Big|_{s=0} = \int_{a_{i-1}}^{a_i} \frac{d}{dt} \langle D_S S, T\rangle \, dt \Big|_{s=0}$$

$$= \langle D_S S(0,a_i), \dot\gamma(a_i)\rangle - \langle D_S S(0,a_{i-1}), \dot\gamma(a_{i-1})\rangle \, .$$

b) $S(s,t) = 0 \quad \forall\, s$ kun $t = a$ tai b

$$\Rightarrow \quad D_S S(s,a) = 0 = D_S S(s,b)$$

c) Lisäksi

$D_S S$ jatkuva $\forall\ (s,t)$

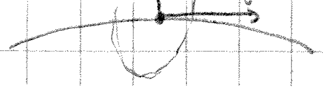
$$\overset{a)b)c)}{\underset{V_t = S(0,t),\ T_0 = \dot\gamma}{\Rightarrow}} \quad \frac{d^2}{ds^2}\, \ell(H_s)\Big|_{s=0} = \int_a^b \left\{ |D_{\dot\gamma} V|^2 - \langle D_{\dot\gamma} V, \dot\gamma\rangle^2 - \langle R_V(\dot\gamma), \dot\gamma\rangle \right\} dt \, .$$

If $V \perp \dot\gamma \Rightarrow \quad 0 = \frac{d}{dt}\langle V, \dot\gamma\rangle = \langle D_{\dot\gamma} V, \dot\gamma\rangle + \langle V, \underbrace{D_{\dot\gamma}\dot\gamma}_{=0}\rangle$

$$\Rightarrow \quad \langle D_{\dot\gamma} V, \dot\gamma\rangle = 0 \quad \Rightarrow \quad (3.1). \ \square \, .$$

Especially $(3.1) \Rightarrow$

$$\frac{d^2}{ds^2}\, \ell(H_s)\Big|_{s=0} = \int_a^b \left(|D_{\dot\gamma} V|^2 - \underbrace{K(V,\dot\gamma)}_{\text{sectional curvature of the plane } \mathrm{sp}\{V,\dot\gamma\}} |V|^2 \right) dt \, .$$

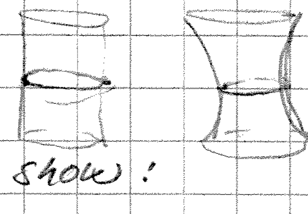


Remark: If $K(v,\dot{\gamma}) \leq 0 \quad \forall t$ then

$$\frac{d^2}{ds^2} \ell(H_s)\Big|_{s=0} > 0$$ and the geodesic γ

locally minimizes length (that is, among <u>close-by curves</u>). Need not be globally

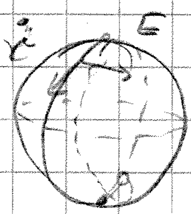
minimizing as seen before.



If $K(v,\dot{\gamma}) > 0 \quad \forall t$ then can show:

every long enough geodesic <u>does not minimize</u> <u>even locally</u>.

<u>Example</u>: S^2 globally situation clear:



when $\ell = \ell(\gamma) > \pi$ γ does not minimize

local minimizing property breaks down as well:

Set $V_t := \sin\left(t\frac{\pi}{\ell}\right) E \qquad |E| = 1, \quad E_0 \perp \dot{\gamma}(0)$

$D_{\dot{\gamma}} E = 0$.

$$\Rightarrow \frac{d^2}{ds^2} \ell(H_s)\Big|_{s=0} = \int_0^\ell \left\{ \left|\frac{dV_t}{dt}\right|^2 - \overbrace{K(\dot{c}, V_t)}^{=1} |V_t|^2 \right\} dt$$

$$= \int_0^\ell \underbrace{\left\{ \left(\frac{\pi}{\ell}\right)^2 \cos^2\left(t\frac{\pi}{\ell}\right) - \sin^2\left(t\frac{\pi}{\ell}\right) \right\}}_{= \cos^2\left(t\frac{\pi}{\ell}\right)\left(\left(\frac{\pi}{\ell}\right)^2 + 1\right) - 1 \, = \, \frac{1+\cos\frac{2t\pi}{\ell}}{2}\left(\left(\frac{\pi}{\ell}\right)^2+1\right) - 1} dt$$

$$= \Big/_0^\ell \frac{1}{2}\left\{ t + \left(\sin\frac{2t\pi}{\ell}\right)\frac{\ell}{2\pi} \right\}\left(\left(\frac{\pi}{\ell}\right)^2 + 1\right) - t$$

$$= \frac{1}{2}\,\ell\left(\left(\frac{\pi}{\ell}\right)^2 + 1\right) - \ell = -\frac{1}{2\ell}\left(\ell^2 - \pi^2\right) < 0 \quad \text{when } \ell > \pi.$$

and variation gives curves that are both shorter than γ and also nearby!