Cristopher B. Croke,
"Rigidity and Distance Between Boundary Points"
J. Differential Geometry 33 (1991) 465-486

Boundary rigidity

(M,g) compact Riemannian manifold w/ C^∞-boundary

$$L(\gamma) = \int_a^b \sqrt{g(\dot\gamma(t),\dot\gamma(t))}^2\,dt \quad , \quad \gamma:[a,b]\to M$$

$$d(x,y) := \inf\{L(\gamma) \mid \gamma(a)=x,\ \gamma(b)=y\}$$

M compact $\Rightarrow \exists$ ~~smooth~~ piecewise $\gamma:[a,b]\to M$ s.t. $d(x,y)=L(\gamma)$.

- If $\gamma(]c,d[)\subset M\setminus\partial M$, $\gamma|_{[c,d]}$ is a geod. of M
- If $\gamma([c,d])\subset\partial M$, $\gamma|_{[c,d]}$ is a geod. of ∂M

Q: For which classes of (M,g) does $d|_{\partial M\times\partial M}$ determine (M,g) uniquely?

Lens rigidity

For any $\xi\in\partial_\pm SM := \{\xi\in SM \mid \pi(\xi)\in\partial M,\ \pm g(\nu,\xi)>0\}$
define $\tau_\pm(\xi) := \inf\{t>0 \mid \Phi_H^{\pm t}(\xi)\in\partial SM\}$

$F:\partial_- SM\to\mathbb{R}\times\partial SM$; $F(\xi) = (\tau_+(\xi), \Phi_H^{+\tau(\xi)}(\xi))$.

Q: For which classes of (M,g) does F determine (M,g) uniquely?

Note: Boundary rigid $\Rightarrow$ Lens rigid

Obstruction #1

If $\exists z\in M$ s.t. $\sup_{x,y\in\partial M} d(x,y) < d(z,\partial M)$, then one can change (M,g) near z without affecting $d|_{\partial M\times\partial M}$ at all.

Note: In this case the geodesics through z do not minimize the distances between their endpoints at the boundary.

Obstruction #2 (surfaces of revolution)

$M = \{(x,y,z) \mid x^2+y^2 = f(z)^2,\ 0 \le z \le L\}$, $f(z) \ge 1$

$$\begin{cases} x = f(z)\cos\varphi \\ y = f(z)\sin\varphi \end{cases} \Rightarrow \begin{cases} dx = -f(z)\sin\varphi\, d\varphi + f'(z)\cos\varphi\, dz \\ dy = f(z)\cos\varphi\, d\varphi + f'(z)\sin\varphi\, dz \end{cases}$$

$$\Rightarrow \boxed{g_M = dx^2+dy^2+dz^2 = f(z)^2 d\varphi^2 + (1+f'(z)^2)dz^2}$$

Geod. eq. $\ddot\gamma^k + \dot\gamma^i\dot\gamma^j\Gamma_{ij}^k = 0$, $\Gamma_{ij}^k = \frac{g^{kl}}{2}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)$

$$\Gamma_{\varphi z}^{\varphi} = \frac{f'(z)}{f(z)},\quad \Gamma_{\varphi\varphi}^{z} = -\frac{f(z)f'(z)}{1+f'(z)^2},\quad \Gamma_{zz}^{z} = \frac{f'(z)f''(z)}{1+f'(z)^2}$$

$$\Rightarrow \begin{cases} \ddot\varphi + 2\dot\varphi\dot z\,\dfrac{f'(z)}{f(z)} = 0, \\ \ddot z - (\dot\varphi)^2\dfrac{f(z)f'(z)}{1+f'(z)^2} + (\dot z)^2\dfrac{f'(z)f''(z)}{1+f'(z)^2} = 0. \end{cases}$$

$$\frac{\ddot\varphi}{\dot\varphi} = -2\frac{f'(z)\dot z}{f(z)} \Rightarrow \frac{d}{dt}\ln\dot\varphi = -2\frac{d}{dt}\ln f(z) \Rightarrow \boxed{\dot\varphi = \frac{c}{f(z)^2}}$$

$$\Rightarrow \boxed{g\left(\dot\gamma, \tfrac{\partial}{\partial\varphi}\right) = f(z)^2\dot\varphi = c.}$$

$$g(\dot\gamma,\dot\gamma) = f(z)^2(\dot\varphi)^2 + (1+f'(z)^2)(\dot z)^2 = 1 \Rightarrow \boxed{(\dot z)^2 = \frac{f(z)^2-c^2}{f(z)^2(1+f'(z)^2)}}$$

If $c^2 < 1$, then $z(t)$ is strictly monotone, and the traveltime is

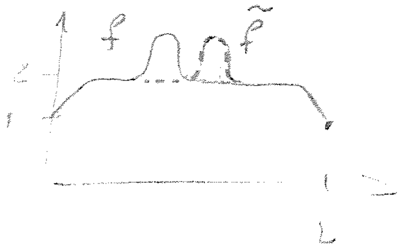
$$T = \int_0^L \frac{dt}{dz}dz = \int_0^L f(z)\sqrt{\frac{1+f(z)^2}{f(z)^2-c^2}}\,dz. \qquad (*)$$

Assume that $f(0) = f(L) = 1$. Then the initial & final angles with the boundary are equal to $\arccos c$.

The final angular coordinate is given by

$$\varphi(T) - \varphi(0) = \int_0^T \dot\varphi(t)dt = \int_0^L \frac{c}{f(z)}\frac{dt}{dz}dz = \int_0^L \frac{c}{f(z)^2}\sqrt{\frac{1+f'(z)^2}{f(z)^2-c^2}}\,dz \qquad (**)$$

Choose f & $\tilde f$ which give the same values for $(*)$ & $(**)$. Then M & $\tilde M$ are equivalent as geodesic lenses.



Note: There exist an infinitely long geodesic, and consequently a grazing ∞-long geod.

Note: All geodesics emanating from the boundary minimize distances between first points of contact ∇

BGM-manifolds

- Loose definition: All geodesics with $\gamma(]a,b[)\cap\partial M=\emptyset$ strongly minimize distances between boundary points and there do <u>not</u> exist (possibly grazing) geodesics of infinite length.
- Can be defined in terms of $d|_{\partial M\times\partial M}$ only!

<u>Def</u>: [M. Gromov "Filling Riemannian Manifolds" J. Diff. Geom. 18 (1983)]
Compact Riemannian (M,g) w/ C^∞-boundary is BGM if $\forall \xi\in T\partial M$ w/ $|\xi|<1$ $\exists!\, y\in\partial M$:

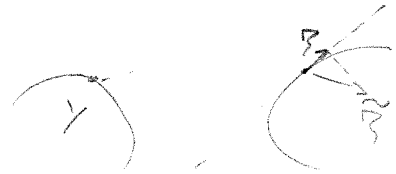
(1) $\operatorname{grad}_{\partial M} d(y,\cdot)=\xi \quad (d(y,\cdot):\partial M\to\mathbb{R})$

(2) $\nexists\, z\in\partial M\setminus\{y,\underbrace{\pi(\xi)}_{=x}\}: d(x,z)+d(z,y)=d(x,y)$

<u>Claim</u>: All geod. w/ $\gamma(]a,b[)\cap\partial M=\emptyset$ and $\gamma(a),\gamma(b)\in\partial M$ strongly minimize distances <u>iff</u> (M,g) is BGM.

<u>Proof</u>: For any $\xi\in T\partial M$ w/ $|\xi|\le 1$, let

$$\tilde\xi=\xi+\sqrt{1-|\xi|^2}\,\nu\in\partial_+SM\cup S\partial M$$



(1-1 correspondence between ξ & $\tilde\xi$)

"$\Rightarrow$": Let $y=\pi\circ\Phi_H^{-\tau_-(\tilde\xi)}(\tilde\xi)$. Then (2) holds by assumption and (1) holds, because $\tilde\xi=\operatorname{grad}_M d(y,\cdot)$ and $\operatorname{grad}_{\partial M}$ is the projection of grad_M into $T\partial M$.

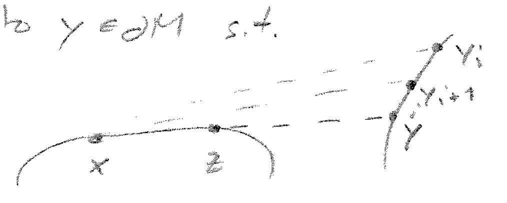
"$\Leftarrow$": Let $\gamma:[a,b]\to M$ be a geod. w/ $\gamma(]a,b[)\cap\partial M=\emptyset$ and $y=\gamma(a)\in\partial M$, $x=\gamma(b)\in\partial M$, $\tilde\xi=\dot\gamma(b)\in\partial_+SM$.
(2) $\Rightarrow$ the minimizing path between x & y is a geod. w/ $\gamma(]a,b[)\cap\partial M=\emptyset$.
(1) $\Rightarrow$ γ must be the <u>unique</u> minimizing geodesic $\square$

<u>Note</u>: Counterexamples in H2 are BGM!

<u>Def</u>: $\gamma:[a,b]\to\partial M$ is a "straight segment of ∂M" if it is a geodesic of M as well as of ∂M, and if $L(\gamma)=d(\gamma(a),\gamma(b))$.

<u>Note</u>: $\gamma:[a,b]\to\partial M$ is a straight segment of ∂M iff $L(\gamma)=d(\gamma(a),\gamma(b))$ and $\exists\{\xi_i\}\in T_x\partial M$ w/ $|\xi_i|<1$ & $\xi_i\to\xi\in S\partial M$ s.t. y_i determined from the def. of BGM converge to $y\in\partial M$ s.t.

$$d(x,z)+d(z,y)=d(x,y).$$



Def: $x, y \in \partial M$ are "straight connected", if the shortest curve between them is a (possibly grazing) geodesic.

Note: $x, y \in \partial M$ are "straight connected", if
$$\{z \in \partial M \mid d(x,z) + d(z,y) = d(x,y)\}$$
is a countable collection of straight segments of ∂M.

Def: (M,g) is SGM if it is BGM and $\nexists \{x_i\}_{i \in \mathbb{Z}} \in \partial M$ s.t.

- x_i is straight connected to x_{i+1},
- $\| \mathrm{grad}_{\partial M}(d(x_{i-1}, \cdot))|_{x_i} \| = 1$, "tangential endpoints"
- $\mathrm{grad}_{\partial M}(d(x_{i-1}, \cdot))|_{x_i} = - \mathrm{grad}_{\partial M}(d(x_{i+1}, \cdot))|_{x_i}$ "same direction at x_i"
- $\sum_{i=-\infty}^{0} d(x_i, x_{i+1}) = \infty$ & $\sum_{i=0}^{\infty} d(x_i, x_{i+1}) = \infty$.

Note: If (M,g) is BGM and there exists a complete geodesic, there must be a complete <u>grazing</u> geodesic, which must (by BGM) hit the boundary tangentially infinitely often, which is prohibited by <u>def. of SGM</u>.

Lemma: If (M,g) & $(\tilde{M}, \tilde{g})$ are BGM and $d|_{\partial M \times \partial M} = \tilde{d}|_{\partial \tilde{M} \times \partial \tilde{M}}$ then (M,g) & $(\tilde{M}, \tilde{g})$ are equivalent as geod. lenses.

Proof: $d|_{\partial M \times \partial M}$ determines the first point of contact $y \in \partial M$ for any $\tilde{\xi} \in \partial_+ SM$ and it is by the unique geodesic between $x = \pi(\tilde{\xi})$ and y. Next we let $\eta \in SyM$ be the unique unit vector which gives this geodesic. Now $F(-\tilde{\xi}) = -\eta \in \partial_+ SM \cup S\partial M$ $\square$