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# Conjugates & cuts

Spse: $(M,g)$ complete Riemannian manifold (1.)

## 1° Cut locus

1905 Poincaré "ligne de partage"

Fix $m \in M$, $\gamma : [0,\infty) \to M$ geodesic $\gamma(0) = m$
$|\dot\gamma| = 1$

Earlier: For small $t > 0$

(1.1.) $d(m, \gamma(t)) = t$ holds ($d$ Riemannian distance induced by $g$)

Denote $t_0 = \sup \{ t > 0 \;/\; d(m, \gamma(t)) = t \}$

1.2. Definition

If $t_0 < \infty$ then $p = \gamma(t_0)$ is the cut point (or: cut value) of $m$ along geodesic $\gamma$.

Hence: cut point is the last point on $|\gamma|$ where (1.1) remains true.

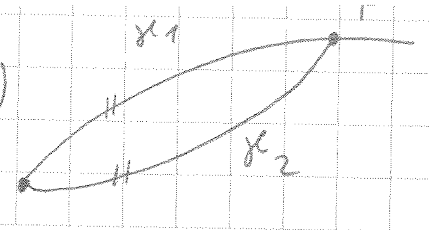
Denote $\mathrm{Cut}(m) := \{ p \in M \;/\; p \text{ is the cut point of } m \text{ along some } \gamma \}$
the cut locus of $m$.

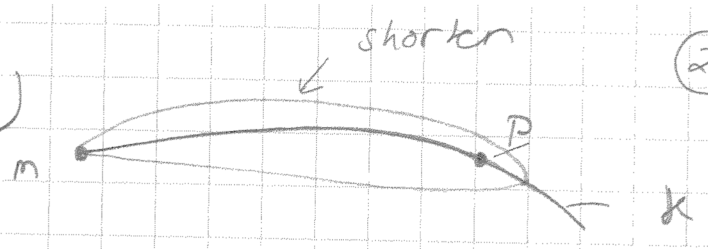
$\mathrm{Cut}^*(m) := \{ X \in T_m M \;/\; p = \exp_m(X) \text{ cut point along } \exp_m(tX) \}$ cut locus in $T_m M$

Main source of confusion: $\exists$ two types of cut points

a) $p \in \mathrm{Cut}(m)$ along $\gamma_1$ and $\gamma_2$, $\gamma_1 \neq \gamma_2$

b) $p \in \mathrm{Cut}(m)$ along single $\gamma$.

a) 

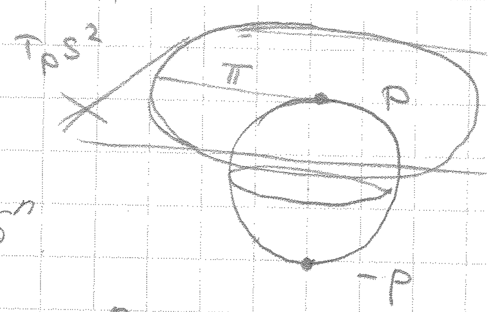

b) 


Can show (F. Warner 1965):

$$Cut(m) = \{p \in M \mid p \text{ cut point of } m \text{ of type a)}\}$$

Basic examples:

1) Round unit sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$:

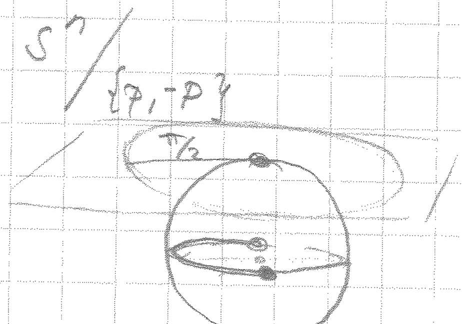



$$Cut(p) = \{-p\} \quad \forall\, p \in S^n$$

$$Cut^*(p) = S^{n-1}(0,\pi) \subset T_pS^n$$

2) Real projective space $\mathbb{R}P^n = S^n / \{p,-p\}$

$$Cut(p) = \pi(S^{n-1}) \approx \mathbb{R}P^{n-1}$$

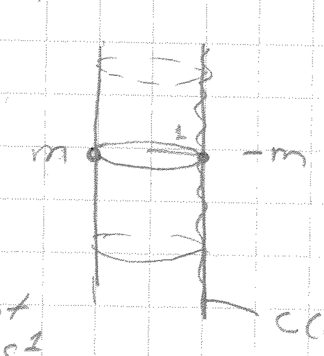



$\pi : S^n \longrightarrow \mathbb{R}P^n$ projection

$S^{n-1} \hookrightarrow S^n$ equator

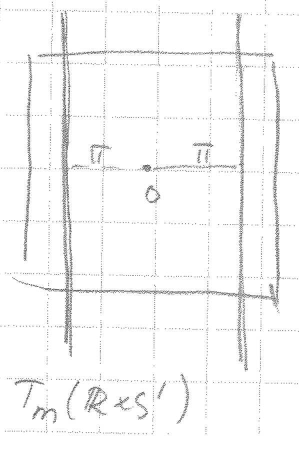
$$Cut^*(p) = S^{n-1}\left(0, \tfrac{\pi}{2}\right) \subset T_p\mathbb{R}P^n$$

3) Infinite cylinder: $\mathbb{R} \times S^1$
induced metric from $\mathbb{R}^3$



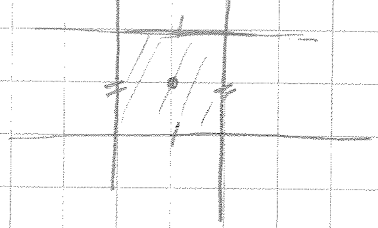

$$Cut(m) = \mathbb{R} \times \underbrace{\{-m\}}_{\text{antipodal pt of } m \text{ in } S^1}$$

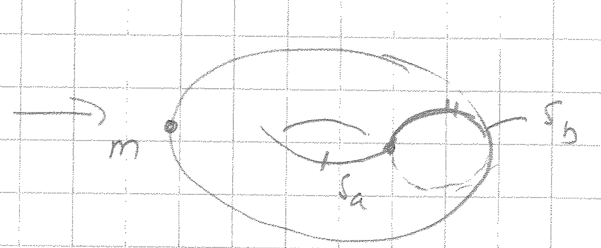
$Cut^*(m)$ = two lines:



4) Flat torus $T^2$ metric induced from $\mathbb{R}^2$

roughly:



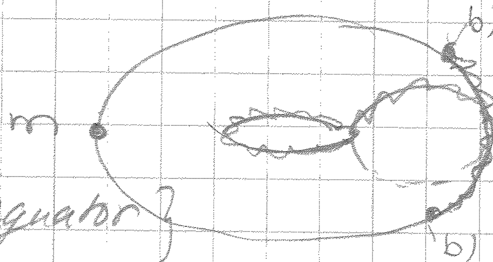



$$cut(m) = S^1_a \cup S^1_b$$

$$cut^k(m) = \partial \text{ Fundamental domain}$$

5) Torus of revolution metric induced from $\mathbb{R}^3$

If $m \in$ outer equator




cut(m) = {inner equator}
∪ {opposite meridian}
∪ {arc of outer equator}

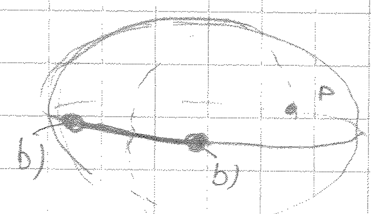
gen. open ?

6) ellipsoid

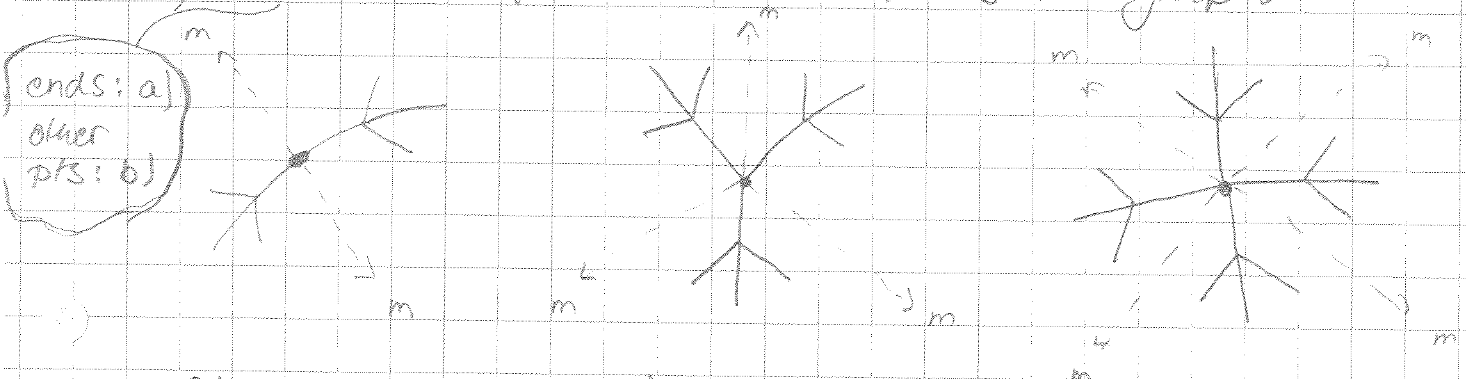
Conjecture: Braunmühl 1878 $\forall m$: $C(m)$ = arc

Proof: Itoh & Kiyohara 2004 (!)

- geodesics of an ellipsoid explicitely given by hyperelliptic functions!




7) Generic surface cut locus ≈ graph

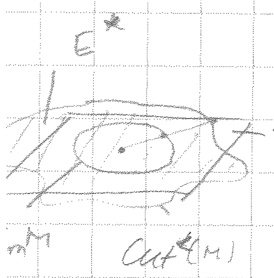



Gluck & Singer (1979): nasty examples (infinite graphs, ends accumulating at one point etc)

Basic properties:

- $Cut(m)$ closed for $\forall\, m \in M$

- $f_m : S_m M \to \mathbb{R}$ , $S_m M = \{ \bar{X} \in T_m M \,/\, |\bar{X}| = 1 \}$




$f_m(\bar{X}) = d(m, t\bar{X})$ , $t\bar{X} \in Cut^*(m)$ for some $t > 0$

is a continuous function

$\Rightarrow$ if $\boxed{M \text{ compact}}$ $\quad f_m(\bar{X}) < \infty \quad \forall\, \bar{X}$ : no geodesics of length $>$ diam$(M)$ can be minimizing!

$Cut^*(m) \approx S^{n-1}$

- Denote $E^* \subset T_m M$ domain bounded by $C^*(M)$ then

$M = E \cup Cut(M) \qquad E = \exp_m(E^*)$ (homeomorphically)

disjoint union , $E$ open and closed $Cut(m)$

especially when $M$ compact

$Cut(m) = \exp_m Cut^*(m) \approx \exp_m S^{n-1}$

connected!

- $Cut(m)$ inherits many topological properties of $M$ : especially $M$ simply connected $\Leftrightarrow$ $Cut(m)$ —"—

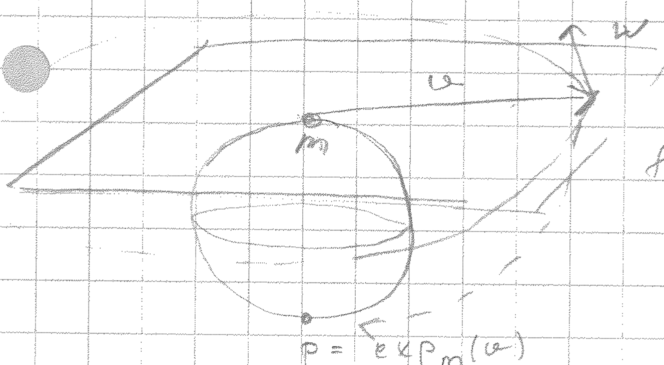
- $\pi_k(M)$ , $\pi_k(Cut(m))$ also related $\forall\, k$

## 2° Conjugate locus

Fix $m \in M$. Point $p \in M$ is <u>a conjugate point of $m$</u> if it is a critical value of the mapping $\exp_m : T_mM \to M$. This means:

mapping $d(\exp_m)_v : \underbrace{T_v(T_mM)}_{\cong T_mM} \to T_pM$

is singular (= not injective) for some $v \in T_mM$




so. $d(\exp_m)_v(w) = 0$

for some $w \in T_v(T_m(M)) \setminus \{0\}$

The <u>order of conjugacy</u> at $v$

$= \dim(\ker d(\exp_m)_v)$

Call $p$ <u>a conjugate point of $m$ along</u> geodesic $\gamma$ if $\dot\gamma(0) \parallel v$ and $\gamma(0) = m$.

<u>Conjugate locus</u> of $m$ $C(m) = \{p \in M \mid p \text{ conjugate point of } m\}$

<u>Conjugate locus of $m$ in $T_mM$</u> : $C^*(m) = \{X \in T_mM \mid p = \exp_m(X) \in C(m),\ d(\exp_m)(tX) \neq 0\ \forall t \in (0,1)\}$

<u>Examples</u>:

1) $S^n \hookrightarrow \mathbb{R}^{n+1}$ round, $\forall m$: $C(m) = \mathrm{Cut}(m) = \{-m\}$

2) $\mathbb{R}P^n$ $C(m) = \{m\}$

$m$ conjugate to itself along any geodesics emanating from $m$

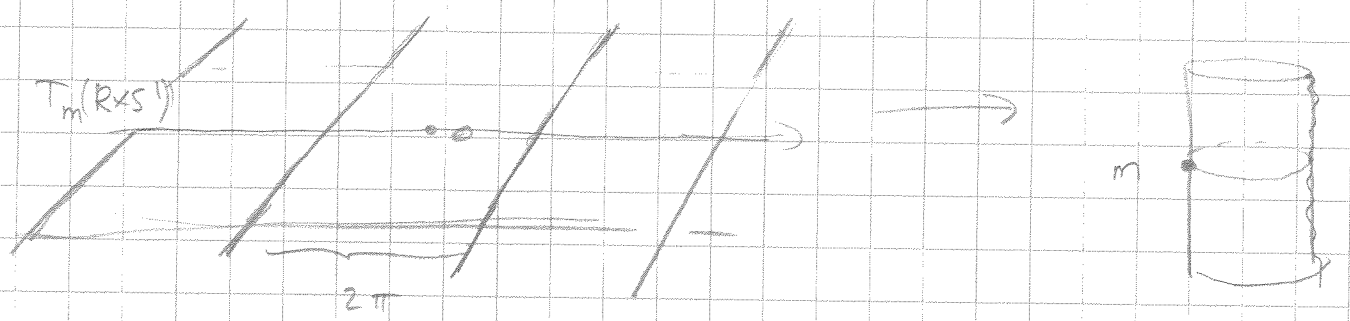
$\pi : S^n \to \mathbb{R}P^n$ riemannian covering (local homeo)

$\exp_m^{\mathbb{R}P^n} : T_m\mathbb{R}P^n \cong T_mS^n \xrightarrow{\exp_m^{S^n}} S^n \xrightarrow{\pi} \mathbb{R}P^n$

esp. $C(m) \cap \mathrm{Cut}(m) = \emptyset$

3) $\mathbb{R}\times S^1$ $C(m)=\emptyset$ $\forall$ $m$.

$\exp_m: \mathbb{R}^2 \to \mathbb{R}\times S^1$ riemannian covering

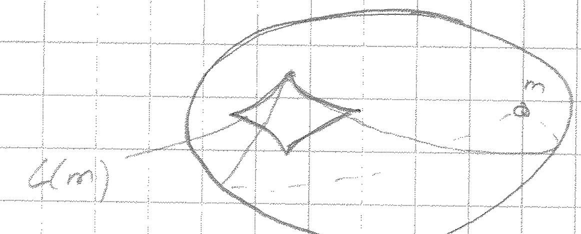



4) flat torus $T^2$ $C(m)=\emptyset$

again: $\exp_m: \mathbb{R}^2 \to T^2$ riemannian covering

5) torus of revolution $C(m)$ ? open question?
at least points of type b) in $Cut(m)$ (ex 1°.5) belong to $C(m)$.

6) ellipsoid:




'Last geometric statement of Jacobi':

The conjugate locus of a non-umbilical point on ellipsoids has exactly four cusps.

proof: Itoh & Kiyohara 2004 (!)

On the relation between $Cut(p)$ & $C(p)$:

Alan Weinstein (1968): $\forall$ compact, smooth manifold $M \not\cong S^2$ $\exists$ riemannian metric and a point $p\in M$ s.t. $Cut^*(p)\cap C^*(p)=\emptyset$

Note: - conjugate locus need not be closed
- along fixed geodesic $\gamma$ from $\gamma(0)=m$ cut points occur before or at $1^{st}$ conjugate points.

- Both types of cut points can be but need not be conjugate points at the same time

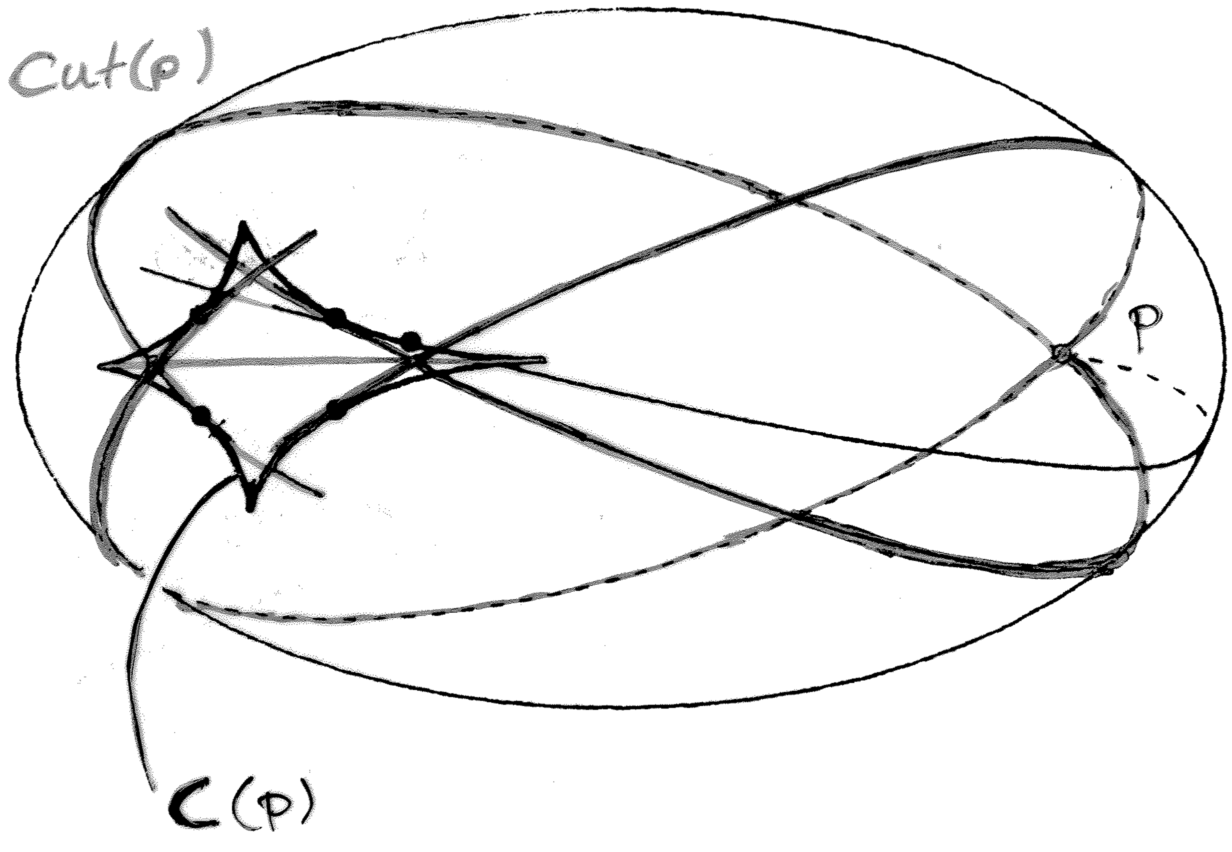
Especially: $Cut(m)=\emptyset$ $\forall m \Rightarrow M$ simply connected and $C(m)=\emptyset$ $\forall m.$ $\Rightarrow$

$\exp_m : \underbrace{T_m M}_{\approx \mathbb{R}^n} \to M$ local homeo between simply connected manifolds $\Rightarrow \exp_m$ homeo so $M \approx \mathbb{R}^n$.

Open (?) question: If $M$ compact, smooth riemannian manifold does there exists a point so that $C(p) \cap Cut(p) \neq \emptyset$.

References: [1] H. Gluck, D. Singer: Scattering of geodesic fields I Ann. of Math Vol 108 (-78) II Ann. of Math. Vol 110 (-79)
[2] A. Weinstein: The cut locus and conjugate locus of a riemannian manifold, Ann. of Math. Vol 87 (1968)

[3] Itoh, Kiyohara: The cut loci and the conjugate loci on ellipsoids, Manuscripta Math 114 (2004)


Cut(p)
p
C(p)