Parallel transport and the geodesic flow

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Theorem Let $c : [0,L] \to M$ be a curve and let $v \in T_{c(0)} M$.
Then there exists a unique $X \in \mathfrak{X}(c,M)$ s.t

$$X(0) = v, \quad D_{\dot c} X = 0$$

Note: $D_{\dot c} X$ says that "X does not change in the direction of $\dot c$". The vector field X is the parallel transport of v along c.

Proof

1° $c(I)$ is contained in a single coord. chart.
In local coordinates; the equation $D_{\dot c} X = 0$ reads:

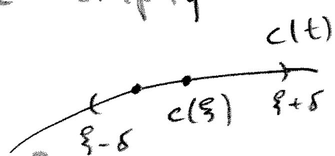
$$\dot X^i(t) = -\Gamma^i_{ab} \circ c(t)\, \dot c^a(t)\, X^b(t) \qquad (*)$$

This is a linear ODE $\Rightarrow$ X exists for all $t \in I$.

2° $c(I)$ not contained in one chart.
Let

$$E = \left\{ s \in [0,L] : \begin{array}{l} \text{There exists a unique} \\ X : [0,s] \to TM \\ X_t \in T_{c(t)} M \\ \text{such that } D_{\dot c} X = 0,\ X(0) = v \end{array} \right\}.$$

$(*)$ locally uniquely solvable $\Rightarrow$ E not empty
Let $\xi = \sup E$. $\xi > 0$.
If $\xi = L$, ~~we~~ claim. Otherwise let



$\delta \in (0,\xi)$ be such that $\{c(t) \mid t \in (\xi-\delta, \xi+\delta)\}$ is in one chart

Let $X : [0, \xi) \to TM$ be the unique soln. to

$$D_{\dot c} X = 0 \qquad X(0) = v.$$

Let $Y : [\xi - \delta/2, \xi + \frac{\delta}{2}) \to TM$ be the unique soln. to

$$D_{\dot c} Y = 0 \qquad Y(\xi - \delta/2) = X(\xi - \delta/2)$$

Now $\begin{cases} X, Y & \text{solve the same ODE} \\ X = Y & \text{at one point} \end{cases}$

$\Rightarrow$ $X = Y$ on their common domain

Let

$$Z(t) = \begin{cases} X(t), & t \in [0, \xi - \delta/2] \\ Y(t), & t \in (\xi - \delta/2, \xi + \delta/2) \end{cases}$$

$\Rightarrow$ Z is smooth at $\xi - \delta/2$

$Z : [0, \xi + \frac{\delta}{2}) \to TM$ is unique soln. to $D_{\dot c} Z = 0$, $Z(0) = v$:

(If $\tilde{Z}$ is another, then $Z = \tilde{Z}$ on $[0, \xi)$ by def. of ξ. Hence $\tilde{Z}(\xi - \delta/2) = X(\xi - \delta/2)$, so $Y = \tilde{Z}$ on $[\xi - \delta/2, \xi + \delta/2)$ as Y is unique; $Z = \tilde{Z}$.)

$\Rightarrow$ ξ not maximal. $\lightning$

Remarks

a) If X, Y are parallel transports on c, then

$$\frac{d}{dt} g(X, Y) = \underbrace{g(D_{\dot c} X, Y) + g(X, D_{\dot c} Y)}_{= 0}$$

Thus, for parallel transports

$$g(X_t, Y_t) = g(X(0), Y(0)) \quad \forall t \in I$$

b) By a), parallel transport can be used to move an ON-basis along a curve.

Geodesic flow

On a compact Riemannian manifold, the **geodesic flow** is the map

$$\varphi: TM \times \mathbb{R} \longrightarrow TM$$
$$(v,t) \longmapsto \dot{c}(t)$$

where $c: \mathbb{R} \longrightarrow M$ is the unique geodesic s.t $\dot{c}(0) = v$.

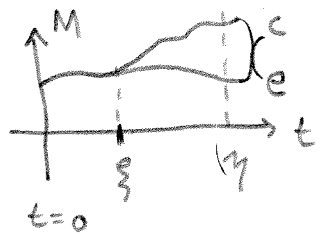
Proof: Existence of c: See previous Lectures.

Uniqueness: Suppose $e, c: \mathbb{R} \to M$ are geodesics with $\dot{e}(0) = \dot{c}(0) = v$, but $e(\eta) \neq c(\eta)$ at $\eta > 0$.
Then

$$E = \{ t \geqslant 0 \;:\; e(t) \neq c(t) \}$$

is not empty.

E open in $\mathbb{R}$ (see ⊛ below)



Let $\xi = \inf E$.

E open $\Rightarrow \xi \notin E$

$\Rightarrow c(t) = e(t) \quad t \leqslant \xi$

$\Rightarrow c(\xi) = e(\xi) \quad \dot{c}(\xi) = \dot{e}(\xi)$

(as c, e are smooth we can calculate derivatives from the left.)

Geodesics are locally unique $\Rightarrow c = e$ near ξ ↯

⊛ $E^C = \{ t \geqslant 0 : e(t) = c(t) \}$ closed in $\mathbb{R}$.

Let $\tau_i \in E^C \quad \tau_i \longrightarrow \tau, \; \tau \in \mathbb{R}$.

If $\varepsilon > 0$, then

$$d(e(\tau), c(\tau)) \leqslant \underbrace{d(e(\tau), e(\tau_i))}_{< \varepsilon/2} + \underbrace{d(e(\tau_i), c(\tau_i))}_{= 0} + \underbrace{d(c(\tau_i), c(\tau))}_{< \varepsilon/2}$$

when i large. $\square$

For $t \in \mathbb{R}$ let $\varphi_t : TM \longrightarrow TM$

$$v \longmapsto \varphi(v,t)$$

<u>Proposition</u> a) $\varphi_0 = id_{TM}$

b) $\varphi_{t+s} = \varphi_t \circ \varphi_s \quad , \quad t,s \in \mathbb{R}.$

c) $\varphi_t^{-1} = \varphi_{-t} \qquad t \in \mathbb{R}$

<u>Proof:</u> a) clear

b) Let $v \in TM$, $t,s \in \mathbb{R}$,

Let c be geodesic s.t. $\dot{c}(0) = v.$

$$\Rightarrow \begin{cases} \varphi_s(v) = \dot{c}(s) \\ \varphi_{s+t}(v) = \dot{c}(s+t) \end{cases}$$

Let e be geodesic s.t. $\dot{e}(0) = \dot{c}(s)$

$$\Rightarrow \quad \varphi_t(\dot{c}(s)) = \dot{e}(t)$$

$\left\{ \begin{array}{l} \text{Geodesics are translation invariant} \\ \text{Geodesics are unique from initial value} \end{array} \right.$

$$\Rightarrow \quad e(\tau) = c(\tau + s) \qquad \tau \in \mathbb{R}$$

$$\Rightarrow \quad \underline{\varphi_{t+s}(v)} = \dot{c}(s+t) = \dot{e}(t) = \varphi_t(\dot{c}(s)) = \underline{\varphi_t \circ \varphi_s(v)}$$

c)

$$\varphi_{-t} \circ \varphi_t = \varphi_0 = id_M \qquad \text{by } a,b$$

□

Geodesic vector field

<u>Definition</u> The <u>geodesic vector field</u> $G \in \mathfrak{X}(TM)$ is the map

$$\begin{aligned} TM &\longrightarrow T(TM) \\ y &\longmapsto G_y := [t \mapsto \varphi_t(y)] \in T_y(TM) \end{aligned}$$

Here $[\varphi_t(y)]$ is the vector induced by the curve $t \mapsto \varphi_t(y) \in TM$.

<u>Note:</u> If $f \in C^\infty(TM)$, then

$$G_y(f) = \frac{d}{dt}\left(f \circ \varphi_t(y)\right)(0).$$

That is, $G_y(f)$ measures how much f changes along a geodesic through y.

<u>Note:</u> If $y \in TM$, $\|y\| = 1$, then $\|\varphi_t(y)\| = 1$ $\forall t$. That is, the unit tangent bundle is invariant under the geodesic flow.

<u>Def.</u> The <u>unit tangent bundle</u> is the $(2n-1)$ manifold

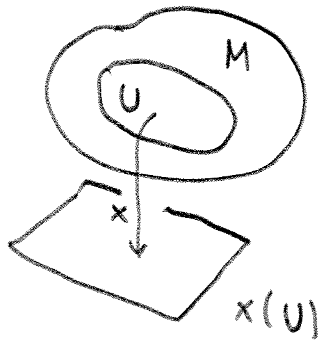
$$\Omega(M) = \{ v \in TM : |v| = 1\}$$

<u>Theorem:</u> If $y \in \Omega(M)$, then $\varphi_t(y) \in \Omega(M)$, so $G \in \mathfrak{X}(\Omega M)$ (G is a smooth map $G : \Omega M \longrightarrow T(\Omega M)$)

Local expression for G

Let x^i be local coordinates for $U \subset M$

With x^i- coordinates we can express elements in $V \in T(U)$ as



$$V = v^a \frac{\partial}{\partial x^a}\Big|_q$$

where $(v^1, \dots, v^n) \in \mathbb{R}^n$, $q \in U$.

This means that v is determined by $2n$ real numbers. The local coordinates for TU are:

$$x^i : U \to \mathbb{R}$$

$$\begin{aligned} x^i &: TU \longrightarrow \mathbb{R} \\ & v^a \frac{\partial}{\partial x^a}\Big|_q \longmapsto x^i \circ q \\ y^i &: TU \longrightarrow \mathbb{R} \\ & v^a \frac{\partial}{\partial x^a}\Big|_q \longmapsto v^i \end{aligned}$$

Notes:

a) Local coordinates for M induce local coordinates for TM.

b) Locally TM look like $x(U) \times \mathbb{R}^n$

c) Elements in $T_y(TM)$ can be written as

$$a^i \frac{\partial}{\partial x^i}\Big|_y + b^i \frac{\partial}{\partial y^i}\Big|_y \qquad a^i, b^i \in \mathbb{R}.$$

where (x^i, y^i) are local coordinates for $TU \ni y$.

d) $$\frac{\partial}{\partial x^i}\Big|_y \in T_y(TM) \text{ for } y \in TM,$$

$$\# \quad \frac{\partial}{\partial x^i}\Big|_q \in T_q(M) \text{ for } q \in M.$$

Let $y = y^i \frac{\partial}{\partial x^i}\Big|_q \in T_q M$,

$\underbrace{G_y}$ = $[\varphi_t(y)]$

Let $c: \mathbb{R} \to M$ geodesic with $\dot{c}(0) = y$

$\Rightarrow \varphi_t(y) = \dot{c}^i(t) \frac{\partial}{\partial x^i}\Big|_{c(t)} \in T_{c(t)} M$

$= \Big(\underbrace{c^1(t), \dots, c^n(t)}_{x^i \text{ coords.}}, \underbrace{\dot{c}^1(t), \dots, \dot{c}^n(t)}_{y^i\text{- coords.}} \Big)$

when x^i, y^i are local coords. induced by x^i.

$= \dot{c}^i(0) \frac{\partial}{\partial x^i}\Big|_{\varphi_0(y)} + \ddot{c}^i(0) \frac{\partial}{\partial y^i}\Big|_{\varphi_0(y)}$

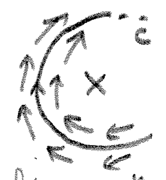
$= y^a \frac{\partial}{\partial x^a}\Big|_y - \Gamma^i_{ab} \circ \underbrace{c(t)}_{=q} \underbrace{\dot{c}^a(t)}_{=y^a} \dot{c}^b(t) \frac{\partial}{\partial y^i}\Big|_y$

$= y^a \Big(\frac{\partial}{\partial x^a}\Big|_y - \Gamma^i_{ab}(q) \, y^b \frac{\partial}{\partial y^i}\Big|_y \Big) \in T_y(TM)$

Let us show how to solve geodesics from the geodesic vector field $\mathbb{G}$.

<u>Def.</u> If $X \in \mathfrak{X}(M)$, then an integral curve of X is a curve $c: I \to M$ such that

$$\dot{c} = X \circ c$$



<u>Idea:</u> c represents the motion of a small particle in a "moving fluid".

ODE-theory: c exists locally.

<u>Theorem</u> a) If $c: I \to M$ is a geodesic, then $\dot{c}: I \to TM$ is an int. curve of $\mathbb{G}$.

b) If $\gamma: I \to TM$ is an integral curve of $\mathbb{G}$, then $\pi \circ \gamma : I \to M$ is a geodesic, and $\gamma = \dot{\overline{\pi \circ \gamma}} = (\pi \circ \gamma)'$

Here $\pi: TM \to M$ is the <u>canonical projection</u>: If $y \in T_xM$, then $\pi(y) = x$.

<u>Proof:</u> Let $\gamma: I \to TM$, $\gamma = (\underset{\substack{\uparrow \\ \text{base point}}}{c}, \underset{\substack{\uparrow \\ \text{vector components.}}}{v})$ (in local coords)

Then γ is an integral curve of $\mathbb{G}$

$$\iff \dot{\gamma} = \mathbb{G} \circ \gamma$$

$$\iff \dot{c}^i \frac{\partial}{\partial x^i}\Big|_\gamma + \dot{v}^i \frac{\partial}{\partial y^i}\Big|_\gamma = v^i \frac{\partial}{\partial x^i}\Big|_\gamma - \Gamma^s_{ij} v^i v^j \frac{\partial}{\partial y^s}\Big|_\gamma$$

$$\iff \begin{cases} \dot{c}^i(t) = v^i(t) \\ \dot{v}^i(t) + \Gamma^i_{ab}(c(t))\, v^a v^b = 0 \end{cases}$$

a) c geodesic $\Rightarrow$ $(c, \dot{c})$ int. curve of $\mathbb{G}$

b) $\gamma = (c, v)$ int. curve of $\mathbb{G}$

$\Rightarrow \quad v^i = \dot{c}^i \quad \Rightarrow \quad \gamma = (c, \dot{c}) = (\pi \circ \gamma)'$

$\Rightarrow \quad \ddot{c}^i + \Gamma^i_{ab} \, \dot{c}^a \dot{c}^b = 0$

$\Rightarrow \pi \circ \gamma = c$ geodesic. $\square$