The covariant derivative and the geodesic flow

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Directional derivative for a scalar function

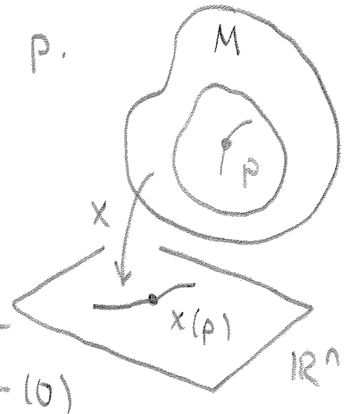
Suppose $y \in T_pM$ and $f \in C^\infty(M)$. Let

$$y(f) := (f \circ c)'(0)$$

where $c : I \longrightarrow M$ is a curve that represents y ; $c(0) = p$, $\dot{c}(0) = y$.

Interpretation: $y(f)$ describes how much f is changing in the direction of y.

Let us express $y(f)$ in local coords.
Let x^i be local coordinates around p.



$$(f \circ c)'(0) = \Big(\underbrace{(f \circ x^{-1})}_{\mathbb{R}^n \to \mathbb{R}} \circ \underbrace{(x \circ c)}_{I \to \mathbb{R}^n} \Big)'(0)$$

$$= \frac{\partial (f \circ x^{-1})}{\partial x^r} \underbrace{(x \circ c(0))}_{= x(p)} \cdot \frac{d(x \circ c)^r}{dt}(0)$$

If $c : I \longrightarrow M$, then $\dot{c} : I \longrightarrow TM$ is defined as

$$\dot{c}(t) = \underbrace{\frac{dc^i}{dt}(t)}_{= \frac{d(x \circ c)^i}{dt}(t)} \frac{\partial}{\partial x^i}\Big|_{c(t)}$$

$$\Longrightarrow \frac{d(x \circ c)^r}{dt}(0) = y^r$$

$$y(f) = \frac{\partial (f \circ x^{-1})}{\partial x^r}(x(p))\, y^r$$

Remarks

a) If $f \in C^\infty(M)$, and x^i are local coords around p, let

$$\frac{\partial f}{\partial x^i}(p) := \left.\frac{\partial}{\partial x^i}\right|_p (f)$$

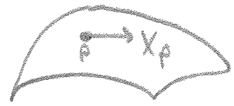
$$= \frac{\partial (f \circ x^{-1})}{\partial x^i}(x(p))$$

b) $v(fg) = v(f) \cdot g + f\, v(g)$

c) $$\frac{\partial^2 f}{\partial x^i \partial x^j}(p) := \left(\frac{\partial}{\partial x^i} \frac{\partial f}{\partial x^j}\right)(p) = \frac{\partial^2 (f \circ x^{-1})}{\partial x^i \partial x^j} \circ x(p).$$

Vector fields

$\mathfrak{X}(M)$ = set of smooth maps $X : M \to TM$ such that $X_p \in T_pM$, $p \in M$



Locally $X \in \mathfrak{X}(M)$ can be written as $X = X^i \frac{\partial}{\partial x^i}$ where $X^i \in C^\infty(M)$.

If $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, then $X(f) \in C^\infty(M)$ is defined pointwise:

$$X(f)|_p = X_p(f) = X^i(p) \frac{\partial f}{\partial x^i}(p)$$

so

$$X(f) = X^i \frac{\partial f}{\partial x^i}$$

Lie bracket for vector fields

Def. Let $X, Y \in \mathfrak{X}(M)$. Then $[X,Y] \in \mathfrak{X}(M)$ is the unique v. field such that

$$[X,Y](f) = X(Y(f)) - Y(X(f)) \quad \forall f \in C^\infty(M). \qquad (*)$$

Proof.

$$\begin{aligned}
\text{RHS} &= X(Y(f)) - Y(X(f)) \\
&= X\left(Y^i \frac{\partial f}{\partial x^i} \right) - Y\left(X^i \frac{\partial f}{\partial x^i} \right) \\
&= X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial f}{\partial x^i} + \cancel{X^j Y^i \frac{\partial^2 f}{\partial x^j \partial x^i}} \\
&\quad - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} - \cancel{Y^j X^i \frac{\partial^2 f}{\partial x^i \partial x^j}} \\
&= \underbrace{\left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}}_{=: Z \in \mathfrak{X}(U)} (f)
\end{aligned}$$

$U \subset M$ chart where x^i are defined.

$\Rightarrow$ In each chart there is a v.f. Z satisfying (*).

Suppose $\tilde{Z} \in \mathfrak{X}(U)$ also satisfies (*)

$\Rightarrow \quad Z(f) = \tilde{Z}(f) \quad \forall f \in C^\infty(M)$

$\Rightarrow \quad Z = \tilde{Z}$

$\Rightarrow$ definition of Z doesn't depend on x^i

$\Rightarrow \quad \exists\, Z \in \mathfrak{X}(M)$ satisfying (*). $\square$

Remarks: a) In local coordinates:

$$[X,Y]_p = \left(\frac{\partial Y^i}{\partial x^j} X^j - Y^j \frac{\partial X^i}{\partial x^j} \right)_p \frac{\partial}{\partial x^i}\Big|_p$$

b) $[X,Y] = -[Y,X]$

c) $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right] = 0$

d) $[\cdot,\cdot]$ is canonical; it doesn't depend on any additional structure on the manifold.

Covariant derivative for vector fields.

Next, we describe the Levi-Civita connection. It is a directional derivative for vector fields on a manifold.

<u>Theorem:</u> On a Riemannian manifold there exists a unique map:

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$
$$(X, Y) \longmapsto \nabla_X Y$$

characterized by conditions:

a) $\nabla_X Y$ is $\mathbb{R}$-linear in Y
$\nabla_X Y$ is $C^\infty(M)$-linear in X

b) $\nabla_X(fY) = X(f)\, Y + f \nabla_X Y$, $\quad f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$

c) If $X, Y \in \mathfrak{X}(M)$, then

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

d) If $X, Y, Z \in \mathfrak{X}(M)$, then

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

<u>Proof.</u> We first derive an implicit eqn. for ∇_Y. This will prove uniqueness. From this eqn, we then derive an explicit expression for ∇ and this will imply existence.

For uniqueness, let us assume that ∇ exists and ∇ satisfies conditions a-d.

Let $X, Y, Z \in \mathfrak{X}(M)$. Then:

$$
\begin{aligned}
X(g(Y,Z)) &= \boxed{g(\nabla_X Y, Z)} + g(Y, \nabla_X Z) \\
Y(g(X,Z)) &= \boxed{g(\nabla_Y X, Z)} + g(X, \nabla_Y Z) \\
\underline{\Sigma} \quad - Z(g(X,Y)) &= -g(\nabla_Z X, Y) - g(X, \nabla_Z Y)
\end{aligned}
$$

$$
\begin{aligned}
&X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\
&= g(\underbrace{\nabla_X Y + \nabla_Y X}_{= \nabla_Y X + [X,Y] + \nabla_Y X}, Z) + g(Y, \underbrace{\nabla_X Z - \nabla_Z X}_{= [X,Z]}) + g(X, \underbrace{\nabla_Y Z - \nabla_Z Y}_{= [Y,Z]}) \\
&= 2g(\nabla_Y X, Z) + g(Z, [X,Y]) + g(Y, [X,Z]) \\
&\qquad + g(X, [Y,Z])
\end{aligned}
$$

Solving for $2g(\nabla_Y X, Z)$ yields Kozul's formula:

$$
2g(\nabla_Y X, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y))
$$
$$
- g(Z, [X,Y]) - g(Y, [X,Z]) - g(X, [Y,Z])
$$
$$
\forall X, Y, Z \in \mathfrak{X}(M)
$$

The RHS depends only on the metric. This is the implicit eqn. for ∇.

Claim: The Kozul formula determines $\nabla_Y X$, $X, Y \in \mathfrak{X}(M)$

Proof. We wish to prove:

'The Kozul formula determines: $2g((\nabla_Y X)_p, u)$, $u \in T_pM$

Let $Z \in \mathfrak{X}(M)$ be an extension of u ($Z_p = u$). Then
(we shortly prove existence...)

$$
2g((\nabla_Y X)_p, u) = 2g(\nabla_Y X, Z)\big|_p = \text{RHS}\big|_p
$$

By inspecting the RHS in Kozul's formula, it may depend on

$$Z^k(p) = u^k \quad \text{and} \quad \frac{\partial Z^k}{\partial x^r}(p).$$

However, as the LHS does not depend on $\frac{\partial Z^k}{\partial x^r}(p)$, the RHS can neighter.

$\Rightarrow$ RHS doesn't depend on the extension of u

$\Rightarrow$ we may solve $(\nabla_Y X)_p$ from Kozul's formula

$\Rightarrow$ $\nabla_Y X$ is uniquely determined by a, b, c, d. $\square$

Existence: We start with a formal calculation where we derive an explicit expression for $\nabla_Y X$ from Kozul's formula.

Let $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}$,

$$\Rightarrow 2g\left(\nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) =$$

$$\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}$$

$$\Rightarrow g_{ak}\left(\nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^i}\right)^a = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \Big| \, g^{bk}$$

$$\Rightarrow \left(\nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^i}\right)^b = \underbrace{\frac{1}{2} g^{kb}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)}_{= \Gamma^b_{ij}}$$

$$\Rightarrow \nabla_{\partial/\partial x^j}\left(\frac{\partial}{\partial x^i}\right) = \Gamma^b_{ij} \frac{\partial}{\partial x^b}$$

$$\nabla_X Y = \nabla_{X^j \frac{\partial}{\partial x^j}} Y$$

$$= X^j \, \nabla_{\frac{\partial}{\partial x^j}} \left(Y^i \frac{\partial}{\partial x^i} \right)$$

$$= X^j \left(\frac{\partial Y^a}{\partial x^j} \frac{\partial}{\partial x^a} + Y^i \underbrace{\nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^i}}_{= \Gamma^a_{ij} \frac{\partial}{\partial x^a}} \right)$$

$$= X^j \left(\frac{\partial Y^a}{\partial x^j} + \Gamma^a_{ij} Y^i \right) \frac{\partial}{\partial x^a} . \qquad (*)$$

In local coordinates, we define ∇ by $(*)$.

Now ∇ satisfies conditions a–d:

a) $\mathbb{R}$-linear in Y

$C^\infty(M)$-linear in X

$\Rightarrow \nabla$ is unique

$\Rightarrow \nabla$ does not depend on local coordinates.

b) $$\nabla_X(fY) = X^j \left(\frac{\partial (f Y^a)}{\partial x^j} + \Gamma^a_{ij} (f Y^i) \right) \frac{\partial}{\partial x^a}$$

$$= \left(X^j \frac{\partial f}{\partial x^j} Y^a + \underbrace{X^j f \frac{\partial Y^a}{\partial x^j} + X^j \Gamma^a_{ij} f Y^j}_{} \right) \frac{\partial}{\partial x^a}$$

$$= X(f) \cdot Y + f \nabla_X Y$$

c) $$\nabla_X Y = X^j \left(\frac{\partial Y^a}{\partial x^j} + \Gamma^a_{ij} Y^i \right) \frac{\partial}{\partial x^a}$$

$$\Sigma \quad - \nabla_Y X = -Y^j \left(\frac{\partial X^a}{\partial x^j} + \Gamma^a_{ij} X^i \right) \frac{\partial}{\partial x^a}$$

$$\nabla_X Y - \nabla_Y X = \underbrace{\left(X^j \frac{\partial Y^a}{\partial x^j} - Y^j \frac{\partial X^a}{\partial x^j} \right) \frac{\partial}{\partial x^a}}_{} = [X, Y]$$

$$+ \underbrace{\left(X^j Y^i \Gamma^a_{ij} - X^i Y^j \Gamma^a_{ij} \right)}_{} \frac{\partial}{\partial x^a}$$

$$= X^i Y^j \underbrace{\left(\Gamma^a_{ji} - \Gamma^a_{ij} \right)}_{= 0 \, ; \; \Gamma^a_{ij} = \Gamma^a_{ji}}$$

d) $g(\nabla_X Y, Z)$

$$= g_{ak} X^j \left(\frac{\partial Y^a}{\partial x^j} + \Gamma^a_{ij} Y^i \right) Z^k$$

$$= g_{ak} X^j \frac{\partial Y^a}{\partial x^j} Z^k + \underbrace{g_{ak} X^j \Gamma^a_{ij} Y^i Z^k}_{= \Gamma_{kij} X^j Y^i Z^k}$$

Exchanging $Y \leftrightarrow Z$

$$g(Y, \nabla_X Z) = g_{ak} X^j \frac{\partial Z^a}{\partial x^j} Y^k + \underbrace{g_{ak} X^j \Gamma^a_{ij} Z^i Y^k}_{= \Gamma_{kij} X^j Z^i Y^k = \Gamma_{ikj} X^j Z^k Y^i}$$

$\Rightarrow$

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$= g_{ak} X^j \underbrace{\left(\frac{\partial Y^a}{\partial x^j} Z^k + Y^a \frac{\partial Z^k}{\partial x^j} \right)}_{= \frac{\partial (Y^a Z^k)}{\partial x^j}} + \underbrace{(\Gamma_{kij} + \Gamma_{ikj})}_{\overset{(*)}{=} \frac{\partial g_{ik}}{\partial x^j}} X^j Y^i Z^k$$

$$= X^j \left(g_{ak} \frac{\partial (Y^a Z^k)}{\partial x^j} + \frac{\partial g_{ak}}{\partial x^j} Y^a Z^k \right)$$

$$= X^j \frac{\partial}{\partial x^j} \left(g(Y, Z) \right) = X(g(Y, Z))$$

(*)

Let $\Gamma_{ijk} := g_{ia} \Gamma^a_{jk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)$.

$$\Rightarrow \quad \Gamma_{ijk} \quad \frac{1}{2} \left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j} \right)$$

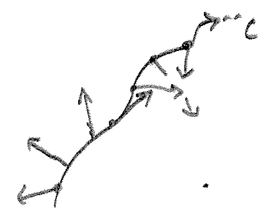
$$\Rightarrow \quad \Gamma_{ijk} + \Gamma_{jik} = \frac{\partial g_{ij}}{\partial x^k}$$

$\square$

<u>Remark</u>: a) ∇ is a map $T_pM \times \mathfrak{X}(M) \longrightarrow T_pM$: $(\nabla_X Y)_p = \nabla_{X_p} Y$, $p \in M$

b) If $\Gamma^i_{jk} = 0$, then $\nabla_X Y|_p = X^j \frac{\partial Y^a}{\partial x^j} \frac{\partial}{\partial x^a} \underset{\uparrow \text{ in } \mathbb{R}^n}{=} \frac{d}{dt}\Big|_{t=0} Y(p + tX)$

Covariant derivative on a curve



Let $c : I \longrightarrow M$ be a curve.

$\mathfrak{X}(c, M)$ = maps $X : I \longrightarrow TM$ s.t $X_t \in T_{c(t)}M$

Example: $\dot{c} \in \mathfrak{X}(c, M)$

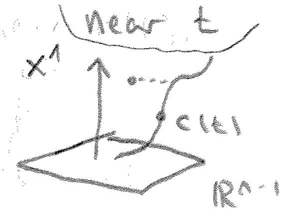
Definition: The covariant der. on c is the map

$$D_{\dot{c}} : I \times \mathfrak{X}(c, M) \longrightarrow TM$$
$$(t, X) \longmapsto (D_{\dot{c}} X)_t$$

defined by $(D_{\dot{c}} X)_t = \nabla_{\dot{c}(t)} \tilde{X}$ where $\tilde{X}$ is an extension of $\tilde{X}$ near $c(t)$.
$\tilde{X} \in \mathfrak{X}(M)$, $\tilde{X}|_c = X$ near t

Claim: We can always construct $\tilde{X}$.

Proof: We may assume that $c^1 : I \longrightarrow \mathbb{R}$ is str. inc. near t
Think of $c^1(t)$ as the "height of $c(t)$"



Near $c(t)$ define
$$\tilde{X}(x^1, \dots, x^n) = X\Big(\underbrace{(c^1)^{-1}(x^1)}_{t \text{ s.t. } c^1(t) = x^1}\Big)$$

We have defined $\tilde{X}$ near $c(t)$. By a suitable cut-off function, $\tilde{X}$ extends to all M. $\square$

Claim: $D_{\dot c} X$ does not depend on $\tilde X$

Proof: $(D_{\dot c} X)_t = \left(\nabla_{\dot c(t)} \tilde X\right)$

$$= \dot c^i(t)\left(\frac{\partial \tilde X^a}{\partial x^i} + \Gamma^a_{ij} \tilde X^j\right)_{c(t)} \frac{\partial}{\partial x^a}\bigg|_{c(t)}$$

$$= \left(\underbrace{\dot c(t)\left(\tilde X^a\right)} + \Gamma^a_{ij}(c(t))\, \dot c^i X^j\right) \frac{\partial}{\partial x^a}\bigg|_{c(t)}$$

$$= \frac{d}{dt}\left(\tilde X^a \circ c\right) \quad (c \text{ represents } \dot c)$$

$$= \frac{dX^a}{dt}$$

$$= \left(\frac{dX^a}{dt} + \Gamma^a_{ij} \circ c \; \dot c^i X^j\right)(t) \frac{\partial}{\partial x^a}\bigg|_{c(t)} \quad \square$$

Properties of $D_{\dot c} X$

a) $D_{\dot c} X$ $\mathbb{R}$-linear in X

b) $D_{\dot c}(fX) = \frac{df}{dt} X + f D_{\dot c} X \qquad f \in C^\infty(I)$

c) $\frac{d}{dt} g(X,Y) = g(D_{\dot c} X, Y) + g(X, D_{\dot c} Y)$

Proof: For c) we have

$$\text{RHS} = g(\nabla_{\dot c(t)} \tilde X, \tilde Y) + g(\tilde X, \nabla_{\dot c(t)} \tilde Y)$$

$$= \dot c(t)\left(g(\tilde X, \tilde Y)\right) = \frac{d}{dt}\left(g(\tilde X, \tilde Y) \circ c\right) = \frac{d}{dt} g(X,Y) \quad \square$$

The next theorem gives a coordinate-free condition for a curve to be a geodesic.

<u>Theorem</u>: Let $c : I \longrightarrow M$ be a curve. Then c is a geodesic $\Leftrightarrow D_{\dot{c}}\dot{c} = 0$.

<u>Proof</u>: For a curve c,

$$D_{\dot{c}}\dot{c} = \frac{d\dot{c}^i}{dt} + \Gamma^i_{ab}(c(t))\,\dot{c}^a(t)\dot{c}^b(t)$$

$\square$