Inverse problems seminar
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Manifolds and geodesics II

Review:

M Riemannian mfld, $\langle \cdot , \cdot \rangle$ Riemannian metric
x chart at p, $v = v^i \frac{\partial}{\partial x^i}$, $w = w^j \frac{\partial}{\partial x^j} \in T_pM$, $\langle v, w \rangle = g_{ij}(x(p)) v^i w^j$
$\gamma : [a,b] \to M$ smooth curve joining p and q, $|\dot\gamma(t)| \equiv 1$

$L(\gamma) \leq L(\tilde\gamma)$ for $\tilde\gamma$ joining p and q

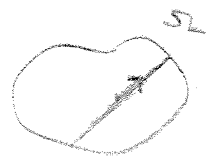
$\Rightarrow$ $\ddot{x}^i(t) + \Gamma^i_{jk}(x(t)) \dot{x}^j(t) \dot{x}^k(t) = 0 \quad (i = 1, \ldots, n)$

$x(t) = x(\gamma(t))$, $\quad \Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{lk,j} + g_{jl,k} - g_{jk,l})$

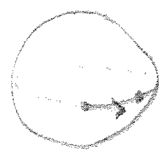
A smooth curve satisfying these equations is geodesic.

Examples

1) $M = \Omega \subseteq \mathbb{R}^n$ open, $g_{ij} = \delta_{ij}$
$\Rightarrow$ $\Gamma^i_{jk} \equiv 0$, geodesics straight lines



2) $M = S^{n-1} \subseteq \mathbb{R}^n$
$\Rightarrow$ geodesics arcs of great circles



Remark Any geodesic is parametrized by arc length
(proof: show $\frac{d}{dt}\langle \dot{x}, \dot{x} \rangle = 0$ using geodesic equations)

Will prove today

Thm M cpt Riemannian mfld $\Rightarrow$ any $p, q \in M$ can be joined by a curve of shortest length, and this curve is a geodesic.

Thm (Existence and uniqueness of geodesics) M Riemannian mfld, $p \in M$, $v \in T_pM$. Then $\exists\ \varepsilon > 0$ and $\exists!$ geodesic $c = c_v : [0, \varepsilon] \to M$ with $c(0) = p$, $\dot{c}(0) = v$. Also, c depends smoothly on p and v.

Pf Use existence and uniqueness for ODE. $\square$

If $x(t)$ solves the geodesic equations then so does $x(\lambda t)$. Therefore, if $c_v(t)$ is the geodesic with $c_v(0)=p$, $\dot c_v(0)=v$, defined for $t\in[0,\varepsilon]$, we have

$$c_{\lambda v}(t)=c_v(\lambda t)$$

and $c_{\lambda v}$ is defined on $[0,\varepsilon/\lambda]$. Since c_v depends smoothly on v, and since $\{v\in T_pM;\ |v|=1\}$ is compact, there is $\varepsilon_0>0$ such that c_v is defined at least on $[0,\varepsilon_0]$ when $|v|=1$. Thus, if $w\in T_pM$ and $|w|\le\varepsilon_0$ c_w is defined at least on $[0,1]$.

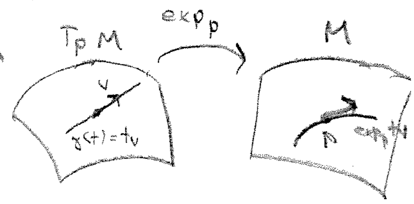
__Def.__ M Riemannian mfld, $p\in M$. Let $V_p=\{v\in T_pM:\ c_v \text{ defined on } [0,1]\}$ and define the __exponential map__

$$\exp_p: V_p\to M,\quad v\mapsto c_v(1)$$
$$\exp_p tv=c_v(t),\quad t\in[0,1]$$

It was shown above that V_p contains a neighborhood of $0\in T_pM$.

__Thm__ $\exp_p$ is a diffeo of nbh of $0\in T_pM$ and nbh of $p\in M$.

__Pf__ $\exp_p: T_pM\to M$, compute $D\exp_p(0): T_0T_pM\to T_pM$



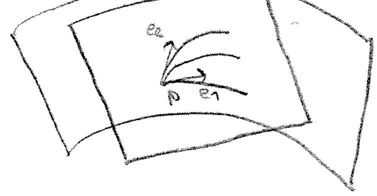
$$D\exp_p(0)v=\frac{d}{dt}(\exp_p tv)\Big|_{t=0}=\frac{d}{dt}c_v(t)\Big|_{t=0}=\dot c_v(0)=v$$

Thus $D\exp_p(0)=\mathrm{id}_{T_pM}$, inverse function theorem $\Rightarrow$ $\exp_p$ local diffeo. □

Let $e_1,\dots,e_n$ be a basis of T_pM which is orthonormal w.r.t. Riemannian metric. Choosing coordinates in T_pM corresponding to this basis, we have that $\exp_p^{-1}$ is a diffeo of a nbh U of p onto a nbh of 0 in $\mathbb{R}^n$.

__Def.__ The local coordinates given by chart $(\exp_p^{-1},U)$ are called (Riemannian) __normal coordinates__ at p.

Normal coordinates $x=(x^1,\dots,x^n): U\to\mathbb{R}^n$.



rays thru 0 are geodesic

__Thm__ In normal coordinates

$$g_{ij}(0)=\delta_{ij}$$
$$\Gamma^i_{jk}(0)=0 \quad (\text{and also } g_{ij,k}(0)=0)\quad \forall\, i,j,k.$$

__Pf__ $g_{ij}(0)=\langle\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\rangle_p=\langle e^i,e^j\rangle_p=\delta_{ij}$

$x(t)=tv$ geodesic, $\ddot x(t)=0$, $\dot x(t)=v$

$\Rightarrow\ \Gamma^i_{jk}(tv)v^jv^k=0$

$\Rightarrow\ \Gamma^i_{jk}(0)=0\quad\forall\, i,j,k$

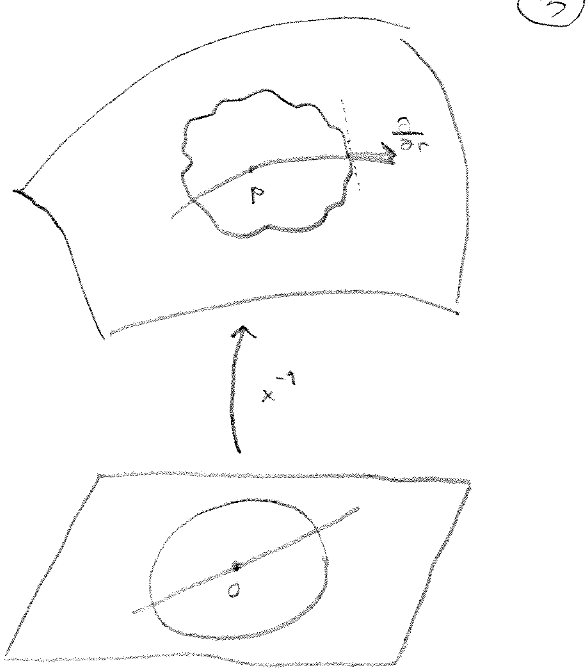
□

Normal coordinates $x : U \to \mathbb{R}^n$

Geodesics $x(r) = r\Theta, \quad \Theta \in S^{n-1}$

Polar coordinates $x = r\Theta(\varphi), \quad \varphi = (\varphi^1, \dots, \varphi^{n-1})$

In new coordinates



$$\frac{\partial}{\partial r} = \frac{\partial x^i}{\partial r}\frac{\partial}{\partial x^i} = \Theta^i \frac{\partial}{\partial x^i}$$

$$\frac{\partial}{\partial \varphi^k} = \frac{\partial x^i}{\partial \varphi^k}\frac{\partial}{\partial x^i} = r\frac{\partial \Theta^i}{\partial \varphi^k}\frac{\partial}{\partial x^i}$$

Metric in polar coordinates $\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix}$

Geodesics $x(t) = (t, \varphi_0), \quad \varphi_0$ fixed

Geodesic equations

$$\Rightarrow \quad \Gamma^i_{rr} = 0 \qquad \forall i$$

$$\Rightarrow \quad g^{il}(2g_{rl,r} - g_{rr,l}) = 0 \qquad \forall i$$

$$\Rightarrow \quad 2g_{rl,r} - g_{rr,l} = 0 \qquad \forall l \qquad (*)$$

If $l = r$, get $g_{rr,r} = 0$

$$\Rightarrow \quad g_{rr} \equiv \lim_{t\to 0} g_{rr}(t\Theta) = \lim_{t\to 0}\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right\rangle_{\delta(t)} = \lim_{t\to 0} g_{ij}(t\Theta)\Theta^i\Theta^j = 1$$

Thus $g_{rr} \equiv 1$, insert in $(*)$ to get

$$g_{r\varphi,r} = 0$$

$$\Rightarrow \quad g_{r\varphi} \equiv g_{r\varphi}(0) = \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}\right\rangle_p = 0$$

<u>Lemma</u> In Riemannian polar coordinates, in a whole chart

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & g_{\phi\phi} \end{pmatrix}$$

Implies Gauss Lemma: $\frac{\partial}{\partial r}$ is orthogonal to geodesic spheres.

<u>Lemma</u> M cpt Riemannian mfld. Then $\exists\, \rho_0 > 0$ s.t. any $p, q \in M$ with $d(p,q) \leq \rho_0$ can be connected by precisely one curve of shortest length. This curve is a geodesic and depends cont. on p, q.

<u>Pf</u> Since $TM \ni (p, v) \mapsto \exp_p v \in M$ smooth and $D\exp_p(0) = id|_{T_pM}$, $\forall\, p \in M\ \exists\, U(p) \subseteq M$ and $\rho > 0$ s.t. $D(\exp_q)$ injective on $B(0,\rho) \subseteq T_qM$. By compactness, $\exists\, \rho_0 > 0$ s.t. Riemannian polar coordinates can be introduced on $B(p, \rho_0)\ \forall\, p \in M$.

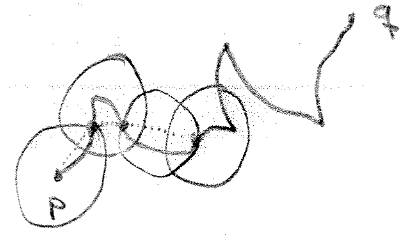
Let $q \in B(p, \rho_0)$ and $\rho = r(q) < \rho_0$. Let $c(t) = (r(t), \varphi(t))$, $[0, t_0] \to \mathbb{R}^n$, be a curve from p to q in $B(p, \rho_0)$. Then

$$L(c) = \int_0^{t_0} \sqrt{\dot r(t)^2 + g_{\phi\phi jk}(c(t))\,\dot\varphi^j(t)\dot\varphi^k(t)}\, dt \geq \int_0^{t_0} \dot r(t)\, dt = r(t_0) = \rho$$

with equality iff $g_{\varphi\varphi}\dot\varphi\dot\varphi \equiv 0$ and $\dot r \geq 0$, i.e. $\varphi \equiv$ const and $c(t)$ line thru 0. $\square$

Thm M cpt Riemannian mfld. Then any $p,q \in M$ can be connected by at least one (piecewise C^1) curve of shortest length, and any such curve is a geodesic. Further, all geodesics exist for all time, and $\exp_p$ is defined on all of T_pM $\forall p \in M$.

Pf Let $p,q \in M$, and let γ_m be piecewise C^1 curves joining p and q such that $d(p,q) = \lim_{m\to\infty} L(\gamma_m)$. Assume all curves are parametrised by arc length.



We may assume each γ_m is piecewise geodesic, namely if $\gamma_m : [0,T] \to M$ we may find $0 = t_0 < t_1 < \dots < t_{k+1} = T$ such that $L(\gamma_m|_{[t_j,t_{j+1}]}) \le \rho_0/2$, ρ_0 as earlier, and we may replace $\gamma_m|_{[t_j,t_{j+1}]}$ by a geodesic without increasing length. Since $L(\gamma_m)$ bounded we may take k independent of m.

Thus, we may assume that for each γ_m there are points $p_{0,m}, \dots, p_{k+1,m} \in M$ s.t. $d(p_{j,m}, p_{j+1,m}) \le \rho_0$ and for which γ_m contains the shortest geodesic arc between $p_{j,m}$ and $p_{j+1,m}$. Since M is cpt, after taking a subsequence we get $p_{j,m} \to p_j$ as $m \to \infty$. The segment of γ_m between $p_{j,m}$ and $p_{j+1,m}$ then converges to the shortest geodesic arc between p_{j-1} and p_j, since geodesics depend continuously on endpoints.

The union of these segments gives a curve γ, with

$$d(p,q) = L(\gamma) = \lim_{m\to\infty} L(\gamma_m).$$

Any such γ has to be a geodesic, since otherwise we could replace some arc shorter than ρ_0 by a geodesic arc without increasing length.

Let $v \in T_pM$, and let

$$\Lambda = \{ t > 0 \ ; \ c_v \text{ defined on } [-t,t] \}.$$

Λ is nonempty, closed since M is cpt, and open since geodesics can be extended for a short time. Thus $\Lambda = \mathbb{R}_+$. □