Inverse problems seminar
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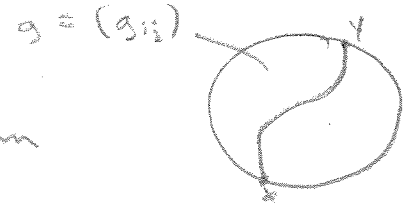
Manifolds and geodesics

Review from last time:



Boundary rigidity problem: (M, g) Riemannian mfld with boundary ∂M. Given the geodesic distances $d_g(x, y)$ $\forall x, y \in \partial M$, can one determine the metric g?

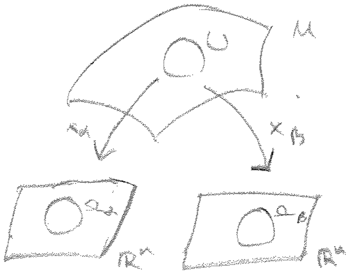
Obstruction: $d_{\psi^* g} = d_g$ if $\psi : M \to M$ diffeo, $\psi|_{\partial M} = \mathrm{id}$.

Thm (Pestov-Uhlmann 2005) If (M, g) is a compact simple 2D mfld, then d_g determines g (up to ψ).

Def. A connected paracompact Hausdorff space M is called a (differentiable) mfld if $\exists\, (U_\alpha, x_\alpha)_{\alpha \in A}$ such that

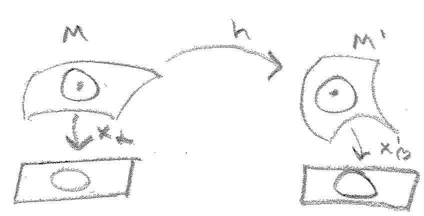
(i) $(U_\alpha)_{\alpha \in A}$ is an open cover of M,
(ii) $x_\alpha : U_\alpha \to \Omega_\alpha$ is a homeomorphism, $\Omega_\alpha \subseteq \mathbb{R}^n$ open set
(iii) $x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$ is C^∞ (if $U_\alpha \cap U_\beta \neq \emptyset$)

x_α are charts, $x(p)$ local coordinates

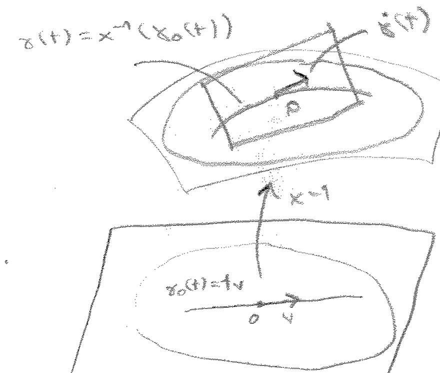


Examples: $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ open
S^n, graphs of smooth functions $h : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^{n+1}$

Def. A map $h : M \to M'$ between two mflds M and M' with chart (U_α, x_α) and (U'_β, x'_β) is differentiable if $x'_\beta \circ f \circ x_\alpha^{-1}$ is C^∞.



Def. Let $p \in M$. If $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, we say $\gamma \sim \hat{\gamma}$ if $(x \circ \gamma)'(0) = (x \circ \hat{\gamma})'(0)$. Equivalence classes $[\gamma]$ are called tangent vectors and they form tangent space $T_p M$. This is an n-dim. vector space spanned by $\frac{\partial}{\partial x^i} = [\gamma_i]$, $\gamma_i = x^{-1}(t e_i)$. $TM = \bigcup_{p \in M} T_p M$ tangent bundle.



If $v = [\gamma] \in T_p M$ and $f : M \to \mathbb{R}$ smooth, then

$v f = (f \circ \gamma)'(0)$ $\qquad v = v^i \frac{\partial}{\partial x^i}$

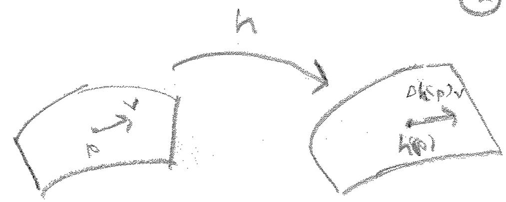
$\frac{\partial}{\partial x^i} f = (f \circ \gamma_i)'(0)$ $\qquad \Rightarrow v f = v^i \frac{\partial f}{\ldots}$

Def. If $h: M \to M'$ smooth, *derivative*

$Dh(p): T_pM \to T_{f(p)}M'$ linear map,

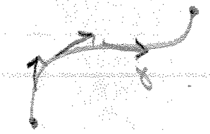
$Dh(p)[\gamma] = [h \circ \gamma]$

$(Dh(p)v)f = v(f \circ h)$



If $\gamma : [a,b] \to \mathbb{R}^n$ curve, then

$$L(\gamma) = \int_a^b |\dot\gamma(t)|\, dt$$



To extend this to mflds, need to measure lengths of tangent vectors.

Def. A *Riemannian metric* on M is a scalar product at each T_pM, depending smoothly on M.

If $v, w \in T_pM$, $v = v^i \frac{\partial}{\partial x^i}$, $w = w^j \frac{\partial}{\partial x^j}$, then

$$\langle v, w \rangle = \langle v^i \tfrac{\partial}{\partial x^i}, w^j \tfrac{\partial}{\partial x^j} \rangle = g_{ij}(x(p))\, v^i w^j$$

where $g_{ij}(x(p)) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$ is a pos. def. symmetric matrix, smooth on $x(U)$.

Length $|v| = \sqrt{\langle v, v\rangle}$.

Remark Any mfld has a Riemannian metric.

If $\gamma : [a,b] \to (M,g)$ smooth curve, define

$$L(\gamma) = \int_a^b |\dot\gamma(t)|\, dt \qquad \left(= \int_a^b \sqrt{g_{ij}(x(\gamma(t)))\, \dot x^i(t)\, \dot x^j(t)}\, dt, \quad \dot x^i(t) = \tfrac{d}{dt}(x(\gamma(t))) \right)$$

$$E(\gamma) = \frac{1}{2} \int_a^b |\dot\gamma(t)|^2\, dt.$$

If $p, q \in M$, define

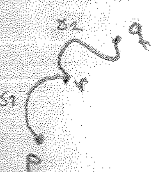
$$d(p,q) = d_g(p,q) = \inf \{ L(\gamma) : \gamma : [a,b] \to M \text{ piecewise smooth curve}, \ \gamma(a) = p, \ \gamma(b) = q \}$$

Note that $\forall p, q \in M$, $\exists$ piecewise smooth curve joining p and q.

Thm (M,d) metric space whose topology is the same as the original topology.

Pf $d(p,q) \geq 0$, $d(p,q) = d(q,p)$ clear

$d(p,q) \leq d(p,r) + d(r,q)$ from picture



Let $p \in M$, (U, x) chart with $x(p) = 0$, $\overline{B_\rho(0)} \subseteq x(U)$

Suppose $\lambda |\xi|^2 \leq g_{ij}(z)\xi^i \xi^j \leq \Lambda |\xi|^2$ for $z \in B_\rho(0)$, $\xi \in \mathbb{R}^n$

If $q \in x^{-1}(B_\rho(0))$ and γ joins p and q then

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t)))\, \dot x^i(t)\, \dot x^j(t)}\, dt$$

$$\Rightarrow \lambda |x(p) - x(q)| \leq d(p,q) \leq \Lambda |x(p) - x(q)|$$

□

Let $\gamma : [a,b] \to M$ smooth curve. ③

<u>Lemma</u> $L(\gamma)^2 \le 2(b-a)E(\gamma)$ with equality iff $|\dot\gamma(t)| \equiv$ const.

<u>Pf</u> $\int_a^b |\dot\gamma(t)|\cdot 1\,dt \le (b-a)^{1/2}\left(\int_a^b |\dot\gamma(t)|^2 dt\right)$, equality iff $|\dot\gamma(t)| \equiv$ const. □

Parametrise γ by arc length:

$s(t) = \int_a^t |\dot\gamma(u)|\,du$, inverse function $t = t(s)$

$\hat\gamma(s) = \gamma(t(s))$, $|\dot{\hat\gamma}(s)| \equiv 1$

<u>Remark</u> $L(\gamma) = L(\hat\gamma) = L(\gamma\circ\phi)$ for any reparametrisation ϕ.

Euler-Lagrange equations for $E(\gamma)$:

$$I(x) = \int_a^b f(t, x(t), \dot x(t))\,dt, \qquad f(t,x,p) = \frac{1}{2} g_{ij}(x) p^i p^j$$

Suppose $I(x) \le I(x+s\varphi)$, $s \in (-\varepsilon,\varepsilon)$, $\varphi : [a,b] \to \mathbb{R}^n$, $\varphi(a) = \varphi(b) = 0$

Then $\frac{\partial}{\partial s} I(x+s\varphi)|_{s=0} = 0$

$$\Leftrightarrow \int_a^b \left(\frac{\partial f}{\partial x^k}(t, x(t), \dot x(t))\,\varphi^k(t) + \frac{\partial f}{\partial p^k}(t, x(t), \dot x(t))\,\dot\varphi^k(t)\right) dt = 0$$

$$\Leftrightarrow \int_a^b \left(\frac{\partial f}{\partial x^k} - \frac{\partial}{\partial t}\frac{\partial f}{\partial p^k}\right) \varphi^k(t)\,dt = 0$$

This is valid $\forall \varphi \Rightarrow \frac{\partial}{\partial t}\frac{\partial f}{\partial p^k} - \frac{\partial f}{\partial x^k} = 0$

$$\frac{\partial}{\partial t}\left(g_{kj}(x(t))\,\dot x^j(t)\right) - \frac{1}{2} g_{ij,k}(x(t))\,\dot x^i(t)\dot x^j(t) = 0$$

$$g_{kj,i}(x(t))\dot x^i(t)\dot x^j(t) + g_{kj}(x(t))\ddot x^j(t) - \frac{1}{2} g_{ij,k}(x(t))\dot x^i(t)\dot x^j(t) = 0$$

$$g_{\ell m}(x(t))\ddot x^m(t) + \frac{1}{2}\left(g_{\ell k,j} + g_{k\ell,j} - g_{jk,\ell}\right)\dot x^j \dot x^k = 0$$

$$\underbrace{g^{i\ell} g_{\ell m}\ddot x^m}_{\ddot x^i} + \underbrace{\frac{1}{2} g^{i\ell}\left(g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell}\right)}_{\Gamma^i_{jk}}\dot x^j \dot x^k = 0$$

<u>Def.</u> A smooth curve $\gamma : [a,b] \to M$, which satisfies (in local coordinates x with $x(t) = x(\gamma(t))$)

$$\ddot x^i(t) + \Gamma^i_{jk}(x(t))\,\dot x^j(t)\dot x^k(t) = 0, \qquad i = 1,\dots,n,$$

is called a <u>geodesic</u>. The Γ^i_{jk} are <u>Christoffel symbols</u>.

<u>Remark</u> Any geodesic is parametrised by arc length (proof: show that $\frac{d}{dt}\langle \dot x, \dot x\rangle = 0$ using geodesic equations).

<u>Thm</u> (existence and uniqueness of geodesics) Let M Riemannian mfld, $p \in M$, $v \in T_pM$. Then $\exists\, \varepsilon > 0$ and precisely one geodesic $c : [0,\varepsilon] \to M$ with $c(0) = p$, $\dot c(0) = v$. In addition, c depends smoothly on p and v.

<u>Pf</u> Existence and uniqueness for ODE