Inverse Problems Seminar
Mikko Salo, 4.12.2006 TKK U322

Reconstructing a sound speed from travel times

We will show how to recover a spherically symmetric sound speed in $\mathbb{R}^2$ from travel times. (Result of Herglotz and Wiechert-Zoeppritz, we follow lecture notes of G. Paul)

1. Geodesic flow as Hamilton flow

Consider a Riemannian metric in $\mathbb{R}^n$, given in Cartesian coordinates as $G(x) = (g_{ij}(x))_{i,j=1}^n$. Geodesic equations are

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0,$$

where Γ^i_{jk} are Christoffel symbols. Writing

$$\xi_i(t) = g_{ij}(x(t))\dot{x}^j(t),$$

the geodesic equations are equivalent to the Hamilton system

$$\begin{cases} \dot{x}(t) = \nabla_\xi f(x(t), \xi(t)), & x(0) = x_0 \\ \dot{\xi}(t) = -\nabla_x f(x(t), \xi(t)), & \xi(0) = \xi_0 \end{cases}$$

where $f(x,\xi) = \sqrt{g^{ij}(x)\xi_i\xi_j}$. This corresponds to kinetic energy $E(x,v) = \frac{1}{2} g_{ij}(x)v^iv^j$. Writing $\gamma(t) = (x(t), \xi(t))$ and using the Hamilton vector field

$$H_f = \nabla_\xi f \cdot \nabla_x - \nabla_x f \cdot \nabla_\xi = (\nabla_\xi f, -\nabla_x f),$$

we may write the Hamilton equations as the flow of H_f,

$$\dot{\gamma}(t) = H_f(\gamma(t)), \qquad \gamma(0) = (x_0, \xi_0).$$

Def. A function $u = u(x,\xi)$ is a conserved quantity if it remains constant along the Hamilton flow.

Now u is conserved $\Leftrightarrow$ $\frac{d}{dt} u(x(t), \xi(t)) = 0$ $\Leftrightarrow$ $H_f u(x(t), \xi(t)) = 0$.
Since $H_f f = (\nabla_\xi f, -\nabla_x f)\cdot(\nabla_x f, \nabla_\xi f) = 0$, the Hamilton function f ("energy") is conserved along H_f.

2. Geodesics in $\mathbb{R}^2$

In $\mathbb{R}^2$, let $g_{ij}(x) = c(x)\delta_{ij}$ where $c \in C^\infty(\mathbb{R}^2)$ and $c > 0$. Then $f(x,\xi) = c(x)|\xi|$ and geodesics are obtained from the flow of H_f, where

$$H_f = c(x)\hat{\xi}\cdot\nabla_x - |\xi|\nabla_x c(x)\cdot\nabla_\xi, \qquad \hat{\xi} = \frac{\xi}{|\xi|}.$$

Define angular momentum

$$g(x,\xi) = \xi\cdot x^\perp, \qquad x^\perp = (-x_2, x_1)$$

When is g conserved? We compute

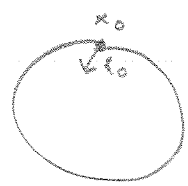
$$H_f g = c(x)\hat{\xi}\cdot(-\xi^\perp) - |\xi|\nabla_x c(x)\cdot x^\perp$$

and $H_f g = 0$ if we assume spherical symmetry

$$c = c(r), \qquad r = |x|.$$

In this case the geodesic flow is completely integrable (under some conditions on c), since it is a Hamilton flow in $\mathbb{R}^2$ with two independent conserved quantities. This implies that the geodesics can be constructed from f and g by quadrature (= rather explicitly).

Consider the flow in $B = B(0,1) \subseteq \mathbb{R}^2$,



$$\begin{cases} \dot{x} = c(r)\hat{\xi} \\ \dot{\xi} = -c'(r)|\xi|\hat{x} \end{cases}$$

with $x(0) \in \partial B$ and $|\xi(0)| = 1$. Write $\xi = (\xi\cdot\hat{x})\hat{x} + (\xi\cdot\hat{x}^\perp)\hat{x}^\perp$. If $r(t) = |x(t)|$,

$$\dot{r} = \frac{x\cdot\dot{x}}{|x|} = \frac{c(r)}{|\xi|}\xi\cdot\hat{x} = \frac{c(r)}{|\xi|}\left(\pm\sqrt{|\xi|^2 - (\xi\cdot\hat{x}^\perp)^2}\right), \qquad \pm\,\xi\cdot\hat{x} \geq 0$$

Now

f conserved $\Rightarrow$ $c(r)|\xi| = c(1)$ $\Rightarrow$ $|\xi| = \dfrac{c(1)}{c(r)}$

g conserved $\Rightarrow$ $\xi\cdot x^\perp = \xi(0)\cdot x(0)^\perp$

Writing $\xi(0) = -\sqrt{1-p^2}\,\hat{x}(0) + p\,\hat{x}(0)^\perp$ where $0 < p < 1$, so $\xi(0)$ points inward, and noting that $\sqrt{1-(\hat{\xi}\cdot\hat{x}^\perp)^2} = \sqrt{1-\left(\frac{p\,c(r)}{r\,c(1)}\right)^2}$, we get

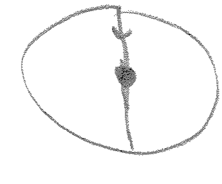
$$\dot{r} = \pm c(r)\sqrt{1-\left(\frac{p\,c(r)}{r\,c(1)}\right)^2}, \qquad \pm\,\xi\cdot\hat{x} \geq 0. \qquad (1)$$

This is an equation for $r(t)$ where all other dependence on t has been eliminated.

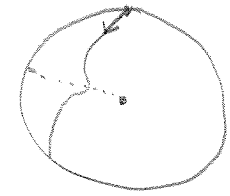
We make some remarks on the validity of (1). One first needs $|\xi(t)| \neq 0$ to divide by $|\xi|$, but since $f = c(r)|\xi|$ is conserved and $|\xi(0)| = 1$, always $|\xi| \neq 0$. One also needs $r(t) \neq 0$ to differentiate $r(t)$, but conservation of g implies

$$r(t) = 0 \Rightarrow x(t) = 0 \Rightarrow g(x(t), \xi(t)) = 0 \Rightarrow p = g(x(0), \xi(0)) = 0.$$

The case $p = 0$ corresponds to lines through 0, which are not so interesting. We assume $0 < p < 1$, and then always $r(t) \neq 0$ and (1) is valid.



Since $\xi(0)$ points inwards, (1) tells that r decreases until some time t_0 for which $\dot r(t_0) = 0$ (if such t_0 exists). Now $\dot r = 0$ if and only if



$$\frac{r}{c(r)} = \frac{p}{c(1)}. \qquad (2)$$

To ensure simple behaviour of geodesics, we assume

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0, \qquad 0 \leq r \leq 1. \qquad (3)$$

This implies that given $p \in (0,1)$, $\exists!$ $r = r_p \in (0,1)$ satisfying (2).

Prop. Assume (3) and let $0 < p < 1$. Then $\exists!$ $t_p > 0$ such that $\dot r(t) < 0$ for $0 \leq t < t_p$, and $\dot r(t_p) = 0$. Further, $r(t_p) = r_p$, and the corresponding geodesic starting at $x(0) \in \partial B$ satisfies $|x(t)| < 1$ for $0 < t < 2t_p$, and $x(2t_p) \in \partial B$. One has $x(t_p + s) = R_p(x(t_p - s))$ where R_p is reflection about $\hat x(t_p)$.

Pf One has $\dot r(t) = 0 \iff x(t) \cdot \xi(t) = 0$. But

$$\frac{d}{dt}(x \cdot \xi) = \dot x \cdot \xi + x \cdot \dot\xi = c(r)|\xi| - rc'(r)|\xi| = |\xi| c(r)^2 \frac{d}{dr}\left(\frac{r}{c(r)}\right) \geq \varepsilon$$

when $0 \leq r \leq 1$. Since $x(0) \cdot \xi(0) < 0$, one must have $x(t_p) \cdot \xi(t_p) = 0$ for some $t_p > 0$ (chosen to be the least such number). It remains to show that $x(t_p + s) = R_p(x(t_p - s))$, but this is a direct computation (both sides satisfy the Hamilton equations). $\square$

Remark The proposition gives a rather complete description of geodesics starting on ∂B and going inwards. The main point is the assumption (3), which implies that any geodesic segment starting and ending on ∂B which otherwise stays inside B is the unique minimizing curve between its endpoints. (Sketch of proof: since ∂B is strictly convex, any minimizing curve in $\bar B$ is a geodesic of $\bar B$, which follows from the first variation formula. The Proposition gives all geodesics, which are unique minimizing since $\frac{\partial t_p}{\partial p} < 0$.) Thus $\bar B$ is SGM

3. Reconstructing the sound speed

The travel times of geodesics starting at $x_0 \in \partial B$, parametrised by $p \in (0,1)$, are given by

$$T(p) = 2t_p = 2\int_{t_p}^{2t_p} dt.$$

Since B is SGM, the boundary distance function determines $T(p)$ by Juha-Matti's talk. The equation (1) reads

$$\frac{dr}{dt} = c(r)\sqrt{1-\left(\frac{pc(r)}{rc(1)}\right)^2}, \qquad t_p \le t \le 2t_p.$$

We use this and change variables to get

$$T(p) = 2\int_{r_p}^{1} \frac{1}{c(r)\sqrt{1-\left(\frac{pc(r)}{rc(1)}\right)^2}}\, dr.$$

Thus, from measurements $T(p)$ with $0<p<1$, we know some integrals involving $c(r)$. We want to recover $c(r)$. Change variables

$$u = \left(\frac{c(1)\, r}{c(r)}\right)^2.$$

This is possible by (3). Then $T(p)$ becomes

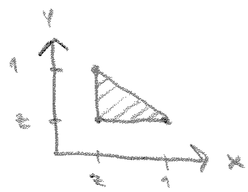
$$T(p) = \frac{2}{c(1)}\int_{p^2}^{1} \frac{dr}{du}\frac{u}{r}\frac{1}{\sqrt{u-p^2}}\, du$$

This is an Abel integral, which can be inverted as shown below. We obtain $f(u) = \frac{dr}{du}\frac{u}{r}$, which gives $r(u) = \exp\left(-\int_u^1 \frac{f(v)}{v}\, dv\right)$. Taking the inverse function, we get $u = u(r)$ and $g(r) = f(u(r)) = \frac{1}{2}\frac{1}{1-\frac{rc'(r)}{c(r)}}$. Thus we know $\frac{rc'}{c}$, so also $\frac{d}{dr}(\log c)$ and $c(r)$.

It remains to prove

Thm Let f be smooth on $(0,1)$. Then the Abel transform

$$g(x) = \int_x^1 \frac{1}{(y-x)^{1/2}} f(y)\, dy$$



has the inverse transform

$$f(y) = -\frac{1}{\pi}\frac{d}{dy}\int_y^1 \frac{g(x)}{(x-y)^{1/2}}\, dx.$$

Pf Compute $\int_z^1 \frac{g(x)}{(x-z)^{1/2}}\, dx = \int_z^1\int_x^1 \frac{f(y)}{(x-z)^{1/2}(y-x)^{1/2}}\, dy\, dx = \int_z^1 k(z,y) f(y)\, dy$, where

$$k(z,y) = \int_z^y \frac{1}{(x-z)^{1/2}(y-x)^{1/2}}\, dx = \int_0^1 \frac{1}{\sqrt{w(1-w)}}\, dw = \int_{-1}^{1}\frac{1}{\sqrt{1-v^2}}\, dv = \pi$$

by taking $x = z + (y-z)w$, $w = \frac{1}{2} + \frac{1}{2}v$, and evaluating the last integral.

Thus $\int_z^1 \frac{g(x)}{(x-z)^{1/2}}\, dx = \pi\int_z^1 f(y)\, dy$, and the result follows by differentiating. □